Remarks on Non-Commutative Interpolation

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We discuss a non-commutative extension of a well-known result of Calderón characterizing interpolation spaces for the couple (L^1, L^∞). The characterization is given in terms of the generalized singular value functions (or decreasing rearrangement) of measurable operators affiliated with a semi-finite von Neumann algebra.

0. Introduction

The theme of this note is that many well-known interpolation theorems in rearrangement invariant function spaces admit extensions to the corresponding spaces of measurable operators affiliated with semi-finite von Neumann algebras. Of course, this theme is not new: a non-commutative extension of the Riesz-Thorin theorem was first given by Kunze [Ku], extensions of the characterization of interpolation spaces for the pair (L^1, L^∞) due to Calderón [Ca1] have been given by Russu [Ru], Ovčinnikov [Ov1] and Yeadon [Ye] and applications of the real interpolation method in the non-commutative setting may be found in Peetre-Sparr [PS]. It is our intention here to briefly survey some of this earlier work and to present extensions to the more general setting of some results for trace ideals due to Arazy [Ar]. These results permit the ready identification of certain operator spaces constructed via the application of an arbitrary exact interpolation functor and consequently reduce the non-commutative versions of the Riesz-Thorin and Marcinkiewicz theorems to the well-known commutative versions. The details, which will appear elsewhere [DDP4], draw on the earlier ideas of Ovčinnikov [Ov1] and Yeadon [Ye] as well as the construction of symmetric operator spaces given elsewhere in this volume [DDP3], and to which we adhere for notation and terminology not further explained in present article.

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1. Spaces intermediate for the couple $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$

We suppose throughout that $\mathcal{M}$, $\mathcal{N}$ are semi-finite von Neumann algebras with (fixed) faithful normal semi-finite traces τ, σ respectively. By $\widetilde{\mathcal{M}}$ (respectively $\widetilde{\mathcal{N}}$) we denote the linear space of operators which are τ-measurable (respectively σ-measurable) in the sense of Nelson [Ne]. If $x \in \widetilde{\mathcal{M}}$ then the *decreasing rearrangement* $\mu_.(x)$ (or *generalized singular value function*) of x is defined to be the right continuous, non-increasing inverse to the (extended) real-valued distribution function

$$s \longrightarrow \tau(\chi_{(s,\infty)}(|x|)), \qquad \text{for } s \in (0, \infty).$$

If $x \in \widetilde{\mathcal{M}}$ and $y \in \widetilde{\mathcal{N}}$ then we say that y is *submajorized by* x and write $y \prec\prec x$ if and only if

$$\int_0^t \mu_s(y)ds \leq \int_0^t \mu_s(x)ds, \qquad t \geq 0.$$

We denote by $\sum(\mathcal{M},\mathcal{N})$ the set of those linear maps T from $L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$ to $L^1(\widetilde{\mathcal{N}}) + \mathcal{N}$ whose restrictions to $L^1(\widetilde{\mathcal{M}})$, $\mathcal{M}$ are continuous linear maps of norm at most one into the spaces $L^1(\widetilde{\mathcal{N}})$, $\mathcal{N}$ respectively and by $\sum(\mathcal{M},\mathcal{N})_+$ the set of all $T \in \sum(\mathcal{M},\mathcal{N})$ such that $Tx \geq 0$ for all $0 \leq x \in L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$. An important example worth noting immediately arises when $\mathcal{N}$ is a von Neumann subalgebra of $\widetilde{\mathcal{M}}$ for which σ is just the restriction of τ (such a subalgebra will be called *proper*). In this case, the conditional expectation of $L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$ onto $L^1(\widetilde{\mathcal{N}}) + \mathcal{N}$, defined initially on $L^1(\widetilde{\mathcal{M}}), \mathcal{M}$ exactly as in the commutative setting via well known duality theory for the pair $(L^1(\widetilde{\mathcal{N}}), \mathcal{N})$, extends to a uniquely determined element of $\sum(\mathcal{M},\mathcal{N})_+$ which maps $L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$ onto $L^1(\widetilde{\mathcal{N}}) + \mathcal{N}$.

Our starting point is the following result.

Theorem 1.1. (i) *If $x \in L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$ and $y \in L^1(\widetilde{\mathcal{N}}) + \mathcal{N}$ then $y \prec\prec x$ if and only if there exists $T \in \sum(\mathcal{M},\mathcal{N})$ such that $y = Tx$.*

(ii) *If* $0 \le x \in L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$ *and* $0 \le y \in L^1(\widetilde{\mathcal{N}}) + \mathcal{N}$ *then* $y \prec\prec x$ *if and only if there exists* $T \in \sum(\mathcal{M}, \mathcal{N})_+$ *such that* $y = Tx$.

