LIE GROUP IMBEDDINGS OF THE FOURIER TRANSFORM AND A NEW FAMILY OF UNCERTAINTY PRINCIPLES

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1. INTRODUCTION

The one-dimensional Fourier-Plancherel operator $F: L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined formally by

$$(1.1) \qquad (Ff)(y) = \hat{f}(y) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-iyx} f(x)\,dx,$$

is a unitary operator; that is

$$(1.2) \qquad \langle f,g\rangle = \langle \hat{f},\hat{g}\rangle \text{ and } \|f\| = \|\hat{f}\|$$

where

$$(1.3) \qquad \langle f,g\rangle = (2\pi)^{-1/2} \int_{\mathbb{R}} \bar{f}(x) g(x)\,dx \text{ and } \|f\| = \langle f,f\rangle^{1/2};$$

also $F^4 = I$, the identity operator, so the integer powers of F form a cyclic group of order 4 [5]. It is natural to contemplate imbedding this finite discrete group of unitary operators in a continuous one. Condon derived a one-parameter group of integral operators $\{F_\theta\}$ ($\theta \in \mathbb{T}$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$) with the appropriate properties in 1937 [2] and Bargmann derived a corresponding one-parameter group for the d-dimensional Fourier operator in 1961 [1]. I have shown [11] the construction of infinitely many distinct imbeddings of the d-dimensional

F into a compact Abelian Lie group of unitary operators that has the d-dimensional torus $\mathbb{T}^d$ as its manifold. A particular "natural" one of these has a subgroup that is the Condon-Bargmann one.

Besides the intrinsic interest in a continuous imbedding there are several areas of application. The recent research [3,4,7,8,9,13,14,15] into inequality relations between a function f and its Fourier transform $\hat{f}$ (relatives of the Heisenberg-Pauli-Weyl uncertainty principle) has applications in quantum mechanics and communication theory. These inequalities can all be put in the form

$$(1.4) \qquad \sigma(f) \geq c, \text{ some constant,}$$

where σ is some measure of overall spread (or "uncertainty") in f and $\hat{f}$. For σ to measure some intrinsic property of the object represented by f, $\hat{f}$ or $F_\theta f$ it ought to be invariant under the continuous group of transforms $\{F_\theta\}$ in which F is naturally imbedded; that is, for all $f \in L^2(\mathbb{R})$ σ should satisfy

$$(1.5) \qquad \forall\theta \in \mathbb{T} \qquad \sigma(f) = \sigma(F_\theta f).$$

In this paper I outline a construction of the Condon-Bargmann group $\{F_\theta\}$, show that Heisenberg's measure of overall uncertainty does <u>not</u> satisfy (1.5), develop a family of measures that <u>do</u> satisfy (1.5) and show that the first one of this family leads to an uncertainty principle that is actually <u>stronger</u> than Heisenberg's.

2. AN IMBEDDING $\{F_\theta\}$ OF F

One can construct a continuous imbedding $\{F_\theta\}$ of F by using a diagonal representation of F. It is well known [5,16] that the Hermite functions $h_n(x)$, where

$$(2.1)\qquad h_n(x) = C_n e^{-x^2/2} H_n(x) \qquad (n \in \mathbf{N}),$$

where H_n is the nth Hermite polynomial and C_n is a normalization constant, form a complete orthonormal set of eigenfunctions of F, satisfying

$$(2.2)\qquad Fh_n = e^{-i\pi n/2} h_n .$$

Each $f \in L^2(\mathbf{R})$ has the Fourier-Hermite series

$$(2.3)\qquad f = \sum_{n \in \mathbf{N}} \langle h_n, f\rangle h_n$$

so its "fractional" Fourier transform $F^\alpha f\,(\alpha \in \mathbf{R})$ is naturally defined by

$$(2.4)\qquad F^\alpha f = \sum_{n \in \mathbf{N}} \langle h_n, f\rangle e^{-i\pi n\alpha/2} h_n ;$$

that is, writing $F_\theta = F^\alpha$ where $\theta = \pi\alpha/2 \qquad (\theta \in \mathbf{T})$,

$$(2.5) \qquad F_\theta f = \sum_{n\in\mathbb{N}} \langle h_n, f\rangle e^{-in\theta} h_n .$$

Provided $\theta/\pi \notin \mathbb{Z}$ the order of summation and integration in (2.5) can be reversed giving

$$(2.6) \qquad (F_\theta f)(x) = \langle K_\theta(s,x), f(s)\rangle$$

where

$$(2.7) \qquad K_\theta(s,x) = \sum_{n\in\mathbb{N}} e^{in\theta} h_n(s)\bar{h}_n(x) .$$

