DISINTEGRATION OF MEASURES ACCORDING TO THE DIMENSION AND ITS RELATION WITH RANDOM COVERINGS AND MULTIPLICATIVE CHAOS

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1. THE DISINTEGRATION THEOREM

Suppose we are given a locally compact metric or pseudometric space (T,dist) (here pseudometric means

$$\mathrm{dist}(x,y) \leq K(\mathrm{dist}(x,z)+\mathrm{dist}(z,y))$$

and metric is the case $K = 1$). A good example is $\mathbb{R}^d$ with the euclidean distance. We write $M^+(T)$ for the set of all positive and bounded Radon measures on T, $M^+_\alpha(T)$ for the subset of $M^+(T)$ which consists of α-Lipschitz measures, that is, measures σ which satisfy

$$\sigma(B) \leq C_\sigma(\mathrm{diam}\ B)^\alpha$$

for all balls B.

Given a Borel set B in T we consider the measures $\sigma \in M^+(T)$ carried by B (that is, $\sigma(B) = \sigma(T)$) and we write $\sigma \in M^+(B)$. We define $M^+_\alpha(B)$ in the same way. The capacitary (or Polya-Szegö) dimension of B is the supremum of the $\alpha > 0$ such that $M^+_\alpha(B) \neq \{0\}$. If T is $\mathbb{R}^d$ with the usual metric the capacitarian dimension is the same as the Hausdorff dimension (theorem of Frostman). In any case we denote it by dim B.

Given $\sigma \in M^+(T)$ we say that σ is unidimensional and has dimension α ($0<\alpha<\infty$) when σ is carried by a Borel set of dimension α (dimension = capacitary dimension) and the σ-measure of any Borel set of dimension $< \alpha$ vanishes. Unidimensional with dimension 0 or dimension ∞ are defined in the obvious way.

THEOREM *Given* $\lambda \in M^+(T)$ *one can write*

$$\lambda = \int_{[0,\infty]} \mu_\alpha d\upsilon(\alpha) \tag{1.1}$$

where $d\upsilon$ *belongs to* $M^+([0,\infty]$ *with* $\int_{[0,\infty]} d\upsilon = \lambda(T)$ *and the* μ_α *are probability measures such that the mapping* $\alpha \to \mu_\alpha$ *is vaguely measurable with respect to* $d\upsilon$ *(so that* (1.1) *has a meaning as a vague integral), and for* $d\upsilon$*-almost each* α *the measure* μ_α *is unidimensional with dimension* α*. Moreover the disintegration* (1.1) *is unique if we identify two mappings* $\alpha \to \mu_\alpha$ *which coincide* $d\upsilon$*-almost everywhere.* (We use *vague* as in Bourbaki. When T is compact, it means weak*.)

Here is a sketch of the proof, where we shall introduce a few notions of potential theory needed in the sequel. For simplicity we restrict ourselves to the case T compact. Given $\lambda \in M^+(T)$ and $\sigma > 0$, the α-potential of λ is the function

$$V_\alpha(t) = \int (\mathrm{dist}(t,s))^{-\alpha} d\lambda(s)$$

and the α-energy of λ is

$$I_\alpha = \int V_\alpha d\lambda = \iint (\mathrm{dist}(t,s))^{-\alpha} d\lambda(s) d\lambda(t).$$

We say that λ is α-singular if its α-potential is infinite λ-a.e. and that λ is α-regular if it is a countable sum of measures with finite α-energies. We say that a Borel set has vanishing α-capacity and we write $\mathrm{Cap}_\alpha B = 0$ if $\lambda(B) = 0$ for all α-regular measures λ. Here are the main steps.

LEMMA 1 *If* λ *is carried by a Borel set* S

$$\sup_{t \in T} V_\alpha(t) \le (2K)^\alpha \sup_{s \in S} V_\alpha(s)$$

LEMMA 2 *For*

$$S(\alpha,\lambda) = \{t : V_\alpha(t) = \infty\}$$

we have $\mathrm{Cap}_\alpha S(\alpha,\lambda) = 0$.

LEMMA 3 *For*

$$\lambda_\alpha = \lambda \, 1_{S(\alpha,\lambda)},$$

λ_α *is* *α-singular and* $\lambda-\lambda_\alpha$ is *α-regular.*

LEMMA 4 *Let us define* $\nu(0) = 0$, $\nu(\alpha) = \lambda_\alpha(T)$ $(0<\alpha<\infty)$, $\nu(\infty) = \lambda(T)$. *The derivatives with respect to* $d\nu$

$$\lim_{h\to 0} \frac{\lambda_{\alpha+h}-\lambda_\alpha}{\nu(\alpha+h)-\nu(\alpha)} = \mu_\alpha$$

$$\lim_{\ell\to\infty} \frac{\lambda-\lambda_\ell}{\nu(\infty)-\nu(\ell)} = \mu_\infty$$

exist $d\nu$*-a.e. as weak limits in* $M^+(T)$ *(in duality with* C(T)) *and define a* ν*-weakly measurable function of* α.