For the special case that the von Neumann algebras $\mathcal{M}$, $\mathcal{N}$ are commutative, the preceding result is due to Calderón ([Ca2], Theorems 1 and 2) for measure algebras which are σ-finite and to Fremlin ([Fr] Theorem 24) for localizable measure algebras. In the non-commutative setting, a restricted form of Theorem 1.1 was (essentially) proved by Ovčinnikov [Ov2] and stated, under the present assumptions, in Yeadon ([Ye2], Proposition 3.4); however, the proof given by Yeadon via [Ye2] Proposition 3.3 fails in general if the trace is not assumed to be finite. A direct proof of Theorem 1.1, following the ideas of Fremlin [Fr], is given in [DDP2] and is based on a separation argument via the following result which is of independent interest.

Proposition 1.2. *If* $0 \le x \in L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$ *and if* $0 \le y \in L^1(\widetilde{\mathcal{N}}) \cap \mathcal{N}$ *then*

$$\int_{[0,\infty)} \mu_t(x)\mu_t(y)dt = \sup\{\sigma(yTx) : T \in \textstyle\sum(\mathcal{M}, \mathcal{N})_+\}.$$

To mention some consequences, we introduce some convenient notation. For each $x \in \widetilde{\mathcal{M}}$ we define

$$\Omega_{\mathcal{N}}(x) = \{y \in \widetilde{\mathcal{N}} : y \prec\prec x\},$$

$$\Omega^+_{\mathcal{N}}(x) = \{0 \le y \in \widetilde{\mathcal{N}} : y \prec\prec x\}.$$

Corollary 1.3. *If* $x \in L(\widetilde{\mathcal{M}}) + \mathcal{M}$, *then each of the sets* $\Omega_{\mathcal{N}}(x), \Omega^+_{\mathcal{N}}(x)$ *are convex and* $\sigma(L^1(\widetilde{\mathcal{N}}) + \mathcal{N}, L^1(\widetilde{\mathcal{N}}) \cap \mathcal{N})$ *compact.*

For the case that $\mathcal{M}, \mathcal{N}$ are commutative, this result is due to Fremlin ([Fr] Corollary 7) and is closely related to earlier results of Luxemburg ([Lu], Theorem 15.3) and Ryff ([Ry], Theorem 2 of section 3). In this connection, we mention the more recent result of F.A. Sukochev [Su] : If $\mathcal{M} = \mathcal{N}$, $\mathcal{M}$ has finite trace and if $x \in L^1(\widetilde{\mathcal{M}})$, then y is an extreme point of the orbit $\Omega_{\mathcal{N}}(x)$ if and only if $\mu(y) = \mu(x)$.

A further simple consequence of Theorem 1.1 extends a commutative result of Lorentz and Shimogaki [LS1].

Corollary 1.4. *If $y \in L^1(\widetilde{\mathcal{N}}) + \mathcal{N}$ and if $x_1, x_2 \in L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$ satisfy $y \prec\prec x_1 + x_2$, then there exist $y_1, y_2 \in L^1(\widetilde{\mathcal{N}}) + \mathcal{N}$ (which may be taken to be positive if y, x_1, x_2 are all positive) such that*

$$y = y_1 + y_2 \qquad \text{and} \qquad y_i \prec\prec x_i,\ i = 1, 2.$$

The importance of Theorem 1.1 in the commutative setting to interpolation theory is, of course, clear from [Ca]. We now formulate Theorem 1.1 as an explicit interpolation theorem. For terminology not otherwise explained we refer to [KPS].

