On the transference theorem of Coifman and Weiss

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Suppose G is a unimodular locally compact group with a compact subgroup K and a polar decomposition with respect to K and a closed unimodular subgroup H. That is, we are assuming that the map $K \times H \times K \to G$, with $(k_1, h, k_2) \mapsto k_1 h k_2$, is surjective and there is a measurable function ω on H such that

$$\int_G f\, dm_G = \int_H \int_K \int_K f(k_1 h k_2)\omega(h) dm_K(k_1) dm_K(k_2) dm_H(h)$$

for all $f \in C_c(G)$. From Fubini's theorem we see that if f is a bi-K-invariant function on G then $\omega f|_H$ is an integrable function on H. We let $Cv_p(G)$ denote the Banach space of all bounded linear operators on the Lebesgue space $L^p(G)$ which commute with right translation by elements of G and we denote by $\|T\|_{Cv_p(G)}$ the norm of such an operator T. An integrable function f on G gives rise to a bounded, right-translation invariant operator $\lambda_G(f)\varphi := f * \varphi$ on each of the Lebesgue spaces $L^p(G)$. The norm of $\lambda_G(f)$ is determined by testing f against elements of the Herz-Figà-Talamanca algebra $A_p(G)$,

$$\| \lambda_G(f) \|_{Cv_p(G)} = \sup \left\{ \left| \int_G f g\, dm_G \right| : g \in A_p(G) \text{ and } \|g\|_{A_p(G)} \le 1 \right\}.$$

The space $A_p(G)$ is defined in the following manner. Fix $1 < p < \infty$ and consider the projective tensor product $L^p(G)\widehat{\otimes}L^{p'}(G)$, where $1/p + 1/p' = 1$. There is a bounded linear map

$$P : L^p(G)\widehat{\otimes}L^{p'}(G) \to C_0(G),$$

given by $P(f \otimes g) = g * f^\vee$. The image of P is called $A_p(G)$ and is equipped with the quotient norm. That is, a function $\varphi \in A_p(G)$ has a series expansion

$$\varphi = \sum_{j=0}^{\infty} g_j * f_j^\vee$$

with $\sum_j \|g_j\|_{p'} \|f\|_p < \infty$, and the norm of φ is the infimum of all these sums. Every bounded linear operator $T : L^p(G) \to L^p(G)$ can be considered to be a bounded linear functional on $L^p(G)\hat{\otimes}L^{p'}(G)$, using the formula

$$\langle T, \sum_{j=0}^{\infty} f_j \otimes g_j\rangle \;=\; \sum_{j=0}^{\infty} g_j * (Tf_j)^{\vee}(1).$$

Herz has shown that $A_p(G)^*$ can be identified with $PM_p(G)$, the weak*-closure of $\lambda_G(C_c(G))$ in $\left(L^p(G)\hat{\otimes}L^{p'}(G)\right)^*$. In general, $PM_p(G)$ is a subspace of $Cv_p(G)$, but $Cv_p(G) = PM_p(G)$ when $p = 2$ or when G is amenable.

In order to find the norm of the operator $\lambda_G(f)$ when f is bi-K-invariant, it is enough to test against ${}^KA_p(G)^K$, the subalgebra of bi-K-invariant elements. Using Herz's theorem on the restrictions of elements of $A_p(G)$ to H, we can give a simple proof of the following version of the transference theorem of Coifman and Weiss [3].

