SOME UNCERTAINTY PRINCIPLES IN ABSTRACT HARMONIC ANALYSIS

John F. Price and *Alladi Sitaram*

The first part of this article is an introduction to uncertainty principles in Fourier analysis, while the second summarizes some recent work by the authors and also by Michael Cowling and the authors.

The following (rather vague) principle is well known to every student of classical Fourier analysis: If a function f is 'concentrated' then its Fourier transform $\hat{f}$ is 'spread out' and vice-versa. After reviewing three precise (and different) formulations of this principle in classical Fourier analysis on $\mathbf{R}^n$, we will describe how it extends to LCA groups and certain nonabelian Lie groups - for instance, semisimple Lie groups and Heisenberg groups.

I **HUP** (Heisenberg Uncertainty Principle) The first one is the celebrated Heisenberg uncertainty principle: For all $f \in L^2(\mathbf{R}^n)$, $a, b \in \mathbf{R}^n$

$$\| |x-a| f \|_2 \| |y-b| \hat{f} \|_2 \geq \frac{n}{4\pi} \|f\|_2^2. \tag{1}$$

A very readable account of this can be found in [4]. Analogous inequalities with different powers of $|x-a|$ and $|y-b|$ and with L^p-norms replacing L^2-norms are given in Cowling and Price [2]. It is easy to see from the above inequality that if we consider compactly supported f supported in an interval around a, with fixed L^2-norm (say equal to 1) and then shrink the support of f to $\{a\}$, then the quantity $\| |y-b| \hat{f} \|_2$ has to blow up, no matter what b is.

II **LUP** (Local Uncertainty Principle) However we would like to say something more. If f is concentrated, not only is $\hat{f}$ spread out, but $\hat{f}$ is not concentrated on <u>any</u> small set E, no matter how E is situated. W.G. Faris [5] and J.F. Price [8,9] have developed a whole host of inequalities to illustrate this. We will just mention one of these due to J.F. Price [8]:

Give $0 < \theta < \frac{1}{2}$, there exists a positive constant k_θ such that for all $f \in L^2(\mathbf{R}^n)$ and measurable $E \subseteq \mathbf{R}^n$,

(2) $$\left(\int_E |\hat{f}(y)|^2 dy\right)^{1/2} \le k_\theta \; m(E)^\theta \; \| |x|^{n\theta} f \|_2$$

Here m denotes Lebesgue measure. (**LUP** is indeed stronger than **HUP** in the sense that inequalities of type (2) imply inequality (1) (though perhaps not with the best possible constant) - see [5,9,12].)

III **QUP** (Qualitative Uncertainty Principle) We next mention a result due to M. Benedicks [1] which can also be viewed as an expression of the principle stated in the beginning.

For $f \in L^1(G)$, let $A_f = \{x: f(x) \neq 0\}$ and $B_f = \{y: \hat{f}(y) \neq 0\}$. (Here we are taking fixed versions of f and $\hat{f}$.) If $m(A_f) < \infty$ and $m(B_f) < \infty$, then $f = 0$ a.e.

(If f is assumed to be compactly supported the above collapses to an easy exercise. However with only the assumption $m(A_f) < \infty$, the result is quite nontrivial. Note also that the supports of f and $\hat{f}$ are $\bar{A}_f$ and $\bar{B}_f$ respectively.)

The first natural question to ask is: How much of the above can be generalized to locally compact abelian (LCA) groups? In 1973, T.

Matolcsi and J. Szücs [7] proved the following restricted version of **QUP**: Let G be an LCA group and $\hat{G}$ its dual group. Let m be a Haar measure and $\hat{m}$ the corresponding dual measure on $\hat{G}$. Let $f \in L^1(G)$ and $\hat{f}$ its Fourier transform (on $\hat{G}$). Retaining earlier notation, if $m(A_f)\hat{m}(B_f) < 1$, then $f = 0$ a.e. However, recently J.A. Hogan [6] has been able to prove **QUP** for general LCA groups G with only the slightly restrictive condition that the identity component of G is noncompact. In [6] Hogan has also been able to extend inequalities of the **HUP** and **LUP** kind to general LCA groups.

We also remark briefly that inequalities of the **HUP** and **LUP** kind can also be considered for compact groups. For the n-torus this has been considered by J.F. Price and P.C. Racki in [10] and for compact Lie groups by the authors in [13]. J.A. Hogan has also some results in this direction.

Let us now turn our attention to noncompact, nonabelian groups. For simplicity, from now on we assume that G is a connected noncompact locally compact unimodular group and $\hat{G}$ its unitary dual, that is, $\hat{G}$ is a maximal set of pairwise inequivalent irreducible unitary representations of G. For each $\pi \in \hat{G}$, let H_π be the corresponding Hilbert space. The Fourier transform $\hat{f}$ of $f \in L^1(G)$ is defined by $\hat{f}(\pi) = \int_G f(x)\pi(x)dx$ for $\pi \in \hat{G}$. Hence $\hat{f}(\pi) \in B(H_\pi)$, the space of bounded linear operators on H_π.

(i) **LUP** Let m be a fixed Haar measure on G and μ the Plancherel measure (or something closely akin to the Plancherel measure) on $\hat{G}$. When G is the Euclidean motion group, a noncompact semisimple Lie group or a Heisenberg group the authors [12] have been able to establish the following analogue of inequality (2): Given $\theta \in (0,1/2)$

there exists a constant k_θ such that for all f in a certain class of functions on G and all measurable $E \subseteq \hat{G}$,

$$\left(\int_E \operatorname{trace}(\pi(f)^* \pi(f))\, d\mu(\pi)\right)^{1/2} \leq k_\theta \mu(E)^\theta \|\phi_\theta f\|_2,$$

where ϕ_θ is a certain weight function on G, measuring the 'concentration' of f, for which an explicit formula is given. (When $G = \mathbf{R}^n$, $\phi_\theta(x) = |x|^{n\theta}$.)

(ii) **HUP** Using the results described above the authors have been able to get versions of **HUP** for symmetric spaces of the noncompact type - see [12].

(iii) **QUP** For a wide variety of groups, including the Heisenberg groups, the Euclidean motion group on the plane, and $K \times \mathbf{R}^n$, where K is compact, the authors have been able to establish results very similar to **QUP** in [11]. In [3] M. Cowling and the authors have established a qualitative uncertainty principle very close to Benedicks' **QUP** for all noncompact connected semisimple Lie groups with finite centre.

In conclusion we would like to say that there are many aspects of uncertainty not mentioned at all in this exposition. For instance in a series of papers W. Schempp has looked at the radar uncertainty principle for the Heisenberg group. See, for example [14]. We have also not gone into any of the many interesting applications of the uncertainty principle, for example, in the theory of partial differential operators. (See "The uncertainly principle" by C. Fefferman in Bull. Amer. Math. Soc. 9 (1983), 129-206. For some aspects of the uncertainty principle on symmetric spaces and its connections with scattering theory see the following preprint of M. Shahshahani: "Poincare inequality, uncertainty principle and scattering theory on symmetric spaces".)

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School of Mathematics
University of New South Wales
Kensington, NSW 2033
Australia

Stat-Math Unit
Indian Statistical Institute
Bangalore 560059
India