POWER-BOUNDED ELEMENTS IN A BANACH ALGEBRA AND A THEOREM OF GELFAND

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1. INTRODUCTION

Most of the unattributed results in §§1–4 are joint work with T.J. Ransford; a fully detailed account, including results not given here, is in [2].

Let A be a (complex) unital Banach algebra and let $x \in A$. We say that:

(i) x is *power-bounded* (pb) if and only if there is $K > 0$ such that $\|x^n\| \leq K$ for all $n \geq 0$;

(ii) x is *doubly power-bounded* (dpb) if and only if x is invertible and there is $K > 0$ such that $\|x^n\| \leq K$ for all $n \in \mathbf{Z}$.

We remark that we may always renorm with an equivalent unital algebra norm so that $K = 1$.

From the spectral radius formula: *if x is pb then $Sp\, x \subseteq \Delta \equiv \{z \in \mathbf{C} : |z| \leq 1\}$; if x is dpb then $Sp\, x \subseteq \mathbf{T} \equiv \{z \in \mathbf{C} : |z| = 1\}$.*

We shall give a very simple proof of the following result of Katznelson and Tzafriri [10].

THEOREM 1. (Katznelson & Tzafriri) *Let A be a complex unital Banach algebra and let $x \in A$ be pb. Then $\|x^n(1-x)\| \to 0$ as $n \to \infty$ if (and only if) $(Sp\, x) \cap \mathbf{T} \subseteq \{1\}$.*

Remarks. (i) The 'only if' is trivial.

(ii) If $(Sp\, x) \cap \mathbf{T} = \emptyset$, then $x^n \to 0$, since $r_A(x) < 1$, so the only case of interest is that in which $(Spx) \cap \mathbf{T} = \{1\}$.

(iii) In [10] the result is phrased in terms of *operators*; we have given an equivalent Banach-algebra statement since that is better suited to our method of proof.

(iv) The special case of Theorem 1 in which $Sp\, x = \{1\}$ was already proved by J. Esterle [5, Theorem 9.1]. Our proof of the general case is much influenced by Esterle's proof.

(v) Some other kinds of generalization of Theorem 1 are contained in [1] and [2].

Our starting point is an early result of I.M. Gelfand [6].

THEOREM 2. (Gelfand) *With A as before let $x \in A$ be dpb and let $Sp\, x = \{1\}$. Then $x = 1$.*

Remark. Theorem 2 may be very easily deduced from Theorem 1, since if $x \in A$ with $Spx = \{1\}$ and if $\|x^n\| \le K$ $(n \in \mathbf{Z})$, then for all $n \ge 1$,

$$\begin{aligned} \|x-1\| &= \|x^{-n}.x^n(x-1)\| \\ &\le \|x^{-n}\|.\|x^n(x-1)\| \\ &\le K\|x^n(x-1)\| \to 0 \quad \text{as } n \to \infty , \end{aligned}$$

by Theorem 1; hence $x = 1$.

However, our purpose here is to give a direct proof of Theorem 2 and then show how to deduce Theorem 1 from Theorem 2.

Proof. Write $x = 1+r$, so that $Sp\, r = \{0\}$. Set $y = \mathrm{Log}\,(1+r) \equiv \sum_{k=1}^{\infty}(-1)^{k-1}r^k/k$, so we have $Sp\, y = \{0\}$ and $x = e^y$. We define $F(z) = e^{zy}$ $(z \in \mathbf{C})$; then F is an entire A-valued function of exponential type. Since $r_A(y) = 0$, F has minimal exponential type: i.e. for every $\epsilon > 0$, there is a constant $C(\epsilon) > 0$ such that

$$\|F(z)\| \le C(\epsilon)e^{\epsilon|z|} \quad (z \in \mathbf{C}) .$$

For every $n \in \mathbf{Z}$,

$$\|F(n)\| \leq \sup_{n\in\mathbf{Z}} \|x^n\| < \infty ,$$

by the assumed double-power-boundedness of x. This is already sufficient to prove that F is constant (by [4, (10.2.1)]). However, in the present case, we may use a more elementary argument since for every $t \in \mathbf{R}$

$$\begin{aligned}\|F(t)\| &= \|e^{([t]+\{t\})y}\| \\ &\leq \big(\sup_{n\in\mathbf{Z}} \|x^n\|\big)\big(\sup_{0\leq t\leq 1} \|F(t)\|\big) < \infty .\end{aligned}$$

Thus F is bounded on $\mathbf{R}$ and is therefore constant by a Phragmén–Lindelöf argument (see e.g. [4, (6.2.13)]).

