THE MINIMAL MARTIN BOUNDARY OF A CARTESIAN PRODUCT OF TREES

Massimo A. Picardello [1)]

and

Peter Sjögren [2)]

Abstract. On a tree, the Martin boundary for positive eigenfunctions of the "Laplacian" or other suitable difference operators is known to coincide with the natural boundary of the tree. In this survey, operators on a finite product of trees are considered. Old and new results are described. In the case when all the trees are homogeneous, we let the operator be a positive linear combination of the Laplacians in the factor trees. If at least one of the trees is not $\mathbb{Z}$, the corresponding Martin boundary is nontrivial for all sufficiently large eigenvalues, and is given as the product of the natural boundaries of the trees times a hypersurface which depends on the eigenvalue. The situation is similar to that of a polydisc. There is a pointwise convergence theorem at the boundary. For $\mathbb{Z}^n$ however, the boundary is a one-point set. To get a nontrivial boundary here, one can consider instead an operator with drift.

1. Minimal Martin boundaries and ends of graphs: an overview

The purpose of this paper is to outline the ideas connected with the search for a concrete, geometric realization of the reproducing boundary of positive harmonic functions for a denumerable Markov chain. For the sake of concreteness, we shall phrase all the statements in terms of transition operators acting on some infinite graph (the graph of the states of the Markov chain). We shall only sketch briefly, in this introductory overview, the successful attempts in this direction. For a more detailed outline of the results, the reader is referred to the survey paper [PW3]. One of our aims is to explain why a naive geometric approach fails to yield the Martin boundary for a

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significant class of graphs: the euclidean lattices.

Denote by P a stochastic nearest-neighbour transition operator on a graph Γ, and let P act on functions on Γ (more precisely, on functions on its set of vertices), by the rule $Ph(x) = \sum_y p(x,y)h(y)$. Moreover, denote by $P^{(n)}$ the n-th iterate of P, and by $p^{(n)}(x,y)$, $x,y \in \Gamma$, its entries. For $t > 0$ and $x,y \in \Gamma$, define the "generalized Green kernel" $G_t(x,y) = \sum_{n\geq 0} p^{(n)}(x,y)/t^{n+1}$. Regarded as an operator, G_t is the resolvent of P: indeed, $(tI-P)G_t = I$. For t sufficiently large, $G_t(x,y)$ exists finite and $G_t(\cdot,y)$ is a positive t-eigenfunction of P outside of y. By a t-harmonic function, we shall mean a t-eigenfunction of P. We are interested in the cone $\mathcal{H}_t$ of positive t-harmonic functions : if $\mathcal{H}_t$ is nontrivial, then G_t is finite, so we restrict attention to those values of t such that $G_t < \infty$.

Positive eigenfunction of P can be expressed as "Poisson integrals" of positive Borel measures over a suitable boundary of Γ. A boundary $\mathcal{M}$ with these properties was constructed by Martin [Ma] for harmonic functions on bounded domains in $\mathbb{R}^n$, and by [Do1] for denumerable Markov chains (which include our present setting): see [Do2, Hl, KSK] for references. This construction makes use of the "Martin kernels" $K_t(x,y) = G_t(x,y)/G_t(o,y)$, where o is a fixed reference vertex. There is a unique compactification $\tilde{\mathcal{M}}_t$ of Γ on which all the functions $K_t(x,\cdot)$ extend continuously and separate points; the Martin boundary is now defined by $\mathcal{M} = \mathcal{M}_t = \tilde{\mathcal{M}}_t \backslash \Gamma$. By abuse of notation, we denote again by $K_t(x,m)$ the extension of $K_t(x,\cdot)$ to $\mathcal{M}$. Now every positive t-eigenfunction of P is of the type $\mathcal{K}_t\mu(x) = \int_{\mathcal{M}} K_t(x,\cdot)d\mu$, for some positive measure μ. In general, μ is not unique: for instance, $\mathcal{K}_t\delta_m = \mathcal{K}_t\mu$ for some measure $\mu \neq \delta_m$, $\mu > 0$, if and only if $K_t(\cdot,m)$ is not an extreme point of the base of the positive cone $\mathcal{H}_t$, i.e., the convex set $\{h>0\colon Ph=th,\ h(0) = 1\}$. The function $K_t(\cdot,m)$ is extremal in this sense if and only if it is a minimal positive t-harmonic function; when this condition is satisfied, m is called a minimal point of $\mathcal{M}$. The subset of minimal points in $\mathcal{M}$ is a Borel set and, for every positive t-eigenfunction h, it carries a unique measure $\mu_h > 0$ such that $\mathcal{K}_t\mu_h = h$. This subset is called the "minimal Martin boundary" with a slightly inaccurate notation, because in general it is not a boundary, not being compact. In this paper, we restrict attention to the minimal Martin boundary, which, by further abuse of notation, will be again denoted by $\mathcal{M}$.

It was shown in [Ca] that, if Γ is a tree, then $\mathcal{M}$ coincides with its natural boundary, i.e., the set of rays emanating from a reference vertex, endowed with its natural topology. This results goes back to [DM] in the particular case of homogeneous trees and group–invariant transition operators. It was extended in [PW1] to transition operators on trees which, rather than being nearest–neighbour, allow jumps of bounded length and satisfy some natural uniformity assumptions (the group–invariant case had been settled in [De]). By modifying the set of edges suitably, these operators can be regarded as being nearest–neighbour on a graph which admits a uniformly spanning tree. For more general graphs Γ, a geometric realization of $\mathcal{M}$ can be given in terms of the space Ω of ends of Γ, that is, equivalence classes of rays in Γ under the equivalence relation which identifies rays that are not separated by finite sets of vertices. The space Ω , with its natural topology, is a boundary of Γ and is a continuous image of $\mathcal{M}$, but it is not always homeomorphic to $\mathcal{M}$. It is called the geometric boundary of Γ. For instance, the lattice $\mathbb{Z}^n$ has only one end, with "infinite diameter", but its Martin boundary may be non–trivial (see below). Other examples of graphs where $\Omega \not\subseteq \mathcal{M}$ are those which admit ends containing two rays such that the probability of hitting a vertex of the first from a vertex of the second without wandering too far vanishes rapidly enough as the two vertices move out to infinity (see [PW2] for details). In this case, the end is said to have poor "transversal conductance" (see [PW3] for references on the analogy between potential theory on graphs and electrical networks). Sufficient conditions for $\mathcal{M} \simeq \Omega$ have be given in [PW2], in terms of diameter and transversal conductance of ends (see also [PW3]).

