ORBIT EQUIVALENCE, LIE GROUPS, AND FOLIATIONS

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In this paper we shall survey some results on orbit equivalence of measure preserving actions of Lie groups and indicate some recent developments concerning the relationship of this classical area in ergodic theory with certain standard problems in the geometry of foliations of compact manifolds.

Suppose G is a separable locally compact group, and that G acts in a measure class preserving way on a (standard) measure space S. Then the orbits of the action define a measurable equivalence relation R(S,G) on S. Two such actions (of possibly different groups) are called orbit equivalent if the equivalence relations are isomorphic (modulo null sets.) I.e., for actions of G on S and G' on S', there is a measure space isomorphism $h:S \to S'$ such that for (almost) all $s \in S$, $h(sG)=h(s)G'$. It is also convenient to consider the more general notion of stable orbit equivalence. The actions of G on S and G' on S' are called stably orbit equivalent if the action of $G \times K$ on $S \times K$ and the action of $G' \times K$ on $S' \times K$ are orbit equivalent, where K is the circle group acting on itself by translation. (In a similar way we can speak of two equivalence relations being stably isomorphic, so that stable orbit equivalence of actions is simply stable isomorphism of the corresponding equivalence relations.) If G and G' are continuous (i.e., non-discrete) groups, and the actions are essentially locally free (i.e., almost every stabilizer is discrete), then by results of [FHM], the actions are stably orbit equivalent if and only if they are orbit equivalent. Moreover, if G and G' are connected continuous groups and the actions have fixed point sets of measure 0, then the actions are orbit equivalent if and only if they are stably orbit equivalent. We say that G is (stably) weakly equivalent to G' if they have (stably) orbit equivalent actions which are properly ergodic, finite measure preserving , and essentially free (i.e., almost every stabilizer is trivial.) We denote this by $G \approx G'$ (or $G \approx_s G'$ in the stable case.) The first question with which we shall be concerned is to determine when $G \approx G'$ (or $G \approx_s G'$) in the case of Lie groups. We remark that weak equivalence and stable weak equivalence are not equivalence relations. As has been pointed out by J. Feldman, it would be of interest to compare groups vis-a-vis the equivalence relations generated by these relations.

We make two remarks of a general nature. Suppose $\Gamma \subset G$ is a lattice subgroup, i.e., Γ is discrete and G/Γ has a finite G-invariant measure. If S is a Γ-space, then one can form the induced G-space by taking $X=(S \times G)/\Gamma$, (where Γ acts on G by left translation) and let G act on X via the action of G on itself by right translation. Then the relations $R(S,\Gamma)$ and $R(X,G)$ are stably isomorphic. The relevant properties of X being inherited from S enables us to assert:

PROPOSTION 1. *If Γ is a lattice in G, then $\Gamma \approx_s G$. Furthermore, if G and G' are continuous groups with lattices Γ, Γ', then $\Gamma \approx_s \Gamma'$ implies $G \approx G'$.*

This of course enables us to transfer non-weak equivalence results from connected groups to lattice subgroups. Suppose now that Γ, Γ' are lattices in the same group G. The action of Γ on G/Γ' is stably orbit equivalent to the action of G on $G/\Gamma \times G/\Gamma'$, and hence by symmetry to that of Γ' on G/Γ. Therefore we have:

PROPOSITION 2. *If $\Gamma, \Gamma' \subset G$ are lattices, then $\Gamma \approx_s \Gamma'$.*

In particular, if F_n is the free group on n generators ($n \geq 2$), then F_n is a lattice in $SL(2,\mathbb{R})$. Thus we deduce that $F_n \approx_s F_m$ for $n,m \geq 2$.

The first major result about orbit equivalence was Dye's theorem, later extended to its natural level of generality by Connes, Feldman, Ornstein, and Weiss.

THEOREM 3. [D],[CFW] *If G and G' are (infinite) amenable groups, then $G \approx_S G'$. If both are discrete or both are continuous then $G \approx G'$.*

Of course, a much stronger result is true, namely that any two finite measure preserving properly ergodic actions of amenable groups are orbit equivalent as long as both have discrete orbits or both have continuous orbits.