In particular $F(1) = F(0)$, i.e. $x = 1$ and the proof is complete.

Remark. In the above proof, we have freely applied to a holomorphic A-valued function, F, theorems proved classically for complex-functions. This is justified, of course, by the usual functional-analytic device of considering $\Lambda \circ F$ for an arbitrary continuous linear functional Λ on A and using the Hahn–Banach theorem.

2. DEDUCTION OF THEOREM 1 FROM THEOREM 2

The most important point of this paper is the following lemma that relates power-boundedness to double-power-boundedness.

LEMMA. *Let $(A; \|\cdot\|)$ be a commutative, unital Banach algebra and let $x \in A$ be such that $\|x\| = r_A(x) = 1$. Then there is a commutative Banach algebra $B \equiv B(x)$ with norm $\|\cdot\|_B$ and a unital homomorphism $\pi : A \to B$ such that:*

(i) $\|\pi(y)\|_B = \lim_{n\to\infty} \|x^n y\| \quad (y \in A)$;

(ii) *$\pi(x)$ is invertible in B and*

$$\|\pi(x)\|_B = \|\pi(x)^{-1}\|_B = 1 ;$$

(iii) $Sp_B(\pi(x)) \subseteq (Sp_A x) \cap \mathbf{T}$.

Proof. Remark first that $\|x^n\| = r_A(x^n) = 1$ for all $n \geq 1$. Define a semi-norm p on A by setting $p(y) = \lim_{n\to\infty} \|x^n y\| \quad (y \in A)$; since $\|x^{n+1}y\| \leq \|x^n y\|$ for all n, p is well-algebra semi-norm on A; also $p(1) = 1$ and $p(x\, y) = p(y) \quad (y \in A)$.

Let $N = p^{-1}(0)$; then N is an ideal and p is constant on each coset of N. Let $B_0 = A/N$ and let $\pi_0 : A \to B_0$ be the quotient homomorphism. There is then a well-defined unital algebra-norm $\|\cdot\|_0$ on B_0 given by $\|\pi_0(y)\|_0 = p(y) \quad (y \in A)$.

For all $\xi \in B_0$ we have $\|\xi\|_0 = \|\xi\pi_0(x)\|_0$ and so, setting $S = \{\pi_0(x)^n : n \geq 0\}$, S is a multiplicative semi-group in B_0 consisting of non-zero-divisors. We may then form the algebra B_1 of all 'fractions' with denominators in S, $B_1 \equiv \{\xi/\pi_0(x)^n : \xi \in B_0\, ,\, n \in \mathbf{Z}^+\}$. Because of the property that $\|\xi\|_0 = \|\xi\pi_0(x)\|_0 \quad (\xi \in B_0)$, we may simply norm B_1 by setting $\|\xi/\pi_0(x)^n\|_1 = \|\xi\|_0 \quad (\xi \in B_0,\ n \in \mathbf{Z}^+)$. Then $(B_1; \|\cdot\|_1)$ is a unital normed algebra and the mapping $\xi \mapsto \xi/1$ is an isometric embedding of $B_0 \to B_1$; we regard B_0 as a subalgebra of B_1 via this embedding. Then $\pi_0(x)$ is a unit of B_1 and $\|\pi_0(x)\|_1 = \|\pi_0(x)^{-1}\|_1 = 1$. Finally we let $(B; \|\cdot\|_B)$ be the completion of $(B_1, \|\cdot\|_1)$ and let $\pi : A \to B$ be the composition of $\pi_0 : A \to B_1$ and the isometric embedding of B_1 in its completion B. Evidently (i) and (ii) are satisfied.

