RIGIDITY, VON NEUMANN ALGEBRAS, AND SEMISIMPLE GROUPS

by Michael Cowling

The material described below is largely contained in three papers, by U. Haagerup [H], M. Cowling and U. Haagerup [CH], and by M. Cowling and R.J. Zimmer [CZ].

1 WHAT IS RIGIDITY? G. Mostow, with a little help from his friends, proved the following result, known as Mostow's rigidity theorem.

THEOREM. *Let Γ_1 and Γ_2 be lattices in the centreless semisimple Lie groups G_1 and G_2 respectively (i.e. Γ_i is a discrete subgroup of G_i and the homogeneous space G_i/Γ_i has finite G_i-invariant measure, for $i = 1,2$). Any isomorphism $\Gamma_1 \to \Gamma_2$ extends to an isomorphism $G_1 \to G_2$.*

This theorem tells us lattices in nonisomorphic semisimple Lie groups cannot be isomorphic. Subsequently, G. Margulis extended this theorem to show that any homomorphism $\Gamma_1 \to \Gamma_2$ extends to a homomorphism $G_1 \to G_2$, provided that G_1 and G_2 are of real rank at least 2, and Γ_1 and Γ_2 are irreducible lattices. This extension, known as Margulis' super-rigidity theorem, gives more information about how different Γ_1 and Γ_2 must be for different G_1 and G_2. The geometric interest of such theorems lies in the fact that the groups Γ arise naturally as the fundamental groups of certain locally symmetric spaces (of the form $K\backslash G/\Gamma$); one deduces that if the spaces are locally different, in their differential geometric structure, then they are globally different, topologically.

A rigidity theorem, then, is one which tells us that objects — in particular lattices — are different.

There are several ways to look at the question of how groups differ, including the von Neumann algebraic and the ergodic theoretic viewpoints.

In studying groups, one frequently encounters the von Neumann algebra $VN(G)$ of the group G. This is the algebra of all bounded operators on $L^2(G)$ (relative to a left-invariant Haar measure) which commute with right translations. If G is abelian, then $VN(G)$ is isomorphic to $L^\infty(\hat{G})$, where $\hat{G}$ is the Pontryagin dual group of G. Since there are nonisomorphic groups with the same von Neumann algebras (for instance, C_4 and $C_2 \times C_2$, where C_n is the cyclic group of order n), but isomorphic groups do have isomorphic von Neumann algebras, the statement that $VN(\Gamma_1)$ and $VN(\Gamma_2)$ are nonisomorphic is stronger than the statement that Γ_1 and Γ_2 are nonisomorphic. It is believed that lattices in nonisomorphic centreless semisimple Lie groups have nonisomorphic von Neumann algebras, except perhaps if the Lie groups are of real rank one, but only a few proofs in this direction have been found.

The first significant rigidity theorem for von Neumann algebras was established by A. Connes (unpublished, but a more general version of the ideas appears in Connes and V.F.R. Jones [CJ]). D.A. Kazhdan had already introduced "property T" for groups, and Connes translated property T into von Neumann algebraic terms. By dividing von Neumann algebras into those with property T, and those without it, Connes showed that, for instance, the von Neumann algebras of lattices in $SL(2, R)$ and in $SL(3, R)$ are nonisomorphic, as only the second lattice has a von Neumann algebra with property T.

The next work in this direction is by U. Haagerup [H] and M. Cowling and U. Haagerup [CH]. In these papers, which will be discussed in some detail below, it is established for instance that the von Neumann algebras of lattices Γ_0 in $SL(2, R)$, Γ_1 in $SL(3, R)$ and Γ_n in $Sp(n, 1)$ $(n \geq 2)$ are all nonisomorphic.