Theorem 1.5. *Let E, F be linear subspaces of $L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$, $L^1(\widetilde{\mathcal{N}}) + \mathcal{N}$ respectively.*
(i) *The following statements are equivalent.*

(a) *Each bounded operator from the couple $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$ to the couple $(L^1(\widetilde{\mathcal{N}}), \mathcal{N})$ maps E into F.*

(b) *Whenever $x \in E, y \in L^1(\widetilde{\mathcal{N}}) + \mathcal{N}$ satisfy $y \prec\prec x$, it follows that $y \in F$.*

(ii) *If, in addition, the spaces E, F are normed spaces, then the following statements are equivalent.*

(a) *Each bounded operator from the couple $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$ into the couple $(L^1(\widetilde{\mathcal{N}}), \mathcal{N})$ of norm at most one maps E into F with norm at most one.*

(b) *Whenever $x \in E, y \in L^1(\widetilde{\mathcal{N}}) + \mathcal{N}$ satisfy $y \prec\prec x$, it follows that $y \in F$ and $\|y\|_F \leq \|x\|_E$.*

For symmetrically normed ideals of compact operators the preceding theorem was first proved by G.I. Russu [Ru] and subsequently generalized by Ovčinnikov [Ov2] to the present setting subject to the restriction that $\mu_t(x), \mu_t(y) \to 0$ as $t \to \infty$ for all $x \in E, y \in F$.

It is now convenient to introduce some further terminology. A Banach space E which is a linear subspace of $L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$ will be called

(i) *rearrangement invariant* if and only if

$$x \in E,\ y \in L^1(\widetilde{\mathcal{M}}) + \mathcal{M} \text{ and } \mu(y) \leq \mu(x) \Longrightarrow y \in E \text{ and } \|y\|_E \leq \|x\|_E;$$

(ii) *symmetric* if and only if

$$x, y \in E \text{ and } y \prec\prec x \Longrightarrow \|y\|_E \leq \|x\|_E;$$

(iii) *fully symmetric* if and only if

$$x \in E,\ y \in L^1(\widetilde{\mathcal{M}}) + \mathcal{M} \text{ and } y \prec\prec x \Longrightarrow y \in E \text{ and } \|y\|_E \leq \|x\|_E.$$

We identify $L^\infty(\mathbb{R}^+)$ throughout as a commutative von Neumann algebra acting by multiplication on $L^2(\mathbb{R}^+)$ with trace given by integration with respect to Lebesgue measure. A Banach space $E(\mathbb{R}^+)$ of almost everywhere finite, measurable functions on $\mathbb{R}^+$ will be called a *rearrangement invariant (symmetric, fully symmetric) Banach function space* on $\mathbb{R}^+$ if the corresponding conditions above hold with respect to the von Neumann algebra $L^\infty(\mathbb{R}^+)$.

We note that the above terminology is perhaps somewhat unfortunate as it differs from that of [KPS]; this, however, should cause no confusion in the sequel.

If $E(\mathbb{R}^+)$ is a rearrangement invariant symmetric Banach function space on $\mathbb{R}^+$, we set

$$E(\widetilde{\mathcal{M}}) = \{x \in \widetilde{\mathcal{M}} : \mu(x) \in E(\mathbb{R}^+)\},$$

and if $x \in E(\widetilde{\mathcal{M}})$, we define

$$\|x\|_{E(\widetilde{\mathcal{M}})} = \|\mu(x)\|_{E(\mathbb{R}^+)}.$$

As is shown in [DDP3], this volume, (see also [DDP1] and [Su]) $E(\widetilde{\mathcal{M}})$ is a Banach space and it now immediately follows that $E(\widetilde{\mathcal{M}})$ is a rearrangement invariant symmetric operator space.

If the Banach space E is intermediate for the Banach couple (X_0, X_1), then E will be called an *interpolation space* for the couple (X_0, X_1) if every linear operator, which acts boundedly from X_i to X_i, $i = 0, 1$, acts boundedly from E to E; E will be called an *exact interpolation space* for the couple (X_0, X_1) (respectively, *exact interpolation space of exponent* θ, $0 \leq \theta \leq 1$) if every linear operator which acts boundedly from X_i to X_i with norm at most one (respectively, at most M_i) $i = 0, 1$, acts boundedly from E to E with norm at most one (respectively, of norm at most $M_0^{1-\theta} M_1^{\theta}$).

Corollary 1.6. *(i) If E is a Banach space which is a linear subspace of $L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$, then the following statements are equivalent.*

(a) E is an exact interpolation space for the couple $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$.

(b) E is fully symmetric.

(ii) If $E(\mathbb{R}^+)$ is an exact interpolation space for the couple $(L^1(\mathbb{R}^+), L^\infty(\mathbb{R}^+))$ then $E(\widetilde{\mathcal{M}})$ is an exact interpolation space for the couple $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$.

The preceding Corollary asserts that the exact interpolation spaces for the couple $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$ are precisely the fully symmetric operator spaces in $\widetilde{\mathcal{M}}$. Moreover, it is not difficult to see, as is noted by [Ye], Proposition 3.6, that any interpolation space for the couple $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$ is a fully symmetric space in an equivalent norm.