For semisimple Lie groups, a great deal is also known. For simplicity, we shall state results in the simple case. We first recall the notion of $\mathbb{R}$-rank. If G is a real linear group, we define the $\mathbb{R}$-rank of G to be the maximal dimension of an abelian subgroup which is conjugate over $\mathbb{R}$ to a subgroup of the group of invertible diagonal matrices. If G is a Lie group and we consider linear real representations then the $\mathbb{R}$-rank of the images for different representations may be different (even for faithful representations). However, for semisimple Lie groups, this number is independent of the representation as long as we assume the kernel is discrete, and thus we may consistently speak of $\mathbb{R}$-rank G where G is a semisimple Lie group. For $G=SL(n,\mathbb{R})$, $\mathbb{R}\text{-rank}(G)=n-1$.

THEOREM 4. [Z1] *Let G,G' be connected simple Lie groups with finite center, and assume $\mathbb{R}\text{-rank}(G)\geq 2$. If $G \approx G'$, then G and G' are locally isomorphic.*

As with Theorem 3, a much stronger assertion is true. Namely, if G has trivial center and acts essentially freely, ergodically, and with finite invariant measure on both S and S', then orbit equivalence of the actions implies that the actions are actually conjugate, modulo an automorphism of G.

Theorems 3,4 leave two broad areas to be investigated to understand weak equivalence for Lie groups (or stable weak equivalence for lattice subgroups.) The first is the issue of simple groups where both have ℝ-rank 1. (ℝ-rank 0 is equivalent to compactness.) These are the groups locally isomorphic to SO(1,n), SU(1,n), Sp(1,n), or one exceptional group. The groups SO(1,n), SU(1,n) fail to have Kazhdan's property, while the other groups have this property.

THEOREM 5.[Z2] *If G, G' are non-compact simple Lie groups with finite center and $G \approx G'$, then G is Kazhdan if and only if G' is Kazhdan.*

This result of course does not go far in resolving the ℝ-rank 1 issue. The following result, which is joint work with M. Cowling, is stated for actions of lattices. Hopefully the techniques of the proof can be extended to give similar results for the actions of the ambient Lie groups.

THEOREM 6. [CZ] *Let $\Gamma_1 \subset Sp(1,n)$ and $\Gamma_2 \subset Sp(1,m)$ be lattices. If $\Gamma_1 \approx \Gamma_2$, then $n=m$.*

The second main issue that needs understanding is the case in which the group is neither semisimple nor amenable, but rather a combination of these cases. Once again, the issue is far from resolved. Here is a small beginning.

THEOREM 7. [Z3] *Let H_1, H_2 be connected Lie groups. Let $N_i \subset H_i$ be the maximal normal amenable subgroup, so that $G_i = H_i/N_i$ is the maximal semisimple adjoint quotient with no compact factors. Assume the ℝ-rank of every simple factor of G_1 is at least two. Suppose $H_1 \approx H_2$. Then G_1 and*

G_2 are isomorphic and N_1 is compact if and only if N_2 is compact.

Before leaving this question, we make some remarks on von Neumann algebras. Given an action of a group G on a measure space S, one can associate a von Neumann algebra A(S,G) by the group measure space construction. Assuming the actions are essentially free, this von Neumann algebra is an invariant of orbit equivalence. For a discrete amenable group acting freely and ergodically with a finite invariant measure, it is classical that A(S,G) will be a hyperfinite II_1 factor, and that all such factors are isomorphic. Theorem 3 can be viewed as asserting that these isomorphisms take place at the level of the equivalence relation rather that just at the algebra level. In the other direction, it is natural to enquire whether the non-equivalence assertion of Theorem 4 can be extended to the algebra level. No serious progress has been made to date on this point. However in the closely related context of Theorem 6, we have:

THEOREM 8. [CZ] *Let $\Gamma \subset Sp(1,n)$, $\Gamma' \subset Sp(1,n')$ be lattices, and suppose S (resp., S') is a profinite group with a dense embedding of Γ (resp., Γ'). Let Γ and Γ' act by translation. If $A(S,\Gamma)$ and $A(S',\Gamma')$ are isomorphic von Neumann algebras, then $n=n'$.*