Theorem 1 (Coifman and Weiss) *Let G,K,H, and p be as above.*

1. *If f is a bi-K-invariant integrable function on G then*

$$\|\,\lambda_G(f)\,\|_{Cv_p(G)} \;\le\; \|\lambda_H(\omega.f|_H\,)\|_{Cv_p(H)}.$$

2. *Suppose that $\{\varphi_\gamma\}_\gamma$ is a net in ${}^KC_c(G)^K$. If there is an operator $T \in PM_p(H)$ such that $\langle T, g\rangle = \lim_\gamma \langle \lambda_H(\omega.\varphi_\gamma|_H), g\rangle$, for every g in $A_p(H)$, then there exists $T' \in {}^KPM_p(G)^K$ such that*

$$\langle T', \psi\rangle = \lim_{\gamma} \langle \lambda_G(\varphi_\gamma), \psi\rangle,$$

for every ψ in $A_p(G)$, and $\|\,T'\,\|_{Cv_p(G)} \;\le\; \|\,T\,\|_{Cv_p(H)}$.

We adopt the approach we used in the paper [8]. Firstly, denote by m_K the Haar measure on K and let $Z : C_0(G) \to C_0(G)$ be the operator

$$Z\varphi \;=\; m_K * \varphi * m_K, \quad \forall \varphi \in C_0(G).$$

Lemma 1 *If $f \in A_p(G)$ then $Zf \in A_p(G)$ and $\|Zf\|_{A_p(G)} \le \|f\|_{A_p(G)}$. Furthermore,*

$${}^KA_p(G)^K \;=\; P(({}^KL^p(G))\hat{\otimes}({}^KL^{p'}(G)).$$

We also let ${}^KCv_p(G)^K$ denote the space of those elements T of $Cv_p(G)$ which satisfy the equation

$$Tf \;=\; m_K * (T(m_K * f))) \quad \forall f \in L^p(G).$$

Clearly, ${}^KL^1(G)^K \subset {}^KCv_p(G)^K$. We can identify ${}^KCv_p(G)^K$ with the space of bounded linear operators on $L^p(K\backslash G)$ which commute with the action of G. Let

$${}^KPM_p(G)^K \;=\; PM_p(G) \cap {}^KCv_p(G)^K.$$

The following lemma is based on Theorem 5 in [5].

Lemma 2 *The space ${}^K PM_p(G)^K$ is the weak*-closure of $\lambda_G\left({}^K C_c(G)^K\right)$ in $PM_p(G)$. If G is amenable and if $S \in {}^K Cv_p(G)^K$ then there is a net $\{\varphi_\gamma\}_\gamma$ contained in ${}^K C_c(G)^K$ and satisfying:*

1. $\|\lambda_G(\varphi_\gamma)\|_{Cv_p(G)} \le \|S\|_{Cv_p(G)}$;
2. $\langle S, \psi\rangle = \lim_\gamma \int_G \psi\varphi_\gamma \, dm_G, \quad \forall \psi \in A_p(G)$.

In the beginning of this note we arranged the Haar measures on G and H in such a way that

$$\int_G f \, dm_G = \int_H (Zf)(h)\omega(h)\, dm_H(h), \quad \forall h \in C_c(G).$$

We will combine this with Theorem 8.7 in [5].

Theorem 2 (Herz) *If G, H, and p are as above and if $f \in A_p(G)$, then $f|_H \in A_p(G)$ and*

$$\|f|_H\|_{A_p(H)} \le \|f\|_{A_p(G)}.$$

The proof of the Theorem 1 depends on the following observation. For a bi-K-invariant function f on G, its norm as a convolution operator on $L^p(G)$ is found by testing against all elements $\varphi \in A_p(G)$ of norm one. That is, we need to estimate

$$\left|\int_G f\varphi \, dm_G\right| = \left|\int_H f|_H(Z\varphi)|_H \omega \, dm_H\right|.$$

Lemma 1 and Theorem 2 combine to show that the right-hand side is dominated by

$$\|\lambda_H(\omega f|_H)\|_{Cv_p(H)} \cdot \|(Z\varphi)|_H\|_{A_p(H)} \le \|\lambda_H(\omega f|_H)\|_{Cv_p(H)} \cdot \|\varphi\|_{A_p(G)}.$$

Part (b) follows in a similar manner, using Lemma 2.