LEMMA 5 μ_α *is unidimensional witih dimension* α.

2. RANDOM COVERING

Random coverings of the circle Π or the torus Π^d are considered in my book Some Random Series of Functions (Health 1968, CUP 1985). Between the 1968 and the 1985 edition the main result was the complete solution by L. Shepp of the Dvoretzky covering problem for the circle. Since 1985 the most important contribution is due to Svante Janson (Acta Mathematica, 1986). Using S. Janson's ideas it is possible to give a new proof of Shepp's result, and to get a complete solution for the covering problem of a subset of the circle.

Here is the problem. We are given a sequence $1>\ell_1\geq\ell_1\geq\ldots>0$ and we consider random intervals

$$I_n = (0,\ell_n) + \omega_n$$

on the circle Π (ω_n are independent random variables with uniform distribution on Π). Given a closed (or Borel) subset F of Π the covering problem is to decide whether or not

$$P(F \subset \bigcup_1^\infty I_n) = 1 \tag{2.1}$$

When (2.1) holds we say that covering holds. We consider the kernel

$$k(t) = \exp \sum_1^\infty (\ell_n - t)^+ \tag{2.2}$$

and we say that F has vanishing k-capacity and write $\mathrm{Cap}_k F = 0$ if F carries no measure of finite energy with respect to k, that is

$$\iint_{F\times F} k(t-s)d\mu(t)d\mu(s) = \infty$$

for all non-zero measures μ carried on F.

THEOREM *Covering holds if and only if*

$$\mathrm{Cap}_k F = 0 \tag{2.3}$$

In particular when the Lebesgue measure of F *is positive*, (2.3) *reads*

(2.4) $$k \notin L^1(\Pi)$$

Actually (2.4) is one of the forms of Shepp's necessary and sufficient condition. Another particular case is $\ell_n = \frac{a}{n}$ $(0<a<1)$. Then (2.3) reads, $\text{Cap}_a F = 0$ (same notation as in part 1).

Though I shall not try to give the proof, let me point out the connection with the decomposition of measures. Let us write $\chi_n(t-\omega_n)$ for the indicator function of I_n and

(2.5) $$Q_N(t) = \prod_1^N \frac{1-\chi_n(t-\omega_n)}{1-\ell_n} .$$

Then

$$Q_N(t) = 0 \Leftrightarrow t \in \bigcup_1^N I_n .$$

Given a measure $\lambda \in M^+(F)$ the random measures $Q_N\lambda$ converge weakly with probability one to a random measure which I denote by $Q\lambda$. If $Q\lambda \neq 0$, F is not covered a.s. : it is the easy part of the proof. In any case it is possible to write

(2.6) $$\lambda = E(Q\lambda) + \lambda - E(Q\lambda)$$

and it is a decomposition of λ into a Q-regular part and a Q-singular part. The operator EQ is a projection, whose image consists of Q-regular measures while the kernel consists of Q-singular measures. In the case $\ell_n = \frac{a}{n}$, Q-regular means exactly a-regular (in the sense of part 1) and Q-singular means a-singular $(0<a<1)$.

The analogue for Π^d $(d>1)$ is still not known.

3. PRODUCTS OF INDEPENDENT WEIGHTS

(2.5) is an example of a more general situation. Suppose that T is as in part 1 and that we are given independent random weights $P_n(t,\omega)(n = 1,2,\ldots;\ \omega \in \Omega$ the probability space; $t \in T)$, that is

1) for almost all ω and all n $P_n(\cdot,\omega)$ is ≥ 0 and Borel
2) for all t and n $P_n(t,\cdot)$ is a positive random variable with expectation 1
3) the σ-fields generated by the random functions $t \to P_n(t,\cdot)$ are independent.

We consider the products

$$Q_N(t,\omega) = \prod_{n=1}^{N} P_n(t,\omega) \tag{3.1}$$

and how they operate on a given measure $\lambda \in M^+(T)$.

THEOREM *With probability 1 the random measures* $Q_N\lambda$ *converge vaguely to a random measure which we denote by* $Q\lambda$. *The operator* EQ *from* $M^+(T)$ *to* $M^+(T)$ *defined by*

$$\text{(3.2)} \qquad EQ\lambda(B) = E((Q\lambda)(B)) \quad (B\text{:Borel})$$

is a projection. If λ *is in the kernel of* EQ *(we say that* λ *is Q-singular) then* $Q\lambda = 0$ *a.o.. If* λ *is in the image of* EQ *(we say that* λ *is Q-regular) we have* $EQ\lambda = \lambda$, *in the sense that both members of* (3.2) *equal* $\lambda(B)$.

The first part of the theorem (the definition of $Q\lambda$) depends only on the fact that $Q_N(t,\omega)$ is a positive martingale for each t, therefore

$$\int f(t)Q_N(t,\omega)d\lambda(t)$$

is a positive martingale for each $f \in C_K^+(T)$ (continuous, compact support, positive). The second part (the projection property of EQ) depends strongly on the definition of $Q_N(t,\omega)$ as (3.1); it is not true for general martingales. For the proofs see [2], [3]..

