A Heat Equation Approach to Boutet de Monvel's Index Theorem for Toeplitz Operators

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§1. The index theorem on the circle

In this talk, we will show how heat-kernel methods can be used to prove Boutet de Monvel's index theorem for Toeplitz operators on a compact strictly pseudo-convex CR manifold. Since this theorem implies the Atiyah-Singer index theorem for an arbitrary elliptic pseudodifferential operator (the proof of this fact makes use of an explicit Fourier integral operator, see the appendix to [4]), this shows how to give a purely analytic proof of the Atiyah-Singer index theorem, without any use of K-theory. In fact, we will only give the first step in the argument, the proof of a McKean-Singer formula for Toeplitz operators; the actual calculation of the index will be found in [6].

Some of the ideas of the proof are already apparent on the simplest of CR manifolds, the circle (where Toeplitz operator and pseudodifferential operators are essentially the same), so in this section, we will give an outline of the calculation in this case.

Let $p(x,\xi) \in S^k(S^1) \otimes M_N$, $k > 0$, be the symbol of an $N \times N$ elliptic pseudodifferential system on the circle (we will need to assume that $[p(x,\xi), p^*(x,\xi)] = 0$ for all $x \in S^1$, $\xi \in \mathbb{R}$), and let $P_\hbar = p(x, iD/\hbar)$ be its quantization. By the McKean-Singer formula, the index of the pseudodifferential operator $P_\hbar$ (which is of course independent of $\hbar > 0$, by homotopy invariance) is equal to the trace

$$\operatorname{ind}(P_\hbar) = \operatorname{Tr}\left(e^{-P_\hbar{}^* P_\hbar} - e^{-P_\hbar P_\hbar{}^*}\right).$$

We can calculate this trace in the limit in which $\hbar \to 0$, by calculating the symbol of the operator $e^{-P_\hbar{}^* P_\hbar} - e^{-P_\hbar P_\hbar{}^*}$. The details are a bit different from the calculation of the index of a Dirac operator in [5], since there, we needed the leading order of the symbol of the heat kernel to calculate the index, whereas here, we will need the subleading order, that is, the coefficient of $\hbar$ in the asymptotic expansion in powers of $\hbar$.

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Let $\{f,g\}$ denote the Poisson bracket of two functions on T^*S^1, given by the formula

$$\{f,g\} = \frac{\partial f}{\partial \xi}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial \xi}.$$

The symbol of the composition $f(x,iD/\hbar)\,g(x,iD/\hbar)$ is given by the asymptotic formula

$$\sum_{n=0}^{\infty} \frac{i^n}{\hbar^n n!}\frac{\partial^n f}{\partial \xi^n}\frac{\partial^n g}{\partial x^n}.$$

It follows easily that the symbol of the operator $e^{-tP_\hbar{}^* P_\hbar} - e^{-tP_\hbar P_\hbar{}^*}$ is equal to

$$te^{-t|p(x,\xi)|^2}\left(\hbar\{p^*,p\}/i + O(\hbar^2)\right).$$

We now use the following formula for the trace of a pseudodifferential operator: if $p(x,\xi)$ is a symbol in $S^k(S^1)\otimes M_N$, where $k<-1$, so that $P_\hbar$ is trace class, then

$$\operatorname{Tr} P_\hbar = \frac{1}{2\pi\hbar}\int_{T^*S^1} \operatorname{Tr} p(x,\xi)\,dx d\xi.$$

Using this formula, we obtain the following result:

THEOREM 1.1. *The index of the pseudodifferential operator $p(x,iD/\hbar)$ is equal to*

$$\frac{1}{2\pi i}\int_{T^*S^1} \operatorname{Tr}\left(\{p^*,p\}e^{-|p|^2}\right) dx d\xi \quad \blacksquare$$

It would be nice to be able to imitate this calculation to obtain the index of an elliptic operator on a higher dimensional manifold. The difficulty is that there, we would have to calculate the coefficient of $\hbar^{\dim M}$ in the asymptotic expansion of the symbols of the operators $e^{-P_\hbar{}^* P_\hbar}$ and $e^{-P_\hbar P_\hbar{}^*}$. However, the index of a Toeplitz operator on a compact strictly pseudo-convex CR manifold can be calculated using the symbol calculus for pseudodifferential operators on Clifford modules [5]; it then suffices to calculate the $O(\hbar)$-term in the asymptotic expansion for the symbol of the heat kernel. In the rest of this talk, we will present the first step in this proof, a McKean-Singer formula for the index of a Toeplitz operator.