As an example of the application of Theorem 1, there is the result of Coifman and Weiss [2] on central multipliers on L^p-spaces on compact semisimple Lie groups. In that paper the transference from the whole group to the maximal torus is achieved by using the Weyl character formula and the Weyl integration formula. Using an identity of Harish-Chandra for Fourier transforms on Lie algebras, we can prove a similar result for $Ad(G)$-invariant multipliers on L^p-spaces on the Lie algebra of a compact simple Lie group G, viewed as a Euclidean space.

Let $\underline{\underline{g}}$ denote this Lie algebra. Fix a maximal torus T in G and denote its Lie algebra by $\underline{\underline{t}}$. The symbol $\mathcal{F}_{\underline{\underline{g}}}$ denotes the Fourier transform on the Euclidean space $\underline{\underline{g}}$, equipped with the Euclidean structure produced by the Killing form, and $\mathcal{F}_{\underline{\underline{t}}}$ denotes that on $\underline{\underline{t}}$. There is the Cartan motion group M given by the semidirect product of G and $\underline{\underline{g}}$, where G acts by the adjoint action on its Lie algebra. In M, let $K = G \times \{0\}$.

Lemma 3 *For $1 < p < \infty$, the map $f \mapsto f|_{\{1\}\times \underline{\underline{g}}}$ provides an isometric isomorphism*

$$^K A_p(M)^K \cong {}^G A_p(\underline{\underline{g}}),$$

where $^G A_p(\underline{\underline{g}})$ is the set of adjoint invariant elements of $A_p(\underline{\underline{g}})$.

See [9], Lemma 3.6. Combining this with the fact that the dual of $^K A_p(M)^K$ is equal to $^K Cv_p(M)^K$, we see that this last space is isometrically isomorphic with the $Ad(G)$–invariant elements of $Cv_p(\underline{\underline{g}})$, as defined in [8],section 3. It is known that $Cv_p(\underline{\underline{g}})$ is equal to $\mathcal{M}_p(\underline{\underline{g}})$, the space of multipliers of $\mathcal{F}_{\underline{\underline{g}}} L^p(\underline{\underline{g}})$. Now we can identify $^G\mathcal{M}_p(\underline{\underline{g}})$ with $^K Cv_p(M)^K$, noting that M is amenable.

Lemma 4 *For $1 < p < \infty$ there are isometric isomorphisms*

$$^K Cv_p(M)^K \;\cong\; {}^G\mathcal{M}_p(\underline{\underline{g}}) \;\cong\; \{\text{the space of } M-\text{invariant operators on } L^p(\underline{\underline{g}})\}.$$

See section 3 of [8].

Lemma 5 *Suppose $1 < p < \infty$ and h is an $Ad(G)$–invariant, essentially bounded measurable function on $\underline{\underline{g}}$. Then h is in $^G\mathcal{M}_p(\underline{\underline{g}})$ if and only if there is a positive constant C such that*

$$\left| \int_{\underline{\underline{g}}} h(\xi).\mathcal{F}_{\underline{\underline{g}}}\phi(\xi)\, dm_{\underline{\underline{g}}}(\xi) \right| \;\leq\; C\|\phi\|_{A_p(\underline{\underline{g}})}$$

for all $\phi \in {}^G A_p(\underline{\underline{g}}) \cap C_c^\infty(\underline{\underline{g}})$.

Fix an ordering of the root system for (G,T) and denote by R_+ the system of positive roots on $\underline{\underline{t}}$. Let $d = \dim(G)$ and $r = \text{rank}(G)$. For each root α denote by H_α the element of $\underline{\underline{t}}$ which satisfies the equation $\alpha(H) = i(H_\alpha|H)$, for all $H \in \underline{\underline{t}}$. In addition, denote by ∂_α the directional derivative in direction H_α. If f is an $Ad(G)$–invariant function in the Schwartz space $\mathcal{S}(\underline{\underline{g}})$ then

$$\int_{\underline{\underline{g}}} f\, dm_{\underline{\underline{g}}} \;=\; c\int_{\underline{\underline{t}}} f(H) \prod_{\alpha\in R_+} (H|H_\alpha)^2\, dm_{\underline{\underline{t}}}(H).$$