We want to understand better why this geometric realization fails for the euclidean lattice $\mathbb{Z}^n$, whose Martin boundary is trivial only if the transition operator has no drift, but is homeomorphic to the sphere S^{n-1} otherwise [Hn, NS]. The graph $\mathbb{Z}^n$ has only one end, and it is the product of n copies of the one–dimensional tree $\mathbb{Z}$, which has two ends. Therefore the Martin boundary of $\mathbb{Z}^n$, in the nontrivial case, is a variety of higher dimension than the product of the geometric boundaries of the factors. This remark suggests considering a collection of graphs Γ_i, $i = 1,\dots,n$, with transition operators P_i and forming their cartesian product Γ, endowed with a transition operator P defined as a convex combination $P = \sum_{i=1}^{n} \alpha_i \tilde{P}_i$. Here $\tilde{P}_i$ denotes the natural lift of P_i from Γ_i to Γ: denoting by $x = (x_1,\dots,x_n)$ the vertices of $\Gamma = \times_i \Gamma_i$, we have $\tilde{p}_i(x,y) = p_i(x_i,y_i)$ if $x_j = y_j$ for every $j \neq i$ and zero otherwise. Thus, if $h(x) = h_1(x_1)\dots h_n(x_n)$, then $\tilde{P}_i h(x) = P_i h_i(x_i) \cdot \prod_{j \neq i} h_j(x_j)$. Therefore, if $P_i h_i = t_i h_i$, then $Ph = \sum \alpha_i t_i h$. In particular, if at least one of the P_i's admits a nontrivial Martin boundary, then so does P, because it has nontrivial positive eigenfunctions. We shall write P_i rather than $\tilde{P}_i$.

It would be interesting to characterize the minimal Martin boundary of (Γ,P) in terms of the corresponding boundaries of the factors. It is advisable, however, to limit attention to the case where Γ_i are homogeneous trees and P_i isotropic nearest neighbour transition operators. In this case, indeed, the Poisson kernels of the factors are explicitly known [Fu, DM]; see [FP] for further references. The minimal Martin boundary of the product of homogeneous trees is studied in the forthcoming paper [PS], in analogy with the approach of [Ka] for symmetric spaces. Also in this setting, the dimension of $\mathcal{M}$ is larger than that of the product of the geometric boundaries. Moreover, the additional dimensions turn out to be related with the rates of escape of the random walk in the individual components. This is shown in [PS] by looking at the asymptotic behaviour of minimal positive t–eigenfunctions along geodesics in the product: limits of this type give rise to a family of "Poisson kernels", parametrized by the "angle of escape" (the ratio of the velocities along each component of the given bi–geodesic). In turn, this approach yields a nontangential Fatou convergence theorem for positive t–eigenfunctions [PS].

In large part, the present paper is a survey of the results of [PS] on Martin boundaries. We consider the product Γ of a finite collection of homogeneous trees T_i, with homogeneity degrees q_i (i.e., with $1+q_i$ edges joining at each vertex), endowed with a convex combination of the isotropic nearest–neighbour transition operators. However, the presentation is aimed to shed light on the Martin boundary theory of euclidean lattices.

In fact, notice that our results hold under the only additional assumption that P be transient. Therefore it is enough to assume that at least one of the P_i is transient; that is, we can allow all but one of the component trees to be isomorphic to $\mathbb{Z}$. In other words, the graph Γ is allowed to be the product of a finite number of trees T_i with homogeneity degrees $q_i > 1$ and of a euclidean lattice $\mathbb{Z}^k$. Restricted to $\mathbb{Z}^k$, the transition operator is symmetric, but not necessarily isotropic. Here symmetry is with respect to sign change in any coordinate, whereas isotropy refers to interchange of coordinates.

If $\Gamma = \mathbb{Z}^k$, then it is easy to show that all positive harmonic functions are constants, and the Martin boundary is a singleton (for $k = 1$ or 2, the transition operator is actually recurrent). Positive t–eigenfunctions exist for $t > 1$. However, the setting of the euclidean lattice $\mathbb{Z}^k$ is simple enough to allow us to deal with asymmetric operators, that is, transition operators with drift. These operators are obviously transient, and in Section 2 we show that their Martin boundary is the sphere S^{k-1}, a result originally obtained in [Hn, NS]. In sections 3 and 4, a similar analysis is carried out for products of trees: this yields the minimal Martin boundary and the integral representation of all the positive t–eigenfunctions of P. In addition, we consider the "Poisson boundary", i.e., the subset $\mathscr{B}$ of $\mathscr{M}$ which supports the representing measures of the bounded harmonic functions. The Poisson boundary turns out to be homeomorphic to the geometric boundary. As a consequence, the bounded P–harmonic functions are jointly P_i–harmonic.

Finally, in section 5 we examine the connection between the "direction of escape" of a trajectory which goes to infinity along a "ray" in $\Gamma = \times_{i=1}^{n} T_i$ and the limit behaviour of the Poisson kernels along this trajectory.

The main references are [NS] for §2, [Ca, FP, MZ] for §3, and [PS] for §4,5. For the sake of clarity, we have often made an effort to cover also the elementary details of the most relevant topics.