§2. THE SZEGÖ PROJECTOR

Let M be a compact $2n+1$-dimensional strictly pseudo-convex CR manifold, and let E be a holomorphic vector bundle on M, with a Hermitian inner product. The subelliptic Dirac operator of the bundle E is the self-adjoint first-order differential operator $\mathcal{D}_b = \bar\partial_b + \bar\partial_b^*$, which acts on sections of the bundle of anti-holomorphic forms $\Lambda^{0,*}M\otimes E$. In this section, following Boutet de Monvel and Guillemin [4], we will define the Szegö projector of

E; roughly speaking, it is the projection onto the kernel of $\mathcal{D}_b$. Our presentation differs from that of Boutet de Monvel and Guillemin in that for us, the Szegö projector acts on the bundle of antiholomorphic differential forms $\Lambda^{0,*}M \otimes E$, which turns out to simplify the formulation of a number of results.

We will denote Hörmander's class of symbols $S^k_{1/2,1/2}$ by $S^k_{1/2}$, and the corresponding space of pseudodifferential operators by $\operatorname{Op} S^k_{1/2}$. We will make frequent use of the fact that $[\operatorname{Op} S^k_{1/2}, \operatorname{Op} S^l] \subset \operatorname{Op} S^{k+l-1/2}_{1/2}$. If E is a vector bundle on M, $S^k(E)$ will be the space of kth order symbols acting on E, whereas S^k is the space of scalar pseudodifferential symbols on M.

DEFINITION 2.1. *Let $L \subset T^*M$ be the line bundle of one-forms which vanish on $T^{1,0}M \oplus T^{0,1}M$. A Szegö projector for the bundle E is a projection in $\operatorname{Op} S^0_{1/2}(\Lambda^{0,*}M \otimes E)$ satisfying the following condition:*

If the symbol $p \in S^k(\Lambda^{0,*}M \otimes E)$ vanishes along L, then the operators $Sp(x,D)$ and $p(x,D)S$ are in $\operatorname{Op} S^{k-1/2}_{1/2}(\Lambda^{0,*}M \otimes E)$. (An example of such an operator is the Dirac operator $\mathcal{D}_b$.) ∎

In this section, we will construct such a Szegö projector for a Hermitian vector bundle E. The proof makes use of the following idea: just as on a Riemannian manifold, the pseudodifferential symbol calculus is modeled on the algebra of Fourier multipliers on $\mathbb{R}^n$, there is a calculus of pseudodifferential operators on an M modeled on the algebra of left-invariant pseudodifferential operators on the Heisenberg group $H_n \cong \mathbb{R}^{2n} \times \mathbb{R}$. We start by discussing this algebra. For a more complete discussion of the Heisenberg pseudodifferential calculus, the reader should refer to the books of Beals and Greiner [1], and its bibliography.

Let X_i $(1 \le i \le 2n)$ and $X_0 = T$ be the left-invariant vector fields on H_n, so that a typical point in $\mathbf{h}_n$ may be denoted by $\sum_{i=0}^{2n} v_i X_i$; here $(v_1, \dots, v_{2n})$ is in the symplectic vector space $\mathbb{R}^{2n}$ and $v_0 \in \mathbb{R}$. We will denote an element of $\mathbf{h}_n^*$ by (ξ, τ), where $\xi \in (\mathbb{R}^{2n})^*$ and $\tau \in \mathbb{R}$. Let $\hat{p} \in \mathcal{D}'(\mathbf{h}_n^*)$ be the Fourier transform of the symbol p. A left-invariant pseudodifferential operator on the Heisenberg group may be written in the form

$$\operatorname{Op} p = (2\pi)^{-n-1} \int_{\mathbf{h}_n} \hat{p}(v) \exp(v.X)\, dv.$$

An example of this quantization rule is that if p is a polynomial function, the corresponding operator is a differential operator; all elements of the enveloping algebra of $\mathbf{h}_n$ are describable in this way.