It is now, of course, appropriate to point out that there exist rearrangement invariant, symmetric operator spaces which are not interpolation spaces for the couple $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$. The first such example was given by Russu [Ru] and independently by Holub [Ho] in the setting of trace ideals. A further example is exhibited in [KPS] Theorem II 5.11 as a closed subspace of a Marcinkiewicz space on $\mathbb{R}^+$. Of course, the preceding corollary permits ready identification of many operator spaces which are exact interpolation spaces for the pair $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$. If $E(\mathbb{R}^+)$ is a rearrangement invariant function space which is either maximal in the sense that the natural embedding of $E(\mathbb{R}^+)$ into its Köthe bidual is a surjective isometry, or if $E(\mathbb{R}^+)$ is separable, then $E(\mathbb{R}^+)$ is an exact interpolation space for the pair $(L^1(\mathbb{R}^+), L^\infty(\mathbb{R}^+))$; the corresponding non-commutative spaces $E(\widetilde{\mathcal{M}})$ are then exact interpolation spaces for the couple $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$. Examples of rearrangement invariant function spaces on $\mathbb{R}^+$ which are maximal include the familiar Orlicz, Marcinkiewicz and Lorentz spaces. An example of a rearrangement invariant function space which is separable but not maximal is given in [KPS] section 5 of Chapter II. We will show later (see Corollary 2.2 below) that the only interpolation spaces for the pair $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$ are those which can be constructed as the non-commutative spaces arising from some fully symmetric function space on $\mathbb{R}^+$. Before proceeding we recall that if (X, Y) is a Banach couple, that is X and Y are Banach spaces continuously embedded in some separated topological linear space, then the Banach space $X+Y$ consists of those elements of the form $x = y+z, y \in X, z \in Y$ equipped with the norm given by

$$\|x\|_{X+Y} = \inf\{\|y\|_X + \|z\|_Y : y \in X, z \in Y, x = y+z\}.$$

We remark that underlying the proofs of the preceding results is the equality

$$(L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+))(\widetilde{\mathcal{M}}) = L^1(\widetilde{\mathcal{M}}) + \mathcal{M}.$$

The norm equality implicit in the preceding identification is a special case of the identity

$$\int_0^t \mu_s(x)ds = \inf\{\|y\|_1 + t\|z\|_\infty : y \in L^1(\widetilde{\mathcal{M}}), z \in \mathcal{M}, x = y+z\}, \qquad t > 0.$$

This identity has been proved by many authors: [Ov2], [PS], [FK], and implicitly in [Ye]. The relevance of this identity to the application of the real method of interpolation in the present setting has been amply demonstrated by Peetre-Sparr [PS] and Kosaki [Ko]. We now state a simple generalization.

Proposition 1.7. *If E_0, E_1 are fully symmetric function spaces on $\mathbb{R}^+$, then*

$$E_0(\widetilde{\mathcal{M}}) + E_1(\widetilde{\mathcal{M}}) = (E_0 + E_1)(\widetilde{\mathcal{M}})$$

with equality of norms.

The preceding Proposition is a simple consequence of Corollary 1.4. In fact, if $x \in E_0(\widetilde{\mathcal{M}}) + E_1(\widetilde{\mathcal{M}})$ and if $x = x_0 + x_1$ with $x_i \in E_i(\widetilde{\mathcal{M}}), i = 0, 1$, then Corollary 1.4 shows that there exist functions $f_0, f_1 \in L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+)$ with $f_i \prec\prec x_i, i = 0, 1$, and

$$\mu(x) = f_0 + f_1.$$

Since $E_0(\mathbb{R}^+), E_1(\mathbb{R}^+)$ are fully symmetric, it follows that $f_i \in E_i(\mathbb{R}^+), i = 0, 1$, and that

$$\|f_i\|_{E_i(\mathbb{R}^+)} \leq \|\mu(x_i)\|_{E_i(\mathbb{R}^+)} = \|x_i\|_{E_i(\widetilde{\mathcal{M}})}, \qquad i = 0, 1.$$

Consequently,

$$\|\mu(x)\|_{E_0(\mathbb{R}^+)+E_1(\mathbb{R}^+)} \leq \|x\|_{E_0(\widetilde{\mathcal{M}})+E_1(\widetilde{\mathcal{M}})}$$

and the proof of the converse inequality is almost identical.