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2. Minimal positive eigenfunctions of transition operators on euclidean lattices

In this section, we determine the Martin boundary of a translation-invariant transition operator P on $\mathbb{Z}^n$. It is not necessary to assume that P is nearest neighbour. To avoid trivialities, we assume that the subgroup generated by $\operatorname{supp} P = \{k \in \mathbb{Z}^n : p(0,k) > 0\}$ is the whole $\mathbb{Z}^n$. We make use of the following Harnack's inequality, which holds under a hypothesis weaker than translation-invariance : it is enough to assume that the non-zero transition probabilities be bounded away from zero.

Proposition 1. *There exists a positive constant* C *such that, for every positive* s*-eigenfunction* h *of* P *on* $\mathbb{Z}^n$ *and every* $k,m \in \mathbb{Z}^n$ *such that* $p(m,k) > 0$, one has $h(k) < C\, h(m)$.

Proof. As h is positive, $h(m) = s^{-1}Ph(m) \geq s^{-1}p(m,k)h(k) > C'h(k)$ for some constant C'. □

The following corollary is well known (see, for instance, [DSW]). In its statement, the dot denotes inner product.

Corollary 1. *Let* P *be translation-invariant on* $\mathbb{Z}^n$ *and let* h *be a minimal positive* s*-eigenfunction of* P *such that* $h(0) = 1$. *Then there exists a* $t \in \mathbb{R}^n$ *with* $\Phi(t) = \sum_k p(0,k)\exp(t\cdot k) = s$ *such that* $h(k) = \exp(t\cdot k)$.

Proof. If $v \in \mathbb{Z}^k$ with $p(0,v) > 0$, the proposition shows that the translated function $h(\cdot+v)$ is dominated by Ch for some $C < \infty$. The minimality of h gives $h(\cdot+v) = c_v h$, which also holds in the case $p(0,-v) > 0$. But there exists a basis of $\mathbb{Z}^n$ as a module over $\mathbb{Z}$ consisting of elements v_i verifying $p(0,v_i) > 0$ or $p(0,-v_i) > 0$. If k has coordinates κ_i in this basis, we see that $h(k) = \exp(\sum a_i\kappa_i)$ for some $a_i \in \mathbb{R}$. A change of basis gives $h(k) = \exp(t \cdot k)$ with $t \in \mathbb{R}^n$, and it is clear that $\Phi(t) = s$. □

The next corollary concerns the existence of nonconstant positive harmonic, i.e., 1–harmonic, functions on $\mathbb{Z}^n$. These functions exist if and only if the transition operator has drift, that is, $\sum_k k\ p(0,k) \neq 0$. For simplicity, we restrict attention to the nearest–neighbour setting, where the condition amounts to saying that P is asymmetric.

Corollary 2. *Let* P *be nearest–neighbour and translation–invariant on* $\mathbb{Z}^n$. *If* P *is symmetric, then the only positive harmonic functions are the constants,* $\Phi(t) = 1$ *only for* $t = 0$, *and the Martin boundary consists of one point. Otherwise, the hypersurface* $\mathcal{D} = \{\Phi(t) = 1\}$ *is non–trivial, and is in one–to–one correspondence with the minimal Martin boundary for positive functions.*

Proof. Let e_i be the standard basis vectors of $\mathbb{Z}^n$. If $p(0,e_i) = p(0,-e_i) = p_i$, then $\Phi(t) = 2\sum_{i=1}^{n} p_i \cosh(t_i)$; therefore $h(t) = 1$ implies $t = 0$. Observe, however, that the hypersurface $\{\Phi(t) = s\}$ is non–trivial for $s > 1$, and is in bijective correspondence with the minimal Martin boundary for positive s–harmonic functions. □

The bijection between $\mathcal{M}$ and $\mathcal{D}$ is a homeomorphism, if $\mathcal{D}$ is endowed with the relative topology of $\mathbb{R}^n$. To see this, however, it is necessary to "glue" $\mathcal{D}$ to $\mathbb{Z}^n$ in an appropriate way (up to homeomorphism), and to show that the Martin kernels extend continuously. We will not discuss the continuous extension of the Martin kernels, because they are difficult to compute (see section 5 below for more comments), and refer the reader to [Hn, NS] for further details. However, we state here the theorem of [Hn] and [NS] that determines the homeomorphic image of $\mathcal{D}$ which compactifies $\mathbb{Z}^n$, in a natural topology. This statement is the prototype of the theorems that we will discuss for products of trees in the sequel. We adopt the following notation: B is the unit ball in $\mathbb{R}^n$, $\mathcal{S} = \partial B$ is the unit sphere, and $\tau : \mathbb{R}^n \to B$ is defined by $\tau(x) = x/(1+\|x\|)$. Observe that, in the relative topology of $\mathbb{R}^n$, the set $Z = \tau(\mathbb{Z}^n) \cup \mathcal{S}$ is compact: $\mathcal{S}$ is the natural geometric boundary of $\tau(\mathbb{Z}^n)$. The topology of Z induces a compact topology on the set $\mathbb{Z}^n \cup \mathscr{S}$; thereby we regard $\mathscr{S}$ as a boundary for $\mathbb{Z}^n$.

Theorem 1 [Hennequin; Ney–Spitzer]. *The Martin boundary (for positive harmonic functions) of a translation–invariant transition operator with drift on* $\mathbb{Z}^n$ *is homeomorphic to the "natural" boundary* $\mathscr{S}$. *Under this homeomorphism, each* s *in* $\mathscr{S}$ *corresponds to the unique point* u *in the hypersurface* $\mathscr{D} = \{t : \Phi(t) = 1\}$ *such that* $\operatorname{grad} \Phi(u)$ *is aligned with* s.