An alternative approach to differences between groups is through ergodic theory. We say that groups G_1 and G_2 have orbit equivalent finite ergodic theory if there exist finite measure spaces X_1 and X_2 on which G_1 and G_2 act freely and ergodically (by measurable transformations), together with a measure-preserving bijection $\phi : X_1 \to X_2$ which maps G_1-orbits onto G_2-orbits. It is known that countably infinite discrete amenable groups have orbit equivalent finite ergodic theory, so that not having orbit equivalent finite ergodic theory is another strong expression of the difference of two groups. R.J. Zimmer, extending the ideas of Margulis, has proved such assertions about lattices in distinct semisimple Lie groups of higher rank. (See Zimmer's book [Z] for a useful discussion of semisimple groups and ergodic theory). In Cowling and Zimmer [CZ], we show (inter alia) that lattices in $Sp(n, 1)$, for different values of n, do not have orbit equivalent finite ergodic theory.

2 MAIN RESULTS. Most of the results of Haagerup [H], Cowling and Haagerup [CH], and Cowling and Zimmer [CZ] can be summarized as follows: numbers $\Lambda(G)$, $\mathrm{M}(\mathcal{M})$, and $\mathrm{N}(\Gamma, X)$ can be associated to a locally compact group G, a von Neumann algebra $\mathcal{M}$, and an ergodic action of a discrete group Γ on a measure space X. These numbers are invariant under group isomorphisms, von Neumann algebra isomorphisms, and orbit equivalence respectively. They can be computed by harmonic analysis for semisimple Lie groups, and hence derived for lattices, for von Neumann algebras of lattices, and for finite ergodic actions of lattices.

The number $\Lambda(G)$ is defined in terms of approximate identities on G. Before defining $\Lambda(G)$, let us first recall that the Fourier algebra $A(G)$ of G is defined to be the set of all coefficients of the left regular representation λ of G on $L^2(G)$: u is in $A(G)$ if and only if there exist h and k in $L^2(G)$ such that

$$u(x) = \langle \lambda(x)h, k \rangle \qquad \forall x \in G, \tag{1}$$

and

$$\|u\|_A = \min\{\|h\|_2\|k\|_2 : \text{ (1) holds }\}. \tag{2}$$

It is known that $A(G)$ is a Banach algebra under pointwise operations, and $A(G)$ has an identity if and only if G is compact, while $A(G)$ contains an approximate identity if and only if G is amenable. Here by approximate identity we mean a net $\{u_i\}$ of $A(G)$-funtions such that, for some constant C,

$$\|u_i\|_A \le C \qquad \forall i, \tag{3}$$

and

$$u_i \to 1 \text{ locally uniformly} \tag{4}$$

$$(\text{or } \|u_i v - v\|_A \to 0 \qquad \forall v \in A(G)).$$

It turns out that, if there exists such a net, then we may take C equal to 1, and we may also suppose that each u_i has compact support. The concept of approximate identity can be weakened, by replacing the condition (3) by the condition

$$\|u_i\|_{M_0} \le C \qquad \forall i. \tag{5}$$

Here $\| \ \|_{M_0}$ denotes the completely bounded multiplier norm, which will be described below. If this relaxation of condition (3) is made, then more groups have approximate identities than before; further, the values of C become critical. In fact,

$$\Lambda(G) = \inf\{C \in [1,\infty) : \text{ (4) and (5) hold for some net } \{u_i\}\}.$$

We still suppose that the functions u_i have compact support.

Before we define $\mathrm{M}(\mathcal{M})$, we need to discuss completely bounded operators. If $\mathcal{M}$ is a von Neumann algebra, and $T : \mathcal{M} \to \mathcal{M}$ is a continuous linear operator, then it may be possible to extend T to a bounded operator $T \otimes I$ on $\mathcal{M} \otimes \mathcal{N}$ where $\mathcal{N}$ is another von Neumann algebra, and $\mathcal{M} \otimes \mathcal{N}$ denotes the spatial tensor product of $\mathcal{M}$ and $\mathcal{N}$. If this is possible, when $\mathcal{N}$ is the set of all bounded linear operators on a separable Hilbert space, then T is called completely bounded, and $\|T\|_{CB}$, its completely bounded norm, is the usual norm of $T \otimes I$ on $\mathcal{M} \otimes \mathcal{N}$. A completely bounded multiplier of $A(G)$ is defined to be a multiplier operator on $A(G)$ whose adjoint operator, which acts on $VN(G)$, is completely bounded, and the completely bounded multiplier norm is the completely bounded norm of this adjoint operator.