2. Main results

A mapping $\mathcal{F}$ from Banach couples to Banach spaces is called an *exact interpolation functor* if
(i) for every Banach couple (X_0, X_1) the Banach space $\mathcal{F}(X_0, X_1)$ is an exact interpolation space for the couple (X_0, X_1),
(ii) for every pair $((X_0, X_1),(Y_0, Y_1))$ of Banach couples, each bounded operator from the couple (X_0, X_1) to the couple (Y_0, Y_1) of norm at most one acts as a bounded linear map from $\mathcal{F}(X_0, X_1)$ to $\mathcal{F}(Y_0, Y_1)$ of norm at most one.

If $\mathcal{F}$ is an exact interpolation functor, and if $0 \leq \theta \leq 1$ then $\mathcal{F}$ will be called an *exact interpolation functor of exponent* θ if, for every pair (X_0, X_1), (Y_0, Y_1) of Banach couples and bounded operator V from the couple (X_0, X_1) to the couple (Y_0, Y_1), V maps $\mathcal{F}(X_0, X_1)$ to $\mathcal{F}(Y_0, Y_1)$ with norm at most $M_0^{1-\theta} M_1^{\theta}$ whenever V maps X_i to Y_i with norm M_i, $i = 0, 1$.

We now extend Proposition 1.7 as follows.

Theorem 2.1. *If $E_0(\mathbb{R}^+), E_1(\mathbb{R}^+)$ are fully symmetric function spaces on $\mathbb{R}^+$ and if $\mathcal{F}$ is an exact interpolation functor, then*

$$\mathcal{F}(E_0(\mathbb{R}^+), E_1(\mathbb{R}^+))(\widetilde{\mathcal{M}}) = \mathcal{F}(E_0(\widetilde{\mathcal{M}}), E_1(\widetilde{\mathcal{M}})).$$

Theorem 2.1 combined with the Aronszajn-Gagliardo theorem ([KPS] Theorem I 4.9, [BL] Theorem 2.5.1) yields the following interesting consequence.

Corollary 2.2. *If $E_i(\mathbb{R}^+)$, $i = 0, 1$, are fully symmetric function spaces on $\mathbb{R}^+$ and if E is an exact interpolation space for the couple $(E_0(\widetilde{\mathcal{M}}), E_1(\widetilde{\mathcal{M}}))$ then there exists a Banach function space $F(\mathbb{R}^+)$ which is an exact interpolation space for $(E_0(\mathbb{R}^+), E_1(\mathbb{R}^+))$ such that $E = F(\widetilde{\mathcal{M}})$.*

When specialized to the couple $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$ the preceding corollary implies that each fully symmetric operator space is derived from a fully symmetric space on $\mathbb{R}^+$ and this answers a question raised by Yeadon [Ye]. Before showing that Corollary 1.6(ii) admits a similar extension, we recall first some relevant terminology.

Let (X_0, X_1), (Y_0, Y_1) be two couples of Banach spaces and let X, Y be intermediate spaces for the couples (X_0, X_1),(Y_0, Y_1), respectively. The pair (X, Y) will be called an *exact interpolation pair* for $((X_0, X_1)$,$(Y_0, Y_1))$ (respectively, *exact interpolation pair of exponent* θ, $0 \leq \theta \leq 1$) if every linear operator which acts boundedly from X_i to Y_i with norm at most one (respectively, at most M_i), $i = 0, 1$, acts boundedly from X to Y with norm at most one (respectively, at most $M_0^{1-\theta} M_1^{\theta}$).

Theorem 2.3. *Suppose that $E_i(\mathbb{R}^+), F_i(\mathbb{R}^+), i = 0, 1$ are fully symmetric function spaces on $\mathbb{R}^+$ and that $E(\mathbb{R}^+), F(\mathbb{R}^+)$ are rearrangement invariant symmetric function spaces on $\mathbb{R}^+$ which are intermediate for the Banach couples $(E_0(\mathbb{R}^+), E_1(\mathbb{R}^+))$, $(F_0(\mathbb{R}^+), F_1(\mathbb{R}^+))$, respectively. If the pair $(E(\mathbb{R}^+), F(\mathbb{R}^+))$ is an exact interpolation pair (respectively, exact interpolation pair of exponent θ) for $((E_0(\mathbb{R}^+), E_1(\mathbb{R}^+))$,$(F_0(\mathbb{R}^+), F_1(\mathbb{R}^+))$ then the* *pair* $(E(\widetilde{\mathcal{M}}), F(\widetilde{\mathcal{N}}))$ *is an exact interpolation pair (respectively, exact interpolation pair of exponent* θ*) for* $((E_0(\widetilde{\mathcal{M}}), E_1(\widetilde{\mathcal{M}})), (F_0(\widetilde{\mathcal{N}}), F_1(\widetilde{\mathcal{N}}))$.