Thus, if we let y_n go to infinity in $\mathbb{Z}^n$ in such a way that $\tau(y_n) \to s$, then $K(x,y_n) \to \exp(u \cdot x)$. In other words, $\exp(u \cdot x)$ is the Poisson kernel which arises from "trajectories to infinity" with asymptotic direction s. The condition that relates u to s is an extremality condition, which will be fully understood in Section 5.

3. The Laplace operator on a homogeneous tree: Poisson kernels and oricycles

In this section, we recall some results on harmonic functions on homogeneous trees. The main reference is [FP]; for oricycles on trees, see [Ca] and [BFP].

Denote by T the homogeneous tree of degree q, and by P the isotropic nearest neighbour transition operator, that is, $p(x,y) = 1/(q+1)$ if x and y are neighbours. P, or, more precisely, the operator $\Delta \equiv P-1$, is called the "Laplace operator" on T. Its Green and Martin kernels are easily computed; for our needs, however, it is enough to consider the Poisson kernels, which are given by the formula

$$K(x,\omega) = q^{h(x,\omega)}$$

where ω is a ray starting at o in T, and the "oricycle index" $h(x,\omega)$ is defined as follows. The ray ω induces an orientation on the edges. For the edges along ω, the positive orientation is the outward direction (with respect to o), and for the others, the positive orientation points towards ω. Consider now the path from o to a vertex x, and give each of its edges the weight $+1$ if it is run through along the positive orientation and -1 otherwise. Then $h(x,\omega)$ is defined as the sum of the weights in the path from o to x. The level sets of $K(\cdot,\omega)$, i.e., the sets $\{x \in T : h(x,\omega) = \text{constant}\}$ are the "oricycles tangent to Ω at ω". The oricycle $\{h(x,\omega) = k\}$ is denoted by $H_k(\omega)$. We have denoted by Ω the space of rays emanating from o, with its natural topology ([C]). The boundary Ω is glued to T as the set of points at infinity: indeed, we recall that $y_n \in T$ converges to $\omega \in \Omega$ if the distance from o to the branching vertex ω_n – between the ray ω and the path from o to y_n – diverges as $n \to \infty$.

For every complex number t, the power $K^{\frac{1}{2}+t} = K^{\frac{1}{2}+t}(x,\omega)$ satisfies $PK^{\frac{1}{2}+t} = \gamma(t)K^{\frac{1}{2}+t}$ with $\gamma(t) = \sigma\cosh(t \log q)$, where $\sigma = 2\sqrt{q_i}/(q_i+1)$ is the spectral radius of P in $\ell^2(T)$. Clearly, $K^{\frac{1}{2}+t}$ is positive if and only if Im t is a multiple of $2\pi/\log q$. Therefore the functions $K^{\frac{1}{2}+t}(x,\omega)$, $t \in \mathbb{R}$, are generalized Poisson kernels. On the other hand, all Poisson kernels are of this type: the Martin boundary is Ω for every eigenvalue $\gamma(t)$, $t \in \mathbb{R}$ [Ca, MZ]. These are all the eigenvalues larger than or equal to σ, and therefore exactly those with positive eigenfunctions. Observe that $K^{\frac{1}{2}+t}$ and $K^{\frac{1}{2}-t}$, $t > 0$, belong to the same eigenspace, with eigenvalue $\gamma(t) = \gamma(-t)$. However, the former is minimal, whereas the latter is not.

Proposition 2. *For $t > 0$, $K^{\frac{1}{2}+t}(\cdot,\omega)$ is a minimal $\gamma(t)$–eigenfunction of P. On the other hand, $K^{\frac{1}{2}-t}$ is not minimal; its integral decomposition is*

$$K^{\frac{1}{2}-t}(x,\omega) = \mathcal{K}_t\mu(x) = \int_\Omega K^{\frac{1}{2}+t}(x,\omega)\, d\mu_t(\omega),$$

where μ_t is a positive Borel measure on Ω.

Proof. For every $t \in \mathbb{C}$, the operator $\mathcal{K}_t$ defined in the statement, called the "Poisson transform", maps $M(\Omega)$ to the $\gamma(t)$–eigenspace of P. If t is real, $\mathcal{K}_t$ is bijective, and one can consider the operator $I_t = \mathcal{K}_t^{-1}\mathcal{K}_{-t} : M(\Omega) \to M(\Omega)$ (see [MZ,FP]). Clearly, $I_t(\delta_\omega) = \mu_t$ is the measure of the statement. We must show that $\mu_t > 0$ if and only if $t > 0$. By a result of [MZ] (see also [FP, Coroll.IV.1.2]), $\mathcal{K}_t$ is an integral kernel operator, with kernel $\kappa_t(\omega,\omega') = (1-\theta(t))q^{-2t}\, q^{(1-2t)N(\omega,\omega')}$, where $\theta(t) = (q-q^{2t})/(q-q^{-2t})$, and $N(\omega,\omega')$ is the distance between o and the branching vertex of the rays ω, ω'. Thus κ_t is positive if and only if $\theta(t) < 1$, i.e., for $t > 0$. □

The next result, originally proved in [BFP], characterizes the group of isometries of T which preserve the oricycle $H_k(\omega)$ for some (hence all) k. We denote by $\mathrm{Aut}(T)$ the group of all isometries of T.

Proposition 3. *Denote by* B *the subgroup of* $\mathrm{Aut}(T)$ *preserving* $H_k(\omega)$ *, and by* B_n *the subgroup of* $\mathrm{Aut}(T)$ *fixing all the vertices in the ray* ω *at distance* $\geq n$ *from* o. *Then* $B = \bigcup_{n=0}^{\infty} B_n$.