We now turn to some questions concerning orbit equivalence that arise in the study of Riemannian and Lie foliations. If (G,X) is a Lie transformation group, we recall that a (G,X) structure on a manifold M is an atlas on M where each coordinate chart is a diffeomorphism with an open subset of X, and the transition functions are restrictions of elements of G. Similarly, if $\mathfrak{F}$ is a foliation of a manifold, a transversal (G,X) structure on $\mathfrak{F}$ is an atlas of foliation charts such that each of the local submersions defining $\mathfrak{F}$ is onto an open subset of X, and the transition functions are restrictions of elements of G. If G acts isometrically on X, we say that $\mathfrak{F}$ has a transverse Riemannian structure, or that $\mathfrak{F}$ is a Riemannian foliation. For many purposes, the work of Molino [M] reduces this case to the study of Lie foliations, i.e., the case in which X=G, and G acts by translations. (We then also speak of a G-foliation.) Given a G-foliation on M, there is a natural homomorphism $h:\pi_1(M)\rightarrow G$; this is called the holonomy homomorphism, and the image, say Γ, is called the holonomy group. The embedding $\Gamma\rightarrow G$ carries most of the transversal information about the foliation. (One can also reduce to the case in which this is a dense embedding.) We shall not recall this construction here, but rather we indicate one consequence pertaining to orbit equivalence. Namely, if we let $R(\mathfrak{F})$ be the measurable equivalence relation defined by $\mathfrak{F}$, then, assuming (as we may always do) that G is simply connected, $R(\mathfrak{F})$ is stably isomorphic to $R(G,\Gamma)$, where Γ acts on G by translations. In any event, the holonomy construction yields, out of a geometric situation, a finitely generated dense subgroup of a connected Lie group, and Haefliger raised the question a number of years ago as to what the pair (G,Γ) could be for a Lie foliation of a compact manifold. In light of the assertion above concerning stable isomorphism, we can ask the following related "measurable Haefliger question."

QUESTION: *Given a foliation $\mathfrak{F}$ of a compact manifold, and a finitely generated dense subgroup Γ of a connected Lie group G, when can we have $R(\mathfrak{F})$ stably isomorphic to $R(G,\Gamma)$?*

An answer to this question of course has immediate application to Haefliger's original question. We now discuss two situations where one has very satisfactory information. Suppose first that $\mathfrak{F}$ is an amenable foliation, i.e., that $R(\mathfrak{F})$ is amenable in the sense of [Z4]. This will always be the case for example if the leaves of $\mathfrak{F}$ have polynomial growth. The question above then reduces to determining when $R(G,\Gamma)$ is amenable. The following result gives a complete picture; this was conjectured by Connes and Sullivan.

THEOREM 9. [Z5] *If Γ is a finitely generated dense subgroup of a connected Lie group, then $R(G,\Gamma)$ is amenable if and only if G is solvable.*

This result, which of course applies to any amenable foliation, was applied in the proof of the following theorem of Carriere who considered Lie foliations with leaves of polynomial growth.

THEOREM 10.[C] *If $\mathcal{F}$ is a G-foliation of a compact manifold and the leaves of $\mathcal{F}$ have polynomial growth, then G is nilpotent.*

The other situation in which we have good control over Γ is again one in which we make a geometric assumption on the leaves of $\mathcal{F}$. Namely we assume that the leaves are locally isometric to the Riemannian symmetric space H/K, where H is a connected semisimple Lie group with finite center and all factors of $\mathbb{R}$-rank at least 2, and K is the maximal compact subgroup. We summarize this by saying that the symmetric space X is of higher rank. (Of course, it also has a purely geometric characterization, i.e., without a priori involvement of the groups H and K.) Up to local isomorphism, H is just the isometry group of X.

THEOREM 11. [Z6] *Suppose $\mathcal{F}$ is a foliation of the compact manifold M, and that the leaves of $\mathcal{F}$ are all locally isometric to a symmetric space X of higher rank. Assume further that there is a dense simply connected leaf. Let H be the identity component of the isometry group of X. Suppose G is a connected Lie group and $\Gamma \subset G$ is a finitely generated dense subgroup, and that $R(\mathcal{F})$ is stably isomorphic to $R(G,\Gamma)$. Then G is semisimple, the* *complexification of any simple factor of G is the complexification of a simple factor of H, and there is a lattice* $\Lambda \subset G \times H$ *such that* Γ *is the projection of* Λ *onto G and this projection is an isomorphism of* Λ *and* Γ.

This immediately applies to Lie foliations as follows.

THEOREM 12. [Z6] *Suppose* $\mathcal{F}$ *is a G-foliation of the compact manifold M, and that the leaves of* $\mathcal{F}$ *are all locally isometric to a symmetric space X of higher rank. Assume further that there is a dense simply connected leaf. Let H be the identity component of the isometry group of X, and* $\Gamma \subset G$ *the holonomy group. Then G is semisimple, the complexification of any simple factor of G is the complexification of a simple factor of H, and there is a lattice* $\Lambda \subset G \times H$ *such that* Γ *is the projection of* Λ *onto G and this projection is an isomorphism of* Λ *and* Γ.

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