Also, since $\pi(x)$ is dpb in B we have $Sp_B(\pi(x)) \subseteq \mathbf{T}$; but also, since π is a unital homomorphism, $Sp_B(\pi(x)) \subseteq Sp_A x$. Thus $Sp_B(\pi(x)) \subseteq (Sp_A x) \cap \mathbf{T}$ and (iii) is proved.

Remarks. (i) Instead of the construction with fractions, we could (and, in [2], *did*) appeal to a more general extension result of Arens [3].

(ii) If $Sp_A(x) \subsetneqq \Delta$ — which is certainly the case for the conditions of Theorem 1 — it is easy to show that $\pi_0(x)$ is already invertible just in the completion of $(B_0; \|\cdot\|_0)$, and we may take $(B; \|\cdot\|_B)$ to be that completion, without any extension by fractions.

Proof of Theorem 1. In fact we may almost as easily deduce the following two-variable version (see [10, Corollary 9]): Write $D \equiv \{(\lambda, \mu) \in \mathbb{C}^2 : \lambda = \mu\}$.

THEOREM 3. *Let A be a commutative unital Banach algebra and let $a, b \in A$. Suppose that both a and b are pb. Then*

$$\|a^n b^n (a - b)\| \to 0 \qquad \text{as } n \to \infty$$

if (and only if) $Sp_A(a, b) \cap \mathbb{T}^2 \subseteq D$.

Remarks. (i) the 'only if' part is trivial — consider any character on A;

(ii) to deduce Theorem 1 from Theorem 3 we just take $a = x$, $b = 1$ (clearly, in Theorem 1, we may suppose A *commutative* by considering the closed subalgebra generated by x.)

Proof of Theorem 3. We may assume that $Sp_A(a, b) \cap \mathbb{T}^2 \neq \emptyset$, since otherwise $r_A(a\,b) < 1$ so that $(ab)^n \to 0$ and the conclusion would follow trivially.

Let $x = ab$; then x is pb and $(Sp_A x) \cap \mathbb{T} \supseteq \{\lambda^2 : (\lambda, \lambda) \in Sp_A(a, b) \cap \mathbb{T}^2\} \neq \emptyset$. Also, by renorming, we may assume that $\|a\| = \|b\| = 1$; in that case $1 = r(x) \leq \|x\| \leq \|a\|.\|b\| = 1$, so also $\|x\| = 1$.

We now form $\pi : A \to B \equiv B(x)$ as in the above Lemma. If χ is any character on B then $\chi \circ \pi$ is a character on A, so that $\big(\chi(\pi(a)), \chi(\pi(b))\big) \in Sp_A(a, b) \subseteq \Delta^2$. But also $\|\pi(x)\|_B = \|\pi(x)^{-1}\|_B = 1$, so that $|\chi(\pi(a))|.|\chi(\pi(b))| \in Sp_A(a, b) \cap \mathbb{T}^2 \subseteq D$, by hypothesis. Thus $\chi(\pi(a)) = \chi(\pi(b)) \in \mathbb{T}$ and so $\chi\big(\pi(a)\pi(b)^{-1}\big) = \{1\}$.

Also, as in the Lemma,

$$\|\pi(a)\pi(b)^{-1}\|_B = \|\pi(a)\pi(b)^{-1}\pi(x)\|_B = \|\pi(a)^2\| = 1\ ,$$

and similarly $\left\|\pi(b)\pi(a)^{-1}\right\|_B = 1$. Hence, by Gelfand's Theorem 2, $\pi(a)\pi(b)^{-1} = \pi(1)$ so $\pi(a) = \pi(b)$, $\pi(a-b) = 0$ so $\left\|(ab)^n(a-b)\right\| \to 0$ as $n \to \infty$, by (i) of the Lemma. This completes the proof.

3. POWER-DOMINATION

In [2] we considered a generalization of the notion of power-boundedness for which the term *power-domination* was coined.

Specifically, let $\boldsymbol{\mu} \equiv \left(\mu(n)\right)_{n\geq 0}$ be a sequence of positive real numbers such that $\mu(n+1)/\mu(n) \to 1$ as $n \to \infty$. We say that the element x of a Banach algebra A is *power-dominated by* $\boldsymbol{\mu}$ if and only if $\|x^n\| \leq C\mu(n) \quad (n \geq 0)$, where C is a positive constant. We observe (see [2 (2.1)]) that x may be power-dominated by such a $\boldsymbol{\mu}$ if and only if $r_A(x) \leq 1$.