Proof. We first show that $B_n \subset B$ for every n . Fix x in T and b in B_n. We want to show that $h(b\cdot x,\omega) = h(x,\omega)$. Denote by $n_+(x)$ the number of edges with positive orientation in the path from o to x, and define $n_-(x)$ similarly. Since B_n increases with n, we can assume that $n > n_+(x)$. Among the points left fixed by B_n, the one which has shortest distance to x is the n-th vertex ω^n of ω, and $d(x,\omega^n) = n - n_+(x) + n_-(x) = n - h(x,\omega)$. Since the same thing is true with $b\cdot x$ instead of x, we conclude that $h(x,\omega) = h(b\cdot x, \omega)$.

Now take $b \in B$. Let $x \in T$ and choose $y \in \omega$ with $d(y,o) > \max(d(x,o), d(b\cdot x,o))$. Among all the points of the oricycle y belongs to, y will then be closest to x and to $b\cdot x$. Hence, $b\cdot y = y$, so that $b \in \cup B_n$. □

We conclude this section with a simple observation on the action of isometries on sequences going to infinity in a homogeneous tree.

Proposition 4. *If* $y_n \to \omega \in \Omega$ *, then there exists isometries* τ_n *which fix* o *and map* y_n *to a vertex of the ray* ω *. The sequence* τ_n *may be chosen so that* τ_n *converges (pointwise on* T*) to the trivial isometry.*

Proof: As before, we denote by ω_n the branching vertex between the path from o to y_n and the ray ω. As $y_n \to \omega$, $\omega_n \to \omega$ too, by the way the topology of $T \cup \Omega$ is defined. Then it is enough to choose τ_n so that it satisfies the following two conditions. First, τ_n interchanges the path from ω_n to y_n with the segment of ω which begins at ω_n and goes forward a number of steps equal to $d(\omega_n,y_n)$. Second, τ_n fixes every vertex that precedes ω_n in the ordering induced by ω. Observe that the τ_n's do not fix ω , but their limit does. □

4. The Minimal Martin boundary of a product of trees

We shall now consider the cartesian product Γ of n trees T_i, homogeneous of degree q_i. As remarked in the introduction, all but one of the q_i's may be 1. If all the q_i's are 1, then $\Gamma = \mathbb{Z}^n$; whose Martin boundary has been studied in Section 2. We consider the transition operator on Γ defined, as in Section 1, as a convex combination $\sum \alpha_i P_i$ of the Laplace operators on the factors. All the results are taken from [PS].

If $\Gamma = \times_{i=1}^{n} T_i$ and $q_k = q_j$ for some k,j, then there exist isometries switching T_k and T_j. The transition operator is invariant under these isometries if and only if $\alpha_k = \alpha_j$. We restrict attention to the subgroup $\mathrm{Aut}_*(\Gamma) \subset \mathrm{Aut}(\Gamma)$ without switches: $\mathrm{Aut}_*(\Gamma) = \times_{i=1}^{n} \mathrm{Aut}(T_i)$.

Proposition 5. *Choose a reference vertex* o *in* Γ *and suppose that the Green kernel* $G_s(x,y)$ *is finite for all* x,y. *For every* $\tau \in \mathrm{Aut}_*(\Gamma)$ *and* $x,y \in \Gamma$, *one has:*

i) $K_s(x,\tau y) = K_s(\tau^{-1}x, y)/ K_s(\tau^{-1}o,y)$

ii) *if* $\sigma \in \mathrm{Aut}_*(\Gamma)$ *fixes* o, $K_s(\tau\sigma\tau^{-1}o,\tau o) = 1$.

More generally, this result holds for all transient operators on an infinite graph Γ which are invariant under the group of isometries $\mathrm{Aut}(\Gamma)$, provided we replace $\mathrm{Aut}_*(\Gamma)$ by $\mathrm{Aut}(\Gamma)$ in the statement.

Proof. As P is invariant under $\mathrm{Aut}_*(\Gamma)$, one has $G_s(\tau x, \tau y) = G_s(x, y)$ for any $\tau \in \mathrm{Aut}_*(\Gamma)$.

$$\begin{aligned} \text{i)} \quad & K_s(x, \tau y) = G_s(\tau^{-1}x,y)/G_s(\tau^{-1}o,y) \\ = \quad & K_s(\tau^{-1}x,y)\cdot G_s(o,y)/G_s(\tau^{-1}o,y) = K_s(\tau^{-1}x,y)/K_s(\tau^{-1}o,y). \end{aligned}$$

ii) $\quad K_s(\tau\sigma\tau^{-1}o,\tau o) = G_s(\sigma\tau^{-1}o,o)/G_s(\tau^{-1}o,o) = 1$

because $\sigma^{-1}o = o$. $\square$

From now on, the statements are meant only for $\Gamma = \times_{i=1}^{n} T_i$. We denote by Ω the space of rays: $\Omega = \times_{i=1}^{n} \Omega_i$. Vertices in Γ are denoted by $x = (x_1,\dots,x_n)$, with $x_i \in T_i$. Similarly, rays in Ω are denoted by $\omega = (\omega_1,\dots,\omega_n)$. The space $\Gamma \cup \Omega$ is endowed with the product topology: x converges to ω if $x_i \to \omega_i$ for every i. The subgroup $\mathrm{Aut}_*(\Gamma)$ is normal and of finite index in $\mathrm{Aut}(\Gamma)$, the quotient being isomorphic to the group of permutations of the factors with the same degree q_i Therefore, in the generic case, $\mathrm{Aut}_*(\Gamma) = \mathrm{Aut}(\Gamma)$. The group $\mathrm{Aut}(\Gamma)$ is endowed with the topology of pointwise convergence on Γ. The subgroup of $\mathrm{Aut}_*(\Gamma)$ which fixes $o \in \Gamma$ is denote by $\mathcal{K}$. Then Proposition 5 yields the following consequence.

Corollary. *Assume that* (y_j) *is a sequence in* Γ *such that* $K_s(\cdot,y_j)$ *converges pointwise to a function* f. *If* $\tau_j \in \mathrm{Aut}_*(\Gamma)$ *and* $\tau_j \to \tau \in \mathcal{K}$, *then* $K_s(x,\tau_j y_j) \to f(\tau^{-1}x)$ *as* $j \to \infty$, *for each* $x \in \Gamma$.