The sum K_θ in (2.7) can be evaluated in closed form [1,11] leading eventually to the theorem :

THEOREM 2.1 *(Condon-Bargmann) A one-parameter Lie group of transforms* $\{F_\theta\}$ $(\theta \in \mathbb{T})$ *in which the Fourier transform on* $L^2(\mathbb{R})$ *is imbedded, (i.e. satisfying* $F_{k\pi/2} = F^k$ $(k \in \mathbb{Z})$*) is given by*

$$(2.8) \qquad (F_\theta f)(x) = A_\theta \int_{\mathbb{R}} \exp\left\{-i\left[\frac{-(x^2+s^2)\cos\theta + 2xs}{2\sin\theta}\right]\right\} f(s)\,ds$$

where $$A_\theta = (2\pi|\sin\theta|)^{-1/2} \exp\left[-\frac{i}{2}\left(\frac{\pi}{2}\operatorname{sgn}\theta - \theta\right)\right]$$

for $0 < |\theta| < \pi$.

3. F_θ AND THE OPERATORS I, J, J^+ AND J^-

Using the operators D and X defined by $(Df)(x) = (d/dx)f(x)$ and $(Xf)(x) = xf(x)$ then define the operators $J^{\pm}$ and J by

$$(3.1) \qquad \begin{cases} J^+ = 2^{-1/2}(D-X); \quad J^- = 2^{-1/2}(-D-X) \\ \text{and } J = J^+J^- = 2^{-1}(-D^2 + X^2 - I). \end{cases}$$

The h_n are well known to be the eigenfunctions of J [6,10,16] and

$$(3.2) \qquad Jh_n = nh_n; \; J^+h_n = -\sqrt{n+1}\, h_{n+1} \text{ and } J^-h_n = -\sqrt{n}\, h_{n-1}.$$

Under the inner product (1.3) J is self-adjoint and J^+ and J^- are adjoints of one another. One notices that J is just the Schrödinger operator for the simple harmonic oscillator (in appropriate units and with subtraction of the zero-point energy).

The operators obey the commutator relations

$$(3.3) \qquad \begin{cases} [J^+,J^-] = -I; \quad [J,J^+] = J^+; \quad [J,J^-] = -J^- \text{ and} \\ [I,J^+] = [I,J^-] = [I,J] = 0 \quad \text{(the additive identity)} \end{cases}$$

so one can see they constitute a basis for an irreducible representation of a complex 4-dimensional Lie algebra.

I have shown [11] that $-iJ$ is the infinitesimal generator of the Lie group $\{F_\theta\}$; that is, $F_\theta = \exp(-i\theta J)$. Setting $\theta = \pi/2$ gives an interesting representation of the Fourier operator F, closely relating it to the quantum mechanical simple harmonic oscillator:

$$(3.4) \qquad F = \exp(-i\tfrac{\pi}{2}J) = \exp\left[-i\tfrac{\pi}{4}(-D^2 + X^2 - I)\right].$$

J, then, commutes with F_θ but J^+ and J^- do not. The following propositions, however, state some invariance relations involving 2-norms and inner products of $J^\pm f$ that I use to construct F_θ-invariant measures of overall spread.

PROPOSITION 3.1 *For all* $k \in \mathbb{N}$

$\|(J^+)^k f\|$ *and* $\|(J^-)^k f\|$ *are* F_θ*-invariant;*

that is,

$$(3.5a) \qquad \forall\theta \in \mathbb{T} \qquad \|(J^+)^k F_\theta f\| = \|(J^+)^k f\|$$

and

$$(3.5b) \qquad \|(J^-)^k F_\theta f\| = \|(J^-)^k f\|.$$

PROPOSITION 3.2 *For all* $k \in \mathbb{N}$

$$(3.6) \qquad \forall\theta \in \mathbb{T} \qquad \langle (J^+)^k F_\theta f, (J^-)^k F_\theta f\rangle = e^{i2k\theta} \langle (J^+)^k f, (J^-)^k f\rangle.$$

I have outlined the proofs of these for k = 1 in [12].