A very similar result to the Lemma of §2 may then be proved and, from Theorem 2 we may then, for example, deduce:

THEOREM 4. *Let A, $\boldsymbol{\mu}$ be as above, let $x \in A$ be power-dominated by $\boldsymbol{\mu}$ and let $(Sp_A x) \cap \mathbf{T} \subseteq \{1\}$. Then $\mu(n)^{-1}\|x^n(1-x)\| \to 0$ as $n \to \infty$.*

For the details of this, and some more general results, in which $Sp_A x$ may intersect $\mathbf{T}$ in an arbitrary closed set, we refer to [2].

4. REMARKS ON HILLE'S EXTENSION OF GELFAND'S THEOREM

In [8] (see also [9]), Hille gave the following extension of Theorem 2; it may be proved in just the way that we have given for Theorem 2.

THEOREM 5. (Hille) *Let A be a unital Banach algebra and let $x \in A$ with $Sp x = \{1\}$. Suppose that*

$$\|x^n\| + \|x^{-n}\| = o(n) \quad \text{as } n \to \infty .$$

Then $x = 1$.

Remark. The '$o(n)$ condition' may *not* be replaced by '$O(n)$' — consider $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in the algebra $M_2(\mathbf{C})$.

However, a straightforward attempt to extend the Katznelson–Tzafriri result in this way does not succeed. A number of rather strong counter-examples are given in §4 of [2]. We do not, however, know whether the additional restriction that $Sp\, x = \{1\}$ would save the situation; i.e. we ask:

Question. Let A be a (commutative) unital Banach algebra and let $x \in A$ with $Sp\, x = \{1\}$. Suppose that $\|x^n\| = o(n)$ as $n \to \infty$. Does it follows that $x = 1$?

5. COMMUTATIVE RADICAL BANACH ALGEBRAS

Let R be a commutative radical Banach algebra and let $A \equiv R_+$ be its unitization. We say that $x \in R$ is *quasi-power-bounded* (qpb) if and only if $1 + x$ is pb in A. We collect some elementary properties in the following theorem. For $x \in R$ we write $x' \equiv (1 + x)^{-1} - 1$, the 'quasi-inverse' of x.

THEOREM 6. *Let R be a non-zero, commutative radical Banach algebra and let Q be its set of qpb elements. Then:*

(i) *Q is convex and $0 \in Q$;*

(ii) *if $x, y \in Q$ then $x \circ y \equiv x + y + xy \in Q$;*

(iii) *if $0 \neq x \in Q$, then $x' \notin Q$;*

(iv) *Q does not include any neighbourhood of 0.*

Proof. (i) Evidently $0 \in Q$, since 1 is trivially pb. Let $x, y \in Q$ and let $a = 1 + x$, $b = 1 + y$; then for some $K > 0$, $\|a^n\| \leq K$, $\|b^n\| \leq K \quad (n \geq 0)$. Let $0 < t < 1$; then $1 + \big(tx + (1-t)y\big) = ta + (1-t)b$ and for $n \geq 1$,

$$\begin{aligned}\|(ta + (1-t)b)^n\| &\leq \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \|a^k\| . \|b^{n-k}\| \\ &\leq K^2 .\end{aligned}$$

Hence $tx + (1-t)y \in Q$, so that Q is convex.

(ii) If $x, y \in Q$ then $(1+x)(1+y) = 1 + x + y + xy$ is pb, so that $x + y + xy$ is qpb.

(iii) Suppose that x, x' both belong to Q.

Then $(1+x)$ and $(1+x)^{-1}$ are both pb, that is, $1+x$ is dpb in $A \equiv R_+$. Since $Sp_A(1+x) = \{1\}$ it follows from Gelfand's theorem (Theorem 2) that $1 + x = 1$, so $x = x' = 0$.