We will now describe a class of symbols suitably generalizing the polynomials, such that the corresponding operators form an algebra.

DEFINITION 2.2. *The symbol class N^k is defined as the space of smooth functions on $\mathbf{h}_n^*$ satisfying the following estimates: each $N \geq 0$,*

$$|\partial^\alpha p(\xi,\tau)| \leq c_\alpha R^{k-\|\alpha\|},$$

where $R(\xi,\tau) = (1+|\xi|^4+|\tau|^2)^{1/4}$; $\|\alpha\|$ is the positive integer obtained by adding 1 for each X in the expression for ∂^α, and 2 for each power of T.

For example, the symbol of a left-invariant differential operator ∂^α on $\mathbf{h}_n$ lies in $N^{\|\alpha\|}$. Note that if $m \geq 0$, then the space of symbols N^m is contained within the space $S^m_{1/2}(\mathbf{h}_n^*)$, while if $m < 0$, $N^m \subset S^{-m/2}_{1/2}(\mathbf{h}_n^*)$.

The space $N^\infty = \bigcup_{k\in\mathbb{Z}} N^k$ is an algebra under the following product:

$$\mathrm{Op}(p \star q) = \mathrm{Op}\, p\, \mathrm{Op}\, q.$$

The composition $p\star q$ thus defined is bounded from $N^k \times N^l$ to N^{k+l}.

It is not so hard to demonstrate the following formula for this composition law (here, ω_{ij} is the standard symplectic form on $\mathbb{R}^{2n}$, obtained by identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and taking the imaginary part of its standard Hermitian form):

$$(p\star q)(\xi,\tau) = (\tau/4\pi)^n \int p(\xi+\alpha,\tau)q(\xi+\beta,\tau)e^{2\omega_{ij}\alpha_i\beta_j/\tau}\, d\alpha\, d\beta.$$

In particular, if p is a polynomial symbol, then there is the more explicit formula:

$$(p\star q)(\xi,\tau) = \exp\left(\tfrac{1}{2} i\tau\omega_{ij}\partial_{\xi_i}\partial_{\eta_j}\right).\, p(\xi,\tau)q(\eta,\tau)\big|_{\xi=\eta}.$$

This should be interpreted as follows: for each $\tau \neq 0$, the symbol $p(\xi,\tau)$ is to be thought of as the Weyl symbol of a pseudodifferential operator on $\mathbb{R}^n$, with Planck's constant τ^{-1}, and the symbol $p\star q$ is just the composition of these Weyl symbols at fixed τ. As an example, the symbol $|\xi|^2$ corresponds for each τ to a harmonic oscillator, and the symbol

$$p_t(\xi,\tau) = (\cosh t\tau)^{-n/2} e^{-\tanh t\tau . |\xi|^2/\tau}$$

corresponds to the heat kernel of this oscillator. It follows that

$$p_\infty(\xi,\tau) = \lim_{t\to\infty} p_t(\xi,\tau) = 2^{n/2} e^{-|\xi|^2/\tau}$$

is the symbol of the projection onto the kernel of the operator with symbol $|\xi|^2$.