Theorems 2.1, 2.3 above extend similar results of Arazy [Ar] for trace ideals. The details of proof will be given elsewhere ([DDP4]). The main tool in the proof is the following result which is based on a modification of the notion of Schmidt decomposition due to Ovčinnikov [Ov1].

Theorem 2.4. *If* $0 \leq x \in \widetilde{\mathcal{M}}$*, there exists a proper von Neumann subalgebra* $M_x \subseteq L^\infty(\mathbb{R}^+)$ *with* $\mu(x) \in \widetilde{M_x}$*, a proper commutative subalgebra* $\mathcal{M}_x \subseteq \mathcal{M}$ *and a positive rearrangement preserving algebra isomorphism* J_x *of* $\widetilde{M_x}$ *onto* $\widetilde{\mathcal{M}_x}$ *whose restriction to the projections of* M_x *is a Boolean algebra isomorphism onto the projections of* $\mathcal{M}_x$ *and for which*

$$\mu(J_x(\mu(x))) = \mu(x).$$

3. Non-commutative Riesz-Thorin and Marcinkiewicz theorems

The effect of Theorems 2.1, 2.3 of the preceding section is to reduce certain non-commutative interpolation results to the special case obtained by taking $\mathcal{M}$ to be $L^\infty(\mathbb{R}^+)$. We illustrate their utility for two familiar interpolation methods.

We recall first the following basic notions from the complex method of interpolation ([KPS], Chapter IV). If (X_0, X_1) is a Banach couple, then $F(X_0, X_1)$ denotes the linear space of all complex functions $z \to f(z)$ defined in the strip

$$\Pi = \{z : 0 \leq Re z \leq 1\}$$

with values in $X_0 + X_1$ with the following properties.

(i) f is continuous and bounded on Π and analytic in the interior of Π.

(ii) The restriction of f to the left (right) hand edge of Π is a bounded continuous function

to X_0 (respectively X_1).
With norm defined by setting

$$\|f\|_{F(X_0,X_1)} = \max\{\sup_t \|f(it)\|_{X_0}, \sup_t \|f(1+it)\|_{X_1}\},$$

the space $F(X_0, X_1)$ is a Banach space.

If $0 < \theta < 1$, the Banach space $[X_0, X_1]_\theta$ consists of those elements $x \in X_0 + X_1$ of the form $x = f(\theta)$ for some $f \in F(X_0, X_1)$ with norm

$$\|x\|_{[X_0,X_1]_\theta} = \|x\|_\theta = \inf\{\|f\|_{F(X_0,X_1)} : x = f(\theta)\}.$$

It may then be shown ([KPS] Theorem IV 1.2) that for any pair $((X_0, X_1),(Y_0, Y_1))$ of Banach couples the pair $([X_0, X_1]_\theta, [Y_0, Y_1]_\theta)$ is an exact interpolation pair of exponent θ for $((X_0, X_1), (Y_0, Y_1))$.

If we specialise to the familiar L^p-spaces, $p \geq 1$, we obtain that, if $1 \leq p_0, p_1 \leq \infty$ and $0 < \theta < 1$, then

$$[L^{p_0}(\mathbb{R}^+), L^{p_1}(\mathbb{R}^+)]_\theta = L^p(\mathbb{R}^+),$$

with equality of norms, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

See, for example, [BL] Theorem 5.1.1.

From Theorem 2.1, we obtain immediately that

$$[L^{p_0}(\widetilde{\mathcal{M}}), L^{p_1}(\widetilde{\mathcal{M}})]_\theta = L^p(\widetilde{\mathcal{M}}),$$

which yields the following non-commutative Riesz-Thorin Theorem, due to Kunze [Ku], (see also [Ov2]).

Theorem 3.1. *Let* $1 \leq p_i, q_i \leq \infty$, $i = 0, 1$, *let* $0 < \theta < 1$ *and define* p, q *by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If V is a bounded linear operator from $L^{p_i}(\widetilde{\mathcal{M}})$ to $L^{q_i}(\widetilde{\mathcal{N}})$ with norm $\|V\|_i, i = 0, 1$, then V acts as a bounded linear operator from $L^p(\widetilde{\mathcal{M}})$ to $L^q(\widetilde{\mathcal{N}})$ with norm at most $\|V\|_0^{1-\theta}\|V\|_1^{\theta}$.