Proof. Making use of Proposition 5.i, we obtain

$$K_s(x,\tau_j y_j) = K_s(\tau_j^{-1}x,y_j)/K_s(\tau_j^{-1}o,y_j) \to f(\tau^{-1}x),$$

since $\tau_j^{-1}x = \tau^{-1}x$ for large j and $K_s(0,y_j) = 1$. $\square$

We are now ready to apply the Martin method and determine the minimal boundary. Take an unbounded sequence (y_j) in Γ for which $K_s(\cdot,y_j)$ converges pointwise to a *minimal* positive eigenfunction f of P. We shall compute f.

Writing $y_j = (y_{j,1},\dots, y_{j,n})$, we may assume that the sequence $(y_{j,i})_j$ is unbounded for $1 \le i \le m$ and bounded for $m < i \le n$, where $m \ge 1$. Passing to a subsequence, we can then assume that $y_{j,i}$ has a limit $\omega_i \in \Omega_i$ for $i \le m$ and that $y_{j,i} = y_i$ is constant for $i > m$. Now write $\Gamma = \Gamma' \times \Gamma''$ with

$$\Gamma' = \mathop{\times}_{1}^{m} T_i \quad \text{and} \quad \Gamma'' = \mathop{\times}_{m+1}^{n} T_i,$$

and analogously $\Omega = \Omega' \times \Omega''$. Then $\omega' = (\omega_1,\ldots,\omega_m) \in \Omega'$ and $y'' = (y_{m+1},\ldots, y_n) \in \Gamma''$.

The first step in the application of the Martin method is to assume the existence of the limit (not the minimality yet), and to reduce the problem to the setting of euclidean lattices.

Proposition 6. *Let* $K_s(\cdot,y_j) \to f$ *as described above. With the assumptions and the notation just introduced, it follows that* f *is constant along the oricycles*

$$H_{k_1\ldots k_m}(\omega') = \mathop{\times}_{i=1}^{m} H_{k_i}(\omega_i)$$

in Γ'.

Proof. We fix a component T_i of T' with $i \le m$ and prove that $f(x)$ does not change if we move the coordinate x_i within an oricycle $H_{k_i}(\omega_i)$. By Proposition 4 and the corollary to Proposition 5, we may assume that $y_{j,i}$ converges "radially" to ω_i, in the sense that each $y_{j,i}$ is on the ray ω_i. Let τ_j for each j be an isometry of T_i whose restriction to ω_i is a shift in the positive direction and which satisfies $\tau_j o = y_{j,i}$.

Let $\beta \in B$, where B is defined with respect to T_i. Because of Proposition 3, $\beta \in B_m$ for large m, in particular for some m of the form $m = d(o, y_{j,i})$. But for this m we have $B_m = \tau_j B_0 \tau_j^{-1}$, so that $\beta = \tau_j \sigma \tau_j^{-1}$ with $\sigma \in B_0 \subset \mathcal{K}$. Now extend τ_j, β, and σ to isometries of Γ by keeping the i' coordinates fixed, $i' \neq i$. Then Proposition 5 gives

$$f(\beta x) = \lim_j K_s(\tau_j \sigma \tau_j^{-1} x, \tau_j o) = \lim_j K_s(x,y_j) = f(x). \qquad \square$$

Finally, we assume that the limit is minimal and we compute it. Recall that, on the i–th component, $\gamma_i(t) = 2\sqrt{q_i}\cosh(t\log q_i)/(q_i+1)$ [see Section 3].

Theorem 2. *Assume that the limit function of Proposition 6 is a minimal positive eigenfunction. Then* $m = n$ *and*

$$f(x) = \prod_{i=1}^{n} K^{\frac{1}{2}+t_i}(x_i, \omega_i)$$

for some $(t_1,\ldots,t_n) \in \overline{\mathbb{R}^n_+}$ *satisfying* $\Phi(t_1,\ldots,t_n) \equiv \sum \alpha_i \gamma_i(t_i) = s$.

Proof. We first assume $m = n$. Then the limit f depends only on the oricycle indices k_i, and we can realize it as a function $f(k_1,\ldots,k_n)$ on $\mathbb{Z}^n$. As the reference vertex o belongs to the oricycle with index zero in each component, we have $f(0,\ldots,0) = 1$. Moreover, for each vertex x_i in T_i, say $x_i \in H_{k_i}(\omega_i)$, the unique neighbour of x_i in the direction of positive orientation (with respect to ω_i) belongs to the oricycle $H_{k_i+1}(\omega_i)$, while all the other neighbours (the predecessors of x_i) belong to the oricycle of index k_i-1. Therefore,

$$
\begin{aligned}
& sf(k_1,\ldots,k_n) = Pf(k_1,\ldots,k_n) = \\
(1) \qquad & \\
& = \sum_{i=1}^{n} \alpha_i[q_i f(k_1,\ldots,k_i-1,\ldots,k_n)+f(k_1,\ldots,k_i+1,\ldots,k_n)]/(q_i+1).
\end{aligned}
$$

Thus f is a minimal positive eigenfunction for a (nearest–neighbour) transition operator with drift in $\mathbb{Z}^n$. By Corollary 1 of Section 2, $f(k_1,\ldots,k_n) = \prod_i \exp(t_i k_i)$, with $\Phi(t_1,\ldots,t_n) = s$. We observed in Section 3 that an exponential function of the oricycle index on a homogeneous tree is a power of the Poisson kernel. This completes the proof in the case $m = n$.