An almost-Heisenberg manifold M is a $2n+1$-dimensional contact manifold with a reduction of its structure group from $\mathrm{Sp}(2n)$ to the maximal compact subgroup $\mathrm{U}(n)$. Thus, an almost-Heisenberg manifold is quite analogous to a Hermitian almost-complex manifold, which is determined by the reduction of the structure group $\mathrm{GL}(n,\mathbb{C})$ to its maximal compact subgroup $\mathrm{U}(n)$. Observe that the complex tangent bundle $T_{\mathbb{C}}M$ of an almost-Heisenberg manifold splits canonically into three sub-bundles:

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{C}T.$$

Since the quotient $\mathrm{Sp}(2n)/\mathrm{U}(n)$ is contractible (indeed, diffeomorphic to $\mathbb{R}^{n^2+n}$), it follows that contact manifolds always have almost-Heisenberg structures. Indeed, one obtains such a reduction by giving a metric on the $2n$-dimensional sub-bundle of TM on which the contact form vanishes. This automatically defines a complex structure on this sub-bundle, since $\mathrm{U}(n) = \mathrm{Sp}(2n) \cap \mathrm{O}(2n)$.

Webster has shown [7] that an almost-Heisenberg manifold has a canonical connection on its tangent bundle compatible with its $\mathrm{U}(n)$-structure, analogous to the Levi-Civita connection on a Riemannian manifold. This connection may be described in terms of the commutation relations between the canonical horizontal vector fields on the principal bundle $\mathrm{U}(M)$, which are X_i $(1 \le i \le 2n)$, corresponding to the directions in which the contact form vanishes, and T, which satisfies $\iota(T)\theta = 1$ and $\iota(T)d\theta = 0$. The formulas for the commutators of these fields are of the following form (here, X_{ij} form a basis for the vertical vector fields on $\mathrm{U}(M)$):

$$\begin{aligned} [X_i, X_j] &= \omega_{ij}T + O(X_{ij}) \quad ; \\ [X_i, T] &= O(X_{ij}) + O(X_j) \quad . \end{aligned}$$

We also need the commutation relations for the vector fields X_i and T acting on a vector bundle E with connection ∇ over M, or rather, the corresponding covariant derivatives acting on the vector bundle $\pi^*(E)$:

$$\begin{aligned} [\nabla_i, \nabla_j] &= \omega_{ij}\nabla_T + O(X_{ij}) + O(1); \\ [\nabla_i, \nabla_T] &= O(X_{ij}) + O(\nabla_j) + O(1). \end{aligned}$$

We will now define the pseudodifferential operator algebra of an almost-Heisenberg manifold, keeping things as similar to the case of the Heisenberg group as possible. It is easiest to do this by working on the principal frame bundle $\mathrm{U}(M)$; there, we have the horizontal vector fields X_i and T which behave as a kind of ersatz Heisenberg algebra, in the sense that their commutation relations are the same as the corresponding vector fields on the Heisenberg group up to lower order corrections involving the curvature of the Webster connection.

We say that $p(x,\xi,\tau)$ is a symbol, and write $p \in N^k(M)$, if it is a $\mathrm{U}(n)$-equivariant map from P to N^k. We can now imitate the definition we gave for quantization of a symbol on the Heisenberg group: acting on the pullback to $\mathrm{U}(M)$ of a function on M, the quantization of $p(x,\xi,\tau)$ ($x \in \mathrm{U}(M)$, $(\xi,\tau) \in \mathbf{h}_n^*$) is given by

$$\operatorname{Op} p = (2\pi)^{-2n-1} \int_{\mathbf{h}_n} \hat{p}(x,v) \exp(v.X)_* \, dv$$

Here, $\hat{p}$ is the Fourier transform along $\mathbf{h}_n$ of the symbol p; thus, it is a $\mathrm{U}(n)$-equivariant map from $\mathrm{U}(M)$ to the space of distributions on $\mathbf{h}_n$. This definition is explicitly invariant under the action of $\mathrm{U}(n)$, so that the operator $\operatorname{Op} p$ descends to an operator on M, which we denote in the same way.