We recall briefly the K-method of real interpolation due to Peetre ([BL], Chapter 3). If (X_0, X_1) is a Banach couple, and if $x \in X_0 + X_1$, we define for every $t > 0$,

$$K(t, x) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_i \in X_i, i = 0, 1\}.$$

For $0 < \theta < 1,\ 1 \leq q \leq \infty$ or for $0 \leq \theta \leq 1,\ q = \infty$, the Banach space $[X_0, X_1]_{\theta,q;K}$ consists of all elements $x \in X_0 + X_1$ for which

$$\Phi_{\theta,q}(K(.,x)) = (\int_0^\infty (t^{-\theta}K(t,x))^q \frac{dt}{t})^{\frac{1}{q}} < \infty,$$

with norm given by

$$\|x\|_{\theta,q;K} = \Phi_{\theta,q}(K(.,x)).$$

The functor $[\ .\ ,\ .\ ]_{\theta,q;K}$ is an exact interpolation functor of exponent θ ([BL], Theorem 3.1.2).

We now specialize to the special cases of Lorentz spaces $L_{pq}(\mathbb{R}^+)$ (see [Ca2]). For $1 < p < \infty,\ 1 \leq q < \infty,\ L_{pq}(\mathbb{R}^+)$ is the class of measurable functions $f \in L^0(\mathbb{R}^+)$ such that

$$\|f\|_{pq} = \{\frac{p-1}{p^2}\int_0^\infty (t^{\frac{1}{p}} f^{**}(t))^q \frac{dt}{t}\}^{\frac{1}{q}} < \infty;$$

for $1 < p < \infty,\ L_{p\infty}(\mathbb{R}^+)$ is the class of measurable functions $f \in L^0(\mathbb{R}^+)$ such that

$$\|f\|_{p\infty} = \sup_t t^{\frac{1}{p}} f^{**}(t) < \infty,$$

where

$$f^{**}(t) = \frac{1}{t}\int_0^t \mu_s(f)ds, \qquad t \geq 0.$$

We define

$$L_{11}(\mathbb{R}^+) = L^1(\mathbb{R}^+), \qquad L_{\infty\infty}(\mathbb{R}^+) = L^\infty(\mathbb{R}^+).$$

The spaces $L_{pq}(\mathbb{R}^+)$ are Banach spaces with respect to the above norms and it is clear that they are fully symmetric spaces on $\mathbb{R}^+$.

It now follows from Theorem 2.1 and [BL] Theorem 5.3.1 that the equality

$$[L_{p_0q_0}(\widetilde{\mathcal{M}}), L_{p_1q_1}(\widetilde{\mathcal{M}})]_{\theta,q;K} = L_{pq}(\widetilde{\mathcal{M}}),$$

holds (up to equivalent norm) where

$$0 < \theta < 1, \quad 1 \leq q_0, q_1, q \leq \infty, \quad \frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1} \quad \text{and} \quad 1 < p_0, p_1 < \infty,$$

$$\text{or,} \qquad p_i = 1 \text{ or } \infty \qquad \text{and} \qquad 1 < p_{1-i} < \infty, \quad i = 0, 1,$$

$$\text{or,} \qquad p_i = 1 \quad \text{and} \quad p_{1-i} = \infty, \quad i = 0, 1.$$

This yields as a special case Kosaki [Ko] Theorem 2.4 (see also [PS]). The special case of the usual Marcinkiewicz Theorem for the spaces $L^p(\mathbb{R}^+)$ ([BP] Theorem 5.3.2) now yields the following non-commutative version of the Marcinkiewicz Theorem.