Let $\Phi(H) \;=\; \prod_{\alpha\in R_+}(H|H_\alpha)$, so that Φ is a polynomial of degree $\frac{1}{2}(d-r)$ on $\underline{\underline{t}}$. When we identify $Ad(G)$–invariant functions on $\underline{\underline{g}}$ with bi–K–invariant functions on the motion group M, then the integration formula above matches that at the beginning of this note, with $\omega = |\Phi|^2$ and H equal to $\{1\}\times\underline{\underline{t}}$. Harish-Chandra proved the following formula relating Fourier transforms on $\underline{\underline{g}}$ and $\underline{\underline{t}}$ for all $Ad(G)$ invariant functions f in the Schwartz space on $\underline{\underline{g}}$,

$$\Phi(H)\mathcal{F}_{\underline{\underline{g}}} f(H) \;=\; c\mathcal{F}_{\underline{\underline{t}}}(f|_{\underline{\underline{t}}}\Phi)(H), \quad \forall H \in \underline{\underline{t}}.$$

Theorem 3 *Suppose that h is an essentially bounded, $Ad(G)-$invariant, measurable function on $\underline{\underline{g}}$. If the distributional derivative*

$$\left(\prod_{\alpha\in R_+}\partial_\alpha\right)\left(h|_{\underline{\underline{t}}}\Phi\right)$$

belongs to $\mathcal{M}_p(\underline{\underline{t}})$, then $h \in {}^G\mathcal{M}_p(\underline{\underline{g}})$. There is a constant $\kappa > 0$ depending on $\underline{\underline{g}}$ such that

$$\left\|\lambda_{\underline{\underline{g}}}\left(\mathcal{F}_{\underline{\underline{g}}}^{-1}h\right)\right\|_{Cv_p(\underline{\underline{g}})} \leq \kappa \left\|\lambda_{\underline{\underline{t}}}\left(\mathcal{F}_{\underline{\underline{t}}}^{-1}\left((\prod_{\alpha\in R_+}\partial_\alpha)(\Phi\, h|_{\underline{\underline{t}}})\right)\right)\right\|_{Cv_p(\underline{\underline{t}})}$$

One could then use various Euclidean space multiplier theorems to provide multiplier theorems for $Ad(G)-$invariant operators on $L^p(\underline{\underline{g}})$, for example Theorem A in [7].

Corollary 1 *Suppose that h is an essentially bounded $Ad(G)-$invariant measurable function on $\underline{\underline{g}}$, that $q \geq 2$, and r^* is the least integer greater than r/q. Furthermore, suppose that*

$$\left(\prod_{\alpha\in R_+}\partial_\alpha\right)\left(h|_{\underline{\underline{t}}}\Phi\right) \in L^\infty(\underline{\underline{t}})$$

and for each $r-$tuple of non-negative integers $\mathbf{n}$, with $|\mathbf{n}| \leq (d-r)/2 + r^*$, the distributional derivative $\partial^{\mathbf{n}}(h|_{\underline{\underline{t}}})$ satisfies

$$\sup_{S>0}\left(S^{-r}\int_{S<|H|\leq 2S}\left|S^{|\mathbf{n}|}\partial^{\mathbf{n}}(h|_{\underline{\underline{t}}})\right|^q dm_{\underline{\underline{t}}}\right)^{1/q} < \infty.$$

Then $h \in \mathcal{M}_p(\underline{\underline{g}})$ for all p satisfying $|(1/2)-(1/p)| < 1/q$.

This can be compared with the Theorem in [10]. In the case when G =SU(2), so that we are actually dealing with radial functions on $\mathbf{R}^3$, it is not as strong as the result of [6].

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