Let $\mathcal{M}$ be a von Neumann algebra. It may be possible to find a net $\{T_i\}$ of operators on $\mathcal{M}$ such that, for some constant C,

$$\|T_i\|_{CB} \leq C \qquad \forall i, \tag{6}$$

$$T_i m \to m \text{ weak-star} \qquad \forall m \in \mathcal{M}, \tag{7}$$

and

$$T_i(\mathcal{M}) \text{ is finite dimensional} \qquad \forall i. \tag{8}$$

The first of these conditions is related to (5), and the second is linked to (4). The last condition is the analogue, for von Neumann algebras, of the condition that each u_i have compact support, at least for discrete groups. We define

$$\mathrm{M}(\mathcal{M}) = \inf\{C \in [1,\infty) : \text{(6), (7) and (8) hold for some net } \{T_i\}\}.$$

These definitions of $\Lambda(G)$ and $\mathrm{M}(\mathcal{M})$ are from Cowling and Haagerup [CH], though they are implicit in Haagerup [H]. In these two papers most of the following results are obtained.

THEOREM. *Let G be a locally compact group, and let $\Lambda(G)$ be as defined above. Then*

(i) if G_1 and G_2 are isomorphic, $\Lambda(G_1) = \Lambda(G_2)$;

(ii) if G_1 is a closed subgroup of G_2, $\Lambda(G_1) \leq \Lambda(G_2)$;

(iii) if $G = G_1 \times G_2$, $\Lambda(G) = \Lambda(G_1) \times \Lambda(G_2)$;

(iv) if Γ *is a lattice in* G, $\Lambda(G) = \Lambda(\Gamma)$;

(v) if Z *is central in* G, $\Lambda(G/Z) = \Lambda(G)$.

THEOREM. *If* G *is an amenable group,* $\Lambda(G) = 1$. *If* G *is a connected noncompact simple Lie group, then* $\Lambda(G)$ *is equal to* 1 *if* G *is locally isomorphic to* $SO(n,1)$ *or* $SU(n,1)$, *to* $2n-1$ *if* G *is locally isomorphic to* $Sp(n,1)$, *to* 21 *if* G *is locally isomorphic to* $F_{4(-20)}$, *and to* $+\infty$ *if* G *is of real rank at least two.*

These two theorems permit the computation of $\Lambda(G)$ for any connected semi-simple Lie group G, and for any lattice in such a group.

THEOREM. *Let* $\mathcal{M}$ *be a von Neumann algebra, and let* $\mathrm{M}(\mathcal{M})$ *be defined as above. Then*

(i) if $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are isomorphic,* $\mathrm{M}(\mathcal{M}_1) = \mathrm{M}(\mathcal{M}_2)$;

(ii) if $\mathcal{M}_1 \subset \mathcal{M}_2$ *and there is a conditional expectation from* $\mathcal{M}_2$ *to* $\mathcal{M}_1$, $\mathrm{M}(\mathcal{M}_1) \leq \mathrm{M}(\mathcal{M}_2)$;

(iii) $\mathrm{M}(\mathcal{M}_1 \otimes \mathcal{M}_2) = \mathrm{M}(\mathcal{M}_1).\mathrm{M}(\mathcal{M}_2)$;

(iv) if Γ *is a discrete group,* $\mathrm{M}(VN(\Gamma)) = \Lambda(\Gamma)$.