(iv) Clearly, given $\epsilon > 0$ there exists $\delta > 0$ such that $\|x\| < \delta$ implies $\|x'\| < \epsilon$. Hence if Q contained a neighbourhood of 0 there would be some non-zero $x \in Q$ such that also $x' \in Q$, contrary to (iii). This concludes the proof.

We now consider some elementary consequences of Theorem 1; in fact we use only Esterle's special case of that result ([5, Theorem 9.1]).

THEOREM 7. *With the notation of Theorem 6, let $0 \neq x \in Q$. Then*

(i) $x(1+x)^n \to 0 \quad (n \to \infty)$;

(ii) *x belongs to the closed linear span of $\{x^k : k \geq 2\}$ — so, in particular, $x \in \overline{Rx}$ and x has finite closed descent;*

(iii) *$\{1 - (1+x)^n : n \geq 1\}$ is a bounded approximate identity for $\overline{Rx}$;*

(iv) $\overline{Rx} = \{y \in R : y(1+x)^n \to 0 \ (n \to \infty)\}$.

Proof. (i) Since $1+x$ is pb and $Sp_A(1+x) = \{1\}$, it follows from Theorem 1 that $x(1+x)^n \to 0$. (ii), (iii) and (iv) now follow almost immediately.

We shall now give some examples to show that both cases $Q = (0)$ and $Q \neq (0)$ may occur.

Example 1. Some commutative radical Banach algebras with $Q \neq (0)$.

Let $A(\Delta)$ be the usual disc algebra and let $M = \{f \in A(\Delta) : f(1) = 0\}$; remark that M is a maximal ideal and is the closed ideal generated by $z-1$, where 'z' denotes the co-ordinate function in $A(\Delta)$. It is well known that, for real $t > 0$, the ideal,

$$I_t \equiv M \exp\left(t(z+1)/(z-1)\right),$$

is a closed ideal of $A(\Delta)$, that M is the *only* maximal ideal of $A(\Delta)$, that contains I_t, and that $I_t \neq M$, since $z-1 \notin I_t$. (See e.g. Hoffman [7].)

Let $R_t = M/I_t$; then R_t is a non-zero commutative radical Banach algebra whose unitization is $A(\Delta)/I_t$. Let $\pi_t : A(\Delta) \to (R_t)_+$ be the quotient map and set $x = \pi_t(z-1)$. Then $x \neq 0$ and $1+x = \pi_t(z)$, which is certainly pb, since z is pb in $A(\Delta)$. Thus R_t contains a non-zero qpb element.

Similar (but different) examples may be constructed by starting with $A^+(\mathbf{T})$ in place of $A(\Delta)$ and quotienting out the closure of the ideal

$$J_t = A^+(\mathbf{T})(z-1)^2 \exp\left(t(z+1)/(z-1)\right).$$

To give a class of algebras having no (non-zero) qpb elements we recall that a (commutative) radical Banach algebra R is called *uniformly radical* if and only if $\|x^n\|^{1/n} \to 0$ uniformly on the unit ball of R. For example the convolution algebra $C_*[0,1]$ of all continuous complex-valued functions on $[0,1]$ with uniform norm, but convolution product, is easily seen to be uniformly radical. By contrast, $L^1[0,1]$ is *not* uniformly radical, as follows from the following simple result.

LEMMA 8. *Let R be a commutative, uniformly radical Banach algebra. Then no non-zero closed subalgebra of R has a bounded approximate identity.*

Proof. Since, evidently, any closed subalgebra of R is itself uniformly radical, it suffices to prove that R has no bounded approximate identity (for $R \neq (0)$).

Suppose that $(e_\lambda)_{\lambda\in\Lambda}$ is a b.a.i. for R, $\|e_\lambda\| \leq K \quad (\lambda \in \Lambda)$. Then it is a simple exercise to see that, for given $n \geq 1$, the net $(e_\lambda^n)_{\lambda\in\Lambda}$ is also a b.a.i., $\|e_\lambda^n\| \leq K^n$. Choose $x \in R$, $\|x\| = 1$ (since $R \neq (0)$). Let n be given; then we may choose $\lambda \in \Lambda$ such that $\|xe_\lambda^n - x\| \leq \frac{1}{2}$. Hence $\|e_\lambda^n\| \geq \|xe_\lambda^n\| \geq \|x\| - \frac{1}{2} = \frac{1}{2}$. But then $\|K^{-1}e_\lambda\| \leq 1$ and $\left\|(K^{-1}e_\lambda)^n\right\|^{1/n} \geq (2K)^{-1}$. Hence, for every $n \geq 1$,

$$\sup_{\|x\|\leq 1} \|x^n\|^{1/n} \geq (2K)^{-1} ,$$

so that R is not uniformly radical — a contradiction.