It follows directly from this definition that the operator $\operatorname{Op} p$, $p \in N^k(M)$ is trace class if $k < -2(n+1)$, in which case its trace is given by the following formula:

$$\operatorname{Tr} \operatorname{Op} p = (2\pi)^{-2n-1} \int_{T^*M} p(x,\xi,\tau) \, dx \, d\xi \, d\tau.$$

The collection of all operators on $\mathcal{D}'(M)$ of the form $\operatorname{Op} p + K$ where K is an infinitely smoothing operator, is denoted by $\operatorname{Op} N^k(M)$. We let $\operatorname{Op} N^{-\infty}(M)$ denote the algebra of smoothing operators on M, while $N^{-\infty}(M)$ denotes $\bigcap_{k\in\mathbb{Z}} N^k(M)$. The following theorem summarizes the main properties of this class of operators (a proof may be found in [1]).

THEOREM 2.3.

1) The map which sends a symbol $p(x,\xi,\tau)$ to its quantization $\operatorname{Op} p$ induces an isomorphism between the spaces $N^k/N^{-\infty}(M)$ and $\operatorname{Op} N^k / \operatorname{Op} N^{-\infty}(M)$.

2) The composition of operators defines a bounded map from $\operatorname{Op} N^k(M) \times \operatorname{Op} N^l(M)$ to $\operatorname{Op} N^{k+l}(M)$. Thus, it induces a product

$$N^k(M)/N^{-\infty}(M) \times N^l(M)/N^{-\infty}(M) \longrightarrow N^{k+l}(M)/N^{-\infty}(M),$$

which is denoted by $p \circ q$.

3) The leading order part of the symbol $p \circ q$ is equal to $p \star q$.

4) If $m \geq 0$, then $\operatorname{Op} N^m(M) \subset \operatorname{Op} S^m_{1/2}(M)$, while if $m < 0$, $\operatorname{Op} N^m(M) \subset \operatorname{Op} S^{m/2}_{1/2}(M)$. In particular, operators in $\operatorname{Op} N^0(M)$ are bounded on $L^2(M)$, while operators in $\operatorname{Op} N^{-m}(M)$ are compact. ∎

This theorem has an obvious generalization to pseudodifferential operators acting on a bundle E on M—the symbol is now taken to be a $\mathrm{U}(n)$-equivariant map from $\mathrm{U}(M)$ to $N^k \otimes \operatorname{End}(\pi^* E)$, and the quantization of a symbol is defined by the formula

$$\operatorname{Op} p.f(x) = (2\pi)^{-2n-1)} \int_{\mathbf{h}_n} \hat{p}(x,v) \exp(v.\nabla)_* f \, dv.$$

The most important example of a pseudodifferential operator in the Heisenberg calculus is the subelliptic Dirac operator $\mathcal{D}_b = \bar{\partial}_b + \bar{\partial}_b^*$, which is in $N^1(\Lambda^{0,*}M \otimes E)$. By the calculation of Folland and Stein, the leading symbol of $\mathcal{D}_b$ on $\Lambda^{0,k}M \otimes E$ equals be

$$|\xi|^2 + (2k-n)\tau.$$

The operator $\mathcal{D}_b$ is not hypoelliptic.

THEOREM 2.4. *Let $p_t(\xi,\tau)$ be as the leading symbol of the heat kernel $e^{-t\mathcal{D}_b^2}$, which by Mehler's formula equals*

$$(\cosh t\tau)^{-n} e^{-\tanh t\tau . |\xi|^2/\tau - t\tau(2k-n)}.$$

There is a projection $S \in \operatorname{Op} N^0(\Lambda^{0,*}M \otimes E)$ with leading symbol equal to

$$\begin{aligned} p_\infty(\xi,\tau) &= \lim_{t\to\infty} p_t(\xi,\tau) \\ &= \begin{cases} 2^n e^{-|\xi|^2/|\tau|} & k=0 \text{ and } \tau>0 \text{ or } k=n \text{ and } \tau<0 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and an operator $Q \in \operatorname{Op} N^{-2}(\Lambda^{0,*}M \otimes E)$ with leading symbol

$$q(\xi,\tau) = \int_0^\infty (p_t(\xi,\tau) - p_\infty(\xi,\tau))\, dt,$$

such that $(Q \circ \mathcal{D}_b^2 + S) - 1$ and $(\mathcal{D}_b^2 \circ Q + S) - 1$ are in $\operatorname{Op} N^{-1}(\Lambda^{0,*}M \otimes E)$.