In the general case, split x as $(x',x'') \in \Gamma' \times \Gamma''$. Then the limit is constant on oricycles in Γ', and we write $f(x',x'') = f(k_1,\ldots,k_m; x'')$. We also decompose P as

$$P = \sum_{1}^{m} \alpha_i P_i + \sum_{m+1}^{n} \alpha_i P_i = P' + P''.$$

Now one has, with $k = (k_1,\ldots,k_m)$,

$$
\begin{aligned}
& Pf(k;x'') = \sum_{i=1}^{m} \alpha_i[q_i f(k_1,\ldots,k_i-1,\ldots,k_m;x'') + \\
(2) \qquad & \\
& + f(k_1,\ldots,k_i+1,\ldots,k_m;x'')]/(q_i+1) + P''f(k;x'') = sf(k;x'').
\end{aligned}
$$

Let e_i be the i–th canonical basis vector. As f and $P''f \geq 0$, one has $\alpha_i f(k+e_i;x'') \leq (q_i+1)P'f(k;x'') \leq sf(k;x'')$. Thus Harnack's inequality (Proposition 1) holds for $f(.;x'')$, and $f(k;x'') = \prod_{i=1}^{m} q_i^{t_i k_i} f(0;x'') = \prod_i K_i^{\frac{1}{2}+t_i}(x_i,\omega_i)f(0;x'')$.

It follows that $f(0;\cdot)$ is a positive eigenfunction of P''. As such, it must be minimal, since f is a minimal positive eigenfunction of P. But P'' is defined on the product of n–m trees. By induction in n, we conclude that

$$f(0;\, x'') = \prod_{m+1}^{n} K^{\frac{1}{2}+t_i}(x_i,\, \omega_i)$$

for some $\omega'' = (\omega_{m+1},\dots,\, \omega_n) \in \Omega''$.

(Observe, by the way, that no indication is necessary when $m = n - 1$, because in that case Proposition 2 applies to $f(0;x'')$). But since y'' is constant in Γ'', $f(o;x'')$ cannot depend on ω'', and we have a contradiction.

The only thing which is left to prove is that $\Phi(t) = s$, and this is immediately checked. The theorem is proved. □

Thus the minimal Martin boundary for the eigenvalue s is contained in $\Omega \times \mathscr{D}_s$, where $\Omega = \times_i \Omega_i$ and $\mathscr{D}_s$ is the hypersurface $\{t \in \mathbb{R}^n : t_i \geq 0$ for $i = 1,\dots,n$, $\Phi(t_1,\dots,t_n) = s\}$. We have not shown yet that the minimal boundary coincides with this set: we must still prove that all the generalized Poisson kernels $K(x,\omega,t) = \prod_i K_i^{\frac{1}{2}+t_i}(x_i,\omega_i)$ are minimal. Anyway, we already have the integral representation that we were looking for:

Theorem 3. *Every positive eigenfunction of* P *in* Γ *with eigenvalue* s *is of the type*

$$h(x) = \int_\Omega \int_{\mathscr{D}_s} K(x,\omega,t)\, d\mu_h(\omega,t)$$

for some positive measure μ_h.

Here one can choose μ_h carried by the set of extreme points of $\Omega \times \mathscr{D}_s$. Then μ_h will be unique. Observe that Theorem 3 yields the correct range of eigenvalues s for which there exist positive eigenfunctions: the condition is

$$s \geq \Phi(0,\dots,0) = \sum_i \alpha_i\, \sigma_i,$$

where σ_i is the spectral radius of P_i in $\ell^2(T_i)$. Notice that $\Phi(0,...,0)$ is the spectral radius of P in $\ell^2(\Gamma)$, because the P_i commute.

We now show that all the generalized Poisson kernels are minimal.

Theorem 4. *All the generalized Poisson kernels* $K(x,\omega,t)$ *are minimal positive eigenfunctions (with eigenvalue* $s = \Phi(t)$*). Therefore the minimal Martin boundary is* $\mathcal{M} = \Omega \times \mathcal{D}_s$. *The joint positive eigenfunctions of the operators* P_i , *with eigenvalues* $s_i = \gamma(t_i)$, *are represented by positive measures on* $\mathcal{M}$ *carried by* $\Omega \times \{t\}$, *with* $t = (t_1,...,t_n)$.

Proof. Observe first that, if $K(\cdot,\omega;t)$ is not minimal, then $K(\cdot,\omega';t)$ cannot be minimal for any $\omega' \neq \omega$, because the action of some $\tau \in \mathscr{K} \subset \mathrm{Aut}_*(\Gamma)$ transforms one kernel into the other, by the corollary to Proposition 5. On the other hand, if $K(x,\omega,t)$ is not minimal and $s = \Phi(t)$, then Theorem 3 yields a unique positive measure ν , carried by the set of minimal points in $\Omega \times \mathcal{D}_s$, such that $K(x,\omega,t) = \int K(x,\xi;u)d\nu(\xi,u)$. By the remark at the beginning of the proof, ν does not charge $\Omega \times \{t\}$. However, $K(x,\omega,t)$ splits as a product and is therefore a joint eigenfunction of the P_i's. The action of P_i gives

$$\gamma(t_i)K(x,\omega,t) = \int_{\Omega\times\mathcal{D}_s} \gamma(u_i)K(x,\xi;u)d\nu(\xi,u).$$

By the uniqueness of ν , we have $\gamma(u_i)d\nu(\xi,u) = \gamma(t_i)d\nu(\xi,u)$. In other words, ν is carried by $\Omega\times\{t\}$, a contradiction. This argument proves also the statement about the joint eigenfunctions of the P_i's. □

Remark. Consider the minimum eigenvalue σ for which there exist positive eigenfunctions of P , that is, the spectral radius of P in $\ell^2(\Gamma)$. Then $\sigma = \Phi(0,...,0) = \Sigma\, \alpha_i\sigma_i$, and $\mathcal{D}_\sigma$ consists only of the origin. Thus every positive σ–eigenfunction of P is a joint σ_i–eigenfunction of the P_i's.