Example 2. If R is any commutative, uniformly radical Banach algebra — for example $R = C_*[0,1]$ — then R has no non-zero qpb element.

For if $x \in R$ is qpb then, by Theorem 7(iii), $\overline{Rx}$ has a bounded approximate identity; so by Lemma 8, $\overline{Rx} = (0)$ — so $x = 0$ (since $x \in \overline{Rx}$).

Our basic problem now is to discover *which* commutative radical Banach algebras contain non-zero qpb elements. (By Theorem 6, if $0 \neq x \in Q$ then at least $[0,x] \subseteq Q$, so that Q is uncountable — and in fact also x^2+2x, $x^3+3x^2+3x, \ldots$ belong to Q as well.) For example we may ask (with R a non-zero commutative radical Banach algebra):

1. If R has a b.a.i., does R have some non-zero qpb element?

2. Do the algebras $L^1[0,1]$, $M_*[0,1]$, $L^1(\mathbf{R}^+, w)$ (with w a radical weight) have any non-zero qpb elements. If so, then what are they?

We mention a few more oddments which help to restrict the search!

PROPOSITION 9. (i) *In any R, if x is a proper nilpotent, then x is not qpb.*

(ii) *If $0 \neq f$ in $L^1[0,1]$ or in $L^1(\mathbf{R}^+ w)$ and if f is qpb then f can not be a.e. non-negative on $[0,\delta]$, for any $\delta > 0$; in particular $\alpha(f) \equiv \inf(\text{supp} f) = 0$.*

(iii) *If w is a radical weight on $\mathbf{Z}^+$, then the unique maximal ideal $\ell_0^1(w)$ of $\ell^1(w)$ has no non-zero qpb element.*

Proof. (i) Let $0 \neq x \in R$ and suppose that $x^{k+1} = 0$, $x^k \neq 0$ for some $k \geq 1$. Then for $n > 2k$ we have $\binom{n}{r} < \binom{n}{r+1}$, $(r = 0, \ldots, k-1)$. Now, for $n > 2k$,

$$\begin{aligned}
\|(1+x)^n\| &= \left\| \sum_{r=0}^{k} \binom{n}{r} x^r \right\| \\
&\geq \binom{n}{k} \|x^k\| - \sum_{r=0}^{k-1} \binom{n}{r} \|x^r\| \\
&\geq \binom{n}{k-1} \left\{ \left(\frac{n-k+1}{k} \right) \|x^k\| - \sum_{r=0}^{k-1} \|x^r\| \right\} \\
&\to \infty \quad \text{as } n \to \infty .
\end{aligned}$$

Hence $(1+x)$ is not pb, so x is not qpb.

(ii) Suppose firstly that $0 \neq f \in L^1[0,1]$ with $f(t) \geq 0$ on [0,1]. Then $\|(1+f)^n\| \geq 1 + n\int_0^1 f(t)\,dt$ for all $n \geq 1$, so that f is not qpb.

The remainder of (ii) is proved by observing that if qpb f (in either $L^1[0,1]$ or $L^1(\mathbf{R}^+, w)$) is a.e. non-negative on some $[0,\delta]$, then by quotienting out the ideal of functions vanishing on $[0,\delta]$, we obtain an element of $L^1[0,\delta] \simeq L^1[0,1]$ that is both a.e. non-negative and also qpb (since, clearly, the continuous homomorphic image of a qpb element is qpb). But this is then reduced to the case already proved.

(iii) Exercise!

Added in proof. In discussion with J. Esterle, it has been remarked that Theorem 6(iv) may be improved to give that 0 *is an extreme point of* Q (using a well-known result of Bohnenblust and Karlin).

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