EQ is a regularizing operator and (2.6) has the same meaning as before; it is a decomposition of λ into a Q-regular part and a Q-singular part. In the example above we had a complete description of a Q-singular measure (carried by a Borel set of vanishing k-capacity) and a Q-regular measure (countable sum of measures of k-finite energy). Here is another example.

4. MULTIPLICATIVE CHAOS

This originated as a random model of turbulence, proposed by B. Mandelbrot in 1972 as an improvement of a very sketchy model by A. Kolmogorov (1961). For the history of the question see [1] or [5].

Now we assume that the random weights $P_n(t,\omega)$ are log-normal, that is,

(4.1) $$P_n(t,\omega) = \exp((X_n(t,\omega)-\frac{1}{2}EX_n^2(t,\cdot))$$

where the X_n are independent gaussian (we mean centered gaussian) processes. The laws of the $X_n(t,\omega)$ (that is, all joint laws of random variables of the form $X_n(t_1,\cdot)$, $X_n(t_2,\cdot),\ldots X_n(t_k,\cdot))$ are well defined by the covariance functions

(4.2) $$p_n(t,s) = E(X_n(t)X_n(s)) \quad (t,s \in T)$$

which are kernels of positive type (=positive definite kernels). The same holds for the laws of the $P_n(t,\omega)$. The law of $Q_N(t,\omega)$ (the normalised exponential of $X_1(t,\omega)+X_2(t,\omega)+..+X_N(t,\omega))$ depends only on

(4.3) $$q_N(t,s) = p_1(t,s)+..+p_N(t,s)$$

This makes the following theorem plausible.

THEOREM *Suppose* $p_n(t,\omega) \geq 0$ *for all* n,t,ω, *and write*

(4.4) $$q(t,s) = \sum_{n=1}^{\infty} p_n(t,\omega).$$

Then the law of the random operator Q *depends only on the function* $q(t,s)$ $(t,s \in T)$.

For the proof see [1],[2].

Here the law of Q means the collection of all joint laws of random variables of the form $Q\lambda_1(B_1), Q\lambda_2(B_2), \ldots, Q\lambda_k(B_k)$. A number of questions can be considered and sometimes solved about the operator Q. When is Q completely degenerate, that is, $Q\lambda = 0$ almost surely, for all λ's? If this is not the case, when is it true that $Q\lambda = 0$ almost surely, (λ is Q-singular) or $EQ\lambda = \lambda$ (λ is Q-regular)? When λ is Q-regular, what can we say about the random measure $Q\lambda$, in particular, its dimensions (in the sense of part 1), and the boundedness of the moments $E(Q\lambda(B))^h$, $(h>1)$?

To have an idea let us consider the case when $T = \mathbb{R}^d$ euclidean and

$$(4.5) \qquad q(t,s) = q_u(t,s) = u \log \frac{1}{\|t-s\|} \quad (\|t-s\| \leq \tfrac{1}{2}).$$

It can be seen easily that $q(t,s)$ can be written in the form (4.4) (in many ways, of course; the theorem above says that the decomposition does not matter). The situation depends on the parameter u $(u>0)$.

THEOREM *When* $u \geq 2d$ Q *is completely degenerate. Suppose now* $0<u<2d$, $\lambda \in M^+(T)$ *and write* λ *in the form* (1.1), *that is*

$$\lambda = \int_{[0,d]} \mu_\alpha d\upsilon(\alpha).$$

Then

$$\int_{]\frac{u}{2},d]} \mu_\alpha d\upsilon(\alpha) \leq EQ\lambda \leq \int_{[\frac{u}{2},d]} \mu_\alpha d\upsilon(\alpha)$$

that is, EQ *kills all components of dimension* $> \frac{u}{2}$ *and keeps all components of dimension* $< \frac{u}{2}$. *Given a non-zero unidimensional component* μ_α *with* $\alpha > \frac{u}{2}$, $Q\mu_\alpha$ *is a.s. unidimensional with dimension* $\alpha - \frac{u}{2}$ *and the moments* $E(Q\mu_\alpha(B))^h$ *(where* B *is any ball, and* $h>1$*) are finite when* $uh<2\alpha$ *and infinite when* $uh>2\alpha$.

For the proof see [1].

REFERENCES

Part 1: none
Part 2: [4]
Part 3: [2],[3]
Part 4: [1],[2],[5]

[1] J.-P. Kahane Sur le chaos multiplicatif, Ann.Sc.Math. Quebec, 1985, vol.9, 105-150. A summary can be found in C.R. Acad. Sc.Paris 301 (1985) 329-332

[2] ———— Positive martingales and random measures, Chinese Ann. of Math. 8B(1) (1987) 1-12

[3] ———— Multiplications aléatoires et dimensions de Hausdorff, Ann.Inst.H.Poincaré (1987), to appear

[4] ———— Intervalles aléatoires et decompositions de mesures, C.R. Acad.Sc.Paris 304 (1987) 551-554

[5] B. Mandelbrot The fractal geometry of nature, Freeman, 1982

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