COROLLARY *For all* $k \in \mathbb{N}$

(3.7) $$|\langle (J^+)^k f, (J^-)^k f\rangle| \text{ is } F_\theta\text{-invariant.}$$

4. THE HEISENBERG MEASURE OF SPREAD, σ_H

In units in which Planck's constant equals 2π Heisenberg's uncertainty principle can be expressed as

(4.1) $$\sigma_H(f) \geq 1/4$$

where the Heisenberg measure $\sigma_H(f)$ of overall spread is the product of the variances of $|f|^2$ and $|\hat{f}|^2$; that is, taking (without loss of generality) both centroids as zero:

(4.2) $$\sigma_H(f) = (\|Xf\|/\|f\|)^2(\|X\hat{f}\|/\|\hat{f}\|)^2.$$

Using the unitarity of F and its basic property that iXF = FD this can be rewritten as

(4.3) $$\sigma_H(f) = \|f\|^{-4}\|Xf\|^2\|Df\|^2.$$

In terms of the set of operators $\{I, J^+, J^-, J\}$ that is clearly the natural one in the context of the fractional Fourier transform F_θ this can be rewritten again [12] as

$$(4.4) \qquad \sigma_H(f) = 4^{-1}\|f\|^{-4}\left\{\left[\|J^+f\|^2+\|J^-f\|^2\right]^2 - 4\left[\Re e\langle J^+f, J^-f\rangle\right]^2\right\}.$$

Using the results of propositions 3.1 and 3.2 (for k=1) one gets theorem 4.1.

THEOREM 4.1 *The Heisenberg measure of overall spread of f and $\hat{f}$, $\sigma_H(f)$, is* <u>*not*</u> *invariant under the fractional Fourier transform F_θ but depends on θ according to the formula:*

$$(4.5) \qquad \sigma_H(F_\theta f) = 4^{-1}\|f\|^{-4}\left\{\left[\|J^+f\|^2+\|J^-f\|^2\right]^2 - 4\left[\Re e\ e^{-i2\theta}\langle J^+f, J^-f\rangle\right]^2\right\}.$$

5. F_θ-INVARIANT MEASURES AND UNCERTAINTY PRINCIPLES

Looking at the Heisenberg measure σ_H in the form (4.4) in the light of the results of theorem 4.1 and propositions 3.1, 3.2 and its corollary leads one to construct a modified and generalized "k-measure", σ_k.

DEFINITION 5.1 The "k-measure" of intrinsic spread of f and $\hat{f}$ is the function $\sigma_k(f)$ ($k \in \mathbb{N}$) where

$$(5.1) \qquad \sigma_k(f)=4^{-1}\|f\|^{-4}\left\{\left[\|(J^+)^kf\|^2+\|(J^-)^kf\|^2\right]^2-4|\langle(J^+)^k f, (J^-)^k f\rangle|^2\right\}.$$

By (3.5) and (3.7) one can see immediately that σ_k is F_θ-invariant.

THEOREM 5.1 *For all f the k-measure of its intrinsic spread $\sigma_k(f)$ satisfies the uncertainty principle:*

(5.2) $$\sigma_k(f) \geq 4^{-1} \| f \|^{-4} \left\{ \| (J^+)^k f \|^2 - \| (J^-)^k f \|^2 \right\}^2.$$

Proof Use the Cauchy-Schwarz-Bunyakovski inequality on the inner-product term in (5.1).

To get a result that can be compared with Heisenberg's it is first convenient to make another definition.

DEFINITION 5.2 The "twistiness" of the function f is the real number v(f) where

(5.3) $$v(f) = \| f \|^{-2} \langle Xf, f \, D \arg f \rangle .$$

(It is zero for functions of constant argument and one can show that $v(\hat{f}) = -v(f)$.)

COROLLARY 5.1 *Heisenberg's uncertainty principle can be both improved and strengthened to:*

(5.4) $$\sigma_1(f) = \sigma_H(f) - v^2(f) \geq 1/4;$$

that is, $\sigma_1(f)$*, the 1-measure of overall spread of* f *and* $\hat{f}$*, is superior not only in being* F_θ*-invariant but also in being a tighter measure, in the sense that*

$$\sigma_H(f) \geq \sigma_1(f) \geq 1/4. \tag{5.5}$$

(In (5.5) there is equality in both places if and only if $f(x) = a \exp(-bx^2)$ $(a \in \mathbf{C},\ b \in \mathbf{R}^+)$*.)*

Proof Put $k = 1$ in definition 5.1 and use definition 5.2 and (4.3) to get $\sigma_1 = \sigma_H - v^2$. On putting $k = 1$ in theorem 5.1 it simplifies to the statement that $\sigma_1(f) \geq 1/4$.

There are many questions here for further research. Looking at the higher values of k should lead to results comparable with Hirschman's extension of Heisenberg's principle to higher order moments [7] and application of the idea of F_θ-invariance to the Landau-Slepian-Pollak work [8,13] would avoid the weaknesses of moment-type measures of spread.

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