PROOF: Let S_0 be the self-adjoint operator $\operatorname{Op} p_\infty$. By the symbol calculus, we see that $S_0^2 - S_0$ is in $\operatorname{Op} N^{-1}(\Lambda^{0,*}M \otimes E)$. It follows that the spectrum of S_0 has as its only two limit points the values 0 and 1, so that by removing two disjoint discs from around 0 and 1 in $\mathbb{C}$, we are left with a domain U on which $S_0 - z$ is invertible. The resolvent $(S_0 - z)^{-1}$ can be shown by standard methods to be a smooth map from U to $\operatorname{Op} N^0(\Lambda^{0,*}M \otimes E)$, so that by taking γ to be a contour in U around $z=1$ that doesn't circle $z=0$, we obtain a projection S with leading symbol $p_\infty(\xi,\tau)$:

$$S = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{S_0 - z}.$$

To finish the proof, we use the symbol calculus to show that any operator Q with symbol $q(\xi,\tau)$ satisfies

$$\begin{aligned} &Q \circ \mathcal{D}_b^2 + S \in I + \operatorname{Op} N^{-1}(\Lambda^{0,*}M \otimes E) \quad \text{and} \\ &\mathcal{D}_b^2 \circ Q + S \in I + \operatorname{Op} N^{-1}(\Lambda^{0,*}M \otimes E) \quad ; \end{aligned}$$

this follows from the corresponding equality at the symbol level, namely that $\sigma(\mathcal{D}_b^2) \circ p_\infty(\xi,\tau) = p_\infty(\xi,\tau) \circ \sigma(\mathcal{D}_b^2) = 0$. ∎

The operator S constructed in this theorem is a Szegö operator for the bundle E. It is clear that any two Szegö operators constructed in this way will differ by an element of $N^{-1}(\Lambda^{0,*}M \otimes E)$.

§3. McKEAN-SINGER FORMULA FOR TOEPLITZ OPERATORS

In this section, we will prove our McKean-Singer formula for Toeplitz operators, which is an adaptation of Boutet de Monvel's idea [2], of converting the index of a Toeplitz operator into the index of a pseudodifferential operator.

Let S be a Szegö projector for the bundle E. An mth-order Toeplitz operator T_P on the bundle E is the compression to $S[\Gamma(\Lambda^{0,*}M) \otimes E]$ of a pseudodifferential operator $P \in \operatorname{Op} S^m(E)$ with scalar symbol, that is,

$$T_P = SPS.$$

For studying the properties of the algebra of Toeplitz operators (for example, to show that they form an algebra) we would need to make more assumptions on the operator S. However, for calculating the index of a Toeplitz operator, all that we need to know is the above definition.

If T_P is an mth-order Toeplitz operator, we define its leading symbol to be the restriction to the line bundle L of the leading symbol of P. We say that the Toeplitz operator T_P is elliptic if its leading symbol is invertible.

PROPOSITION 3.1. *An elliptic Toeplitz operator T_P is Fredholm on the locally convex space $S[C^\infty(\Lambda^{0,*}M \otimes E)]$.*

PROOF: Since the Toeplitz operator T_P is elliptic, it follows that the operator $P \in \operatorname{Op} S^m(E)$ is elliptic along the line bundle L. Let $Q \in \operatorname{Op} S^{-m}(E)$ be a parametrix for P along this set; that is, the symbol of Q on L is the inverse of the symbol of P. Then we see that

$$T_P T_Q = SPSQS = S + S(PQ - I)S + SP[S,Q]S.$$

The operators $S(PQ-I)S$ and $SP[S,Q]S$ are in $\operatorname{Op} S^{-1/2}_{1/2}(\Lambda^{0,*}M \otimes E)$, the first because the symbol of $PQ-I$ vanishes along L, the second because the operator $[S,Q]$ is in $\operatorname{Op} S^{-1/2}_{1/2}(\Lambda^{0,*}M \otimes E)$. ∎