This theorem allows us to compute $\mathrm{M}(\mathcal{M})$ for certain $\mathcal{M}$ — the von Neumann algebras of lattices in semisimple Lie groups. It is worth noting explicitly that (iv) fails for connected groups. Indeed, if G is $Sp(n,1)$, then $VN(G)$ is a direct integral of I_∞ factors, associated to the irreducible unitary representations of G, and it is easy to show $\mathrm{M}(VN(G)) = 1$, although $\Lambda(G) = 2n-1$. The point is that operators on $VN(G)$ with finite-dimensional range spaces need not give rise to pointwise multiplier operators on $VN(G)$ or on $A(G)$, and it is rather surprising that (iv) holds at all.

It will be clear how to combine these theorems to obtain the following corollary.

COROLLARY. *Let* Γ_1 *and* Γ_2 *be lattices in* $Sp(n_1,1)$ *and* $Sp(n_2,1)$ *respectively, where* $n_1 \neq n_2$. *Then* $VN(\Gamma_1)$ *and* $VN(\Gamma_2)$ *are nonisomorphic.*

The generalisation of these ideas to the ergodic theoretic context requires some more notation and definitions. Suppose that G is a locally compact group, acting ergodically on the measure space X. Then we consider the Hilbert space $L^2(G \times X)$, on which G acts by the formula

$$(\pi(g)f)(g',x) = f(g^{-1}g', g^{-1}x) \qquad \forall x \in X, \quad \forall g, g' \in G,$$

for any f in $L^2(G \times X)$, and on which $L^\infty(X)$ acts by the rule

$$(\pi(a)f)(g,x) = a(x)f(g,x) \qquad \forall x \in X, \quad \forall g \in G,$$

for any a in $L^\infty(X)$ and any f in $L^2(G \times X)$. We denote by $\mathcal{A}$ the commutative algebra of operators on $L^2(G \times X)$ generated by $\pi(L^\infty(X))$, and by $\mathcal{M}$ the larger algebra generated by $\mathcal{A}$ together with the operators $\pi(g)$ with g in G. In ergodic theory, it is natural to deal with the pair $(\mathcal{M}, \mathcal{A})$. Unfortunately, we have difficulty in doing this, and we deal with the pair (Γ, X), where Γ is discrete and X has finite invariant measure; however, we also need to use $\mathcal{M}$ and $\mathcal{A}$.

The definition of $\mathrm{N}(\Gamma, X)$ is related to the existence of nets of completely bounded operators $\{T_i\}$ on $\mathcal{M}$, having the properties that

(9) $$\|T_i\|_{CB} \leq C \qquad \forall i,$$

(10) $$T_i m \to m \text{ weak-star} \qquad \forall m \in \mathcal{M},$$

and for each i, there exists a finite subset S_i of Γ such that

(11) $$T_i(\mathcal{M}) \subseteq \sum_{\gamma \in S_i} \pi(\gamma)\mathcal{A}.$$

These three conditions are analogues of conditions (6), (7) and (8) for von Neumann algebras. We define

$$\mathrm{N}(\Gamma, X) = \inf\{C \in [1,\infty) : \text{ (9), (10) and (11) hold for some net } \{T_i\}\}.$$

It is quite easy to modify the techniques developed for von Neumann algebras to deal with $\mathrm{N}(\Gamma, X)$ and prove the following result.

THEOREM. *Let Γ be a discrete group acting on a finite measure space X. Then*

(i) if (Γ_1, X_1) and (Γ_2, X_2) are orbit equivalent, $\mathrm{N}(\Gamma_1, X_1) = \mathrm{N}(\Gamma_2, X_2)$;

(ii) $\Lambda(\Gamma) = \mathrm{N}(\Gamma, X)$.

This theorem, combined with the previous ones, proves the statements made earlier about the nonorbit equivalence of the finite ergodic theories of lattices in different groups $Sp(n,1)$. This, and other rigidity results on actions of lattices in $Sp(n,1)$, will appear in [CZ].

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