We conclude this section by determining the Poisson boundary of Γ, i.e., the support of the reproducing measure μ_1 of the harmonic function 1. Denote by $d\omega_i$ the normalized measure on Ω_i which is invariant under the stability subgroup $\mathcal{K}_i$ of o in $\mathrm{Aut}(T_i)$ (see [FP] for more details). Then $\int_{\Omega_i} K(x_i,\omega_i)d\mu_i = 1$. Hence $d\mu_1 = \otimes_i d\omega_i \otimes \delta_r$, where $r = \frac{1}{2}(1,1,...,1)$, because $K(x,\omega,r) = \prod_i K_i(x_i,\omega_i)$. Therefore:

Theorem 5. *For every bounded* P*–harmonic function* h *there exists* $\varphi \in L^\infty(\Omega)$, *with* $\|\varphi\|_\infty = \|h\|_\infty$, *such that*

$$h(x) = \int_\Omega K(x,\omega,r)\varphi(\omega)d\omega_1...d\omega_n.$$

Thus the Poisson boundary $\Omega\times\{r\}$ is a proper subset of $\mathcal{M}$. In particular, bounded harmonic functions are jointly harmonic with respect to P_i.

5. Asymptotic behaviour of Poisson kernels and rate of escape along trajectories to infinity.

This section is a report of results of [PS]. We omit the details, and present only the ideas. We are interested in finding a connection between the asymptotic behaviour of Martin and Poisson kernels on trajectories moving to infinity along a "multiray" in $\Gamma = \times_{i=1}^{n} T_i$ and the "asymptotic direction" of the trajectory. The existence of a connection is suggested by the results for euclidean lattices (see the comments after Theorem 1).

Let $s > \Phi(0,...,0)$. We would like to determine the limit behaviour of the Martin kernels $K_s(x,y_j)$ as $y_j \to \omega \in \Omega$. However, this is a difficult task. Indeed, the Martin kernels are difficult to compute. They do not split as products (otherwise they could not be harmonic except at one point only), and their construction relies on the delicate combinatorics of path composition in cartesian products. We know that any limit of $K_s(x,y_j)$, for a sequence y_j tending to the boundary, must be either of type $x \to K(x,\omega,t)$ for some $\omega \in \Omega$, $t \in \mathcal{D}_s$ or an integral of such functions. Our explicit formula for $K(x,\omega,t)$ can be used to obtain results about the asymptotic behaviour of Poisson integrals. This we shall now describe.

A point $\omega = (\omega_1,\dots, \omega_n) \in \Omega$ determines a multiray in Γ, also denoted ω and defined as the product of the rays ω_i in T_i. Associating a point $x = (x_1,\dots, x_n)$ in this multiray with the n–tuple $(|x_i|)_1^n$, we get a bijection between the multiray and $\mathbb{N}_0^n$. As x tends to ∞ within the multiray, the direction of the vector $(|x_i|) \in \mathbb{N}_0^n \subset \mathbb{R}^n$ (called the direction of x) will be important.

Clearly, any finite measure μ on the Martin boundary $\Omega \times \mathscr{D}_s$ has a Poisson integral

$$K_s\mu(x) = \int K(x,\omega,t)d\mu(\omega,t).$$

There is a canonical measure $d\omega dt$ on $\Omega \times \mathscr{D}_s$. Here $d\omega = d\omega_1\dots d\omega_n$ was introduced at the end of the last section, and dt is the area measure in $\mathscr{D}_s$. If $f \in L^1(d\omega dt)$, we can define its Poisson integral as $K_sf = K_s(fd\omega dt)$. The corresponding normalized Poisson integral is $\mathscr{K}_sf(x) = K_sf(x)/K_s1(x)$, where 1 denotes the constant function on $\Omega \times \mathscr{D}_s$.

Comparing with the situation in the bidisc [S], we expect pointwise convergence $\mathscr{K}_sf(x) \to f(\omega_0,t_0)$ as $x \to \omega_0 \in \Omega$ and the direction of x approaches grad $\Phi(t_0)$. Here we assume that $t_0 \in \mathscr{D}_s$ is not on the boundary of this manifold with boundary. As will be shown in [PS], this convergence indeed holds if f is continuous in $\Omega \times \mathscr{D}_s$. This means that the functions $(\omega,t) \to K(x,\omega,t)/K_s1(x)$, which are defined in $\Omega \times \mathscr{D}_s$ and positive with integrals 1, tend to a unit mass at (ω_0,t_0) as x tends to ω and the direction of x tends to grad $\Phi(t_0)$. In particular, the map $(\omega,t) \to K(x,\omega,t)$ roughly speaking takes its largest values in $\Omega \times \mathscr{D}_s$ when ω is close to x and t is such that grad $\Phi(t)$ is nearly proportional to $(|x_i|)_1^n$.

For $f \in L^1(d\omega dt)$, there is a Fatou theorem ([PS]) saying that $\mathscr{K}_sf(x) \to f(\omega_0,t_0)$ for a.a. (ω_0,t_0). Here it is required that x approach the boundary within a "tube" associated with (ω_0, t_0): there must exist a $C > 0$ such that $x \in \Gamma$ is for some $r > 0$ at distance at most C from the point of the multiray ω_0 corresponding to $([r\partial\Phi(t_0)/\partial t_i])_{i=1}^{n} \in \mathbb{N}_0^n$. Here the brackets denote the integer part. The proof of this theorem goes via a weak type (1,1) estimate for the corresponding maximal function.

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Massimo A. Picardello
Centre for Mathematical Analysis
Australian National University
GPO Box 4
Canberra ACT 2601 Australia

and

Università dell'Aquila
67100 AQ, ITALY

P.Sjögren
Department of Mathematics
Chalmers Institute of Technology
and University of Göteborg
S–412 96 Göteborg, Sweden