From now on, we will assume that the bundle E has a $\mathbb{Z}_2$-grading, that is, it is a supabundle; it follows that the bundle $\Lambda^{0,*}M \otimes E$ is supabundle with respect to the total $\mathbb{Z}_2$-grading. If T is an odd self-adjoint elliptic Toeplitz operator (in other words, maps $\Gamma(\Lambda^{0,k}M \otimes E^{\pm})$ to $\Gamma(\Lambda^{0,k}M \otimes E^{\mp})$), we define its index as follows:

$$\operatorname{ind}(T) = \operatorname{Str}(P_{\ker T}),$$

where $P_{\ker T}$ is the projection onto the kernel of T. This definition is related to the ordinary definition of index: if we write $T^{\pm}$ for the piece of T that maps $\Gamma((\Lambda^{0,*}M \otimes E)^{\pm})$ to $\Gamma((\Lambda^{0,*}M \otimes E)^{\mp})$, then we see that $T^- = (T^+)^*$, so that

$$\begin{aligned}\operatorname{ind}(T) &= \dim\ker T^+ - \dim\ker T^- \\ &= \dim\ker T^+ - \dim\operatorname{coker} T^+.\end{aligned}$$

Thus, the index of T is the same thing as the Fredholm index of the operator T^+. It follows that $\operatorname{ind}(T)$ is a homotopy invariant function of the leading symbol of T, such that $\operatorname{ind}(T_0T_1) = \operatorname{ind}(T_0)+\operatorname{ind}(T_1)$. In this way, we obtain a homomorphism

$$\operatorname{ind} : \pi_0(\mathrm{GL}(\Gamma(E))) \oplus \pi_0(\mathrm{GL}(\Gamma(E))) \to \mathbb{Z},$$

defined by choosing a non-vanishing section θ of the line bundle L, and considering the map

$$\sigma(T)(x,\theta) \oplus \sigma(T)(x,\theta) \mapsto \operatorname{ind}(T).$$

This map does not depend on the degree k of the operator T used to define it, by the same argument as for pseudodifferential operators.

This index map was evaluated by Boutet de Monvel [**2**], who observed that the index of the Toeplitz operator T_P is equal to the index of the pseudodifferential operator $\mathcal{D}_b+P$. In this way, the calculation of $\operatorname{ind}(T_P)$ is reduced to an application of the Atiyah-Singer index theorem. He and Malgrange later gave a proof, using D-modules, that is very close to Grothendieck's proof of the Riemann-Roch theorem [**3**].

Let $T_P = SPS$ be an odd, self-adjoint elliptic first-order Toeplitz operator on $\Lambda^{0,*}M \otimes E$, where P is an odd, self-adjoint elliptic first-order pseudodifferential operator on E. We now present a McKean-Singer formula for the index of T_P which will be used in [**6**] to give a purely analytic proof of Boutet de Monvel's index theorem.

THEOREM 3.2. *The index of T_P is given by the following formula:*

$$\operatorname{ind}(T_P) = \operatorname{Str}(e^{-t(\mathcal{D}_b+P)^2}), \qquad t>0.$$

PROOF: We start by introducing a family of operators $\mathcal{P}_s$, parametrized by $s \in [0,1]$, such that $\mathcal{P}_{s=1} = \mathcal{D}_b + P$ (here, $S^\perp = 1-S$):

$$\mathcal{P}_s = s\mathcal{D}_b + (1-s)S^\perp\mathcal{D}_bS^\perp + sP + (1-s)SP^*S.$$

For $0 \le s \le 1$, we will show that the quantity $\operatorname{Str}(e^{-t\mathcal{P}_s^2})$ equals $\operatorname{ind}(T_P)$.

LEMMA. *For each sufficiently small positive ε, there is a constant $c(\varepsilon)$ such that uniformly in s,*

$$\mathcal{P}_s^2 + c(\varepsilon) \ge \varepsilon\Delta,$$

where Δ is the Laplacian on the bundle $\Lambda^{0,*}M \otimes E$.

PROOF: Let us denote equality up to an element of $\operatorname{Op} S^{3/2}_{1/2}(\Lambda^{0,*}M\otimes E)$ by $A \sim B$. By the calculus for pseudodifferential operators and the fact that $\mathcal{D}_bS$ and $S\mathcal{D}_b$ lie in $S^{1/2}_{1/2}(\Lambda^{0,*}M\otimes E)$, we obtain the equation

$$\mathcal{P}_s^2 \sim S^\perp\mathcal{D}_b^2S^\perp + SP^2S.$$

Similarly, we can decompose the Laplacian into blocks corresponding to the projections S and $S^{\perp}$:
$$\Delta \sim S^{\perp}\Delta S^{\perp} + S\Delta S.$$
It follows that
$$\mathcal{P}_s^2 - \varepsilon\Delta \sim (1-\varepsilon)S^{\perp}\Delta S^{\perp} + S(P^2 - \varepsilon\Delta)S,$$
since by Weitzenböck's formula,
$$S^{\perp}\mathcal{D}_b^2 S^{\perp} \sim S^{\perp}\Delta S^{\perp}.$$

Let $Q \in \operatorname{Op} S^1(\Lambda^{0,*}M \otimes E)$ be a positive pseudodifferential operator such that the leading symbol of
$$Q^2 - (P^2 - \varepsilon\Delta)$$
vanishes on L; this is possible since the leading symbol of $P^2 - \varepsilon\Delta$ is positive on L for small enough ε. It follows that
$$\mathcal{P}_s^2 \sim \varepsilon\Delta + (1-\varepsilon)(S^{\perp}\Delta^{1/2}S^{\perp})^2 + (SQS)^2.$$
Since the last two terms of the right-hand side of this equation are positive, the lemma follows from Gårding's inequality. ∎

In particular, as operators on $L^2(\Lambda^{0,*}M \otimes E)$, we have the inequality
$$e^{-\mathcal{P}_s^2} \le e^{tc(\varepsilon)} e^{-\varepsilon t\Delta}.$$
This shows that the heat kernel $e^{-t\mathcal{P}_s^2}$ is trace class for all $t > 0$, uniformly in $0 \le s \le s$.

The next step is to show that $\operatorname{Str}(e^{-t\mathcal{P}_s^2})$ is independent of s. This is shown in the conventional way, by taking a derivative with respect to s and rewriting the resulting quantity as the supertrace of a supercommutator, which we know to vanish:
$$\begin{aligned}\frac{\partial a(s)}{\partial s} &= -t \operatorname{Str}\left(\frac{d\mathcal{P}_s^2}{ds} e^{-t\mathcal{P}_s^2}\right) \\ &= -t \operatorname{Str}\left[((\mathcal{D}_b - S^{\perp}\mathcal{D}_b S^{\perp}) + (P - SPS))e^{-t\mathcal{P}_s^2/2}, \mathcal{P}_s e^{-t\mathcal{P}_s^2/2}\right].\end{aligned}$$

Since $\operatorname{Str}(e^{-t\mathcal{P}_s^2})$ is a constant, we may set s equal to 0. In other words, we must calculate
$$\operatorname{Str}\exp\left(-t(S^{\perp}\mathcal{D}_b S^{\perp})^2 - tT_P^2\right).$$
This may be separated into the sum of the supertrace over the image of S, namely
$$\operatorname{Str}(Se^{-tT_P^2}S),$$
and the supertrace over the image of $S^{\perp}$, which equals zero. Finally, we are left with calculating $\operatorname{Str}(Se^{-tT_P^2}S)$. But by the same method as we used above, we see that it is independent of t. As $t \to \infty$, it converges to the index of T_P, since $Se^{-tT_P^2}S$ converges to the projection onto the kernel of T_P. ∎

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