Theorem 3.2. *If V is a bounded linear operator from $L_{p_iq_i}(\widetilde{\mathcal{M}})$ to $L_{r_is_i}(\widetilde{\mathcal{N}})$, $i = 0, 1$, where*

$$p_0 \neq p_1, \quad r_0 \neq r_1, \quad 1 < p_i, r_i < \infty \quad \textit{and} \quad 1 \leq q_i, s_i \leq \infty, \quad i = 0, 1,$$

then V is a bounded linear operator from $L_{pt}(\widetilde{\mathcal{M}})$ to $L_{rt}(\widetilde{\mathcal{N}})$ for $1 \leq t \leq \infty$, where

$$\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}, \quad \frac{1}{r} = (1-\theta)\frac{1}{r_0} + \theta\frac{1}{r_1}, \quad \textit{with} \quad 0 \leq \theta \leq 1.$$

The result holds also if

$$p_i = r_i = 1 \quad \textit{or} \quad \infty \qquad \textit{and} \quad 1 < p_{1-i}, r_{1-i} < \infty, \quad i = 0, 1,$$

$$\textit{or if} \quad p_i = r_i = 1 \quad \textit{and} \quad p_{1-i} = r_{1-i} = \infty, \quad i = 0, 1.$$

4. Interpolation for the couple $(L^p(\widetilde{\mathcal{M}}), \mathcal{M})$

We turn now to a characterization of interpolation spaces for the couple $(L^p(\widetilde{\mathcal{M}}), \mathcal{M})$, $1 \leq p < \infty$, which is due to Lorentz and Shimogaki [LS2] in the commutative setting and to Arazy ([Ar], Corollary 2.12) for separable trace ideals. Following [LS2], if $x, y \in L^p(\widetilde{\mathcal{M}}) + \mathcal{M}$, we define $y \prec^p x$ if and only if for each decomposition

$$x = x_1 + x_2, \quad x_1 \in L^p(\widetilde{\mathcal{M}}), \quad x_2 \in \widetilde{\mathcal{M}},$$

there exists a decomposition

$$y = y_1 + y_2, \quad y_1 \in L^p(\widetilde{\mathcal{M}}), \quad y_2 \in \widetilde{\mathcal{M}},$$

such that

$$\|y_1\|_p \leq \|x_1\|_p \quad \text{and} \quad \|y_2\|_\infty \leq \|x_2\|_\infty.$$

Lemma 4.1. *If $x, y \in L^p(\widetilde{\mathcal{M}}) + \mathcal{M}$ then $y \prec^p x$ if and only if $\mu(y) \prec^p \mu(x)$.*

Proof. Assume first that $y \prec^p x$ and suppose that

$$\mu(x) = f_1 + f_2, \quad f_1 \in L^p(\mathbb{R}^+), \; f_2 \in L^\infty(\mathbb{R}^+).$$

By Theorem 1.1 (i), there exists $T \in \sum(L^\infty(\mathbb{R}^+), \mathcal{M})$ such that

$$x = T\mu(x) = Tf_1 + Tf_2.$$

Consequently there exists a decomposition

$$y = y_1 + y_2, \quad y_1 \in L^p(\widetilde{\mathcal{M}}), \quad y_2 \in \mathcal{M}$$

with

$$\|y_1\|_p \leq \|Tf_1\|_p \leq \|f_1\|_p$$

$$\|y_2\|_\infty \leq \|Tf_2\|_\infty \leq \|f_2\|_\infty.$$

Again by Theorem 1.1 (i), there exists $S \in \sum(\mathcal{M}, L^\infty(\mathbb{R}^+))$ such that

$$\mu(y) = Sy = Sy_1 + Sy_2$$

and

$$\|Sy_1\|_p \leq \|y_1\|_p \leq \|f_1\|_p$$

$$\|Sy_2\|_\infty \leq \|y_2\|_\infty \leq \|f_2\|_\infty.$$

The proof of the converse assertion is identical.

For $x, y \in L^p(\widetilde{\mathcal{M}}) + \mathcal{M}$, $1 \leq p < \infty$, we write $y \prec\prec^p x$ if and only if

$$\int_0^t \mu_s^p(y)ds \leq \int_0^t \mu_s^p(x)ds, \quad \text{for all } t > 0.$$

The preceding Lemma 4.1 combined with [LS] Lemma 2 yields immediately the following.

Lemma 4.2. *If* $x, y \in L^p(\widetilde{\mathcal{M}}) + \mathcal{M}$, $1 \leq p < \infty$, *then*
(i) if $y \prec\prec^p x$ *then* $y \prec^p x$;
(ii) for each $p > 1$, *there exists a smallest constant* λ_p, *with* $1 < \lambda_p \leq 2^{\frac{1}{q}}$, *where* $\frac{1}{p} + \frac{1}{q} = 1$, *such that if* $x, y \in L^p(\widetilde{\mathcal{M}}) + \mathcal{M}$ *and* $y \prec^p x$ *then* $y \prec\prec^p x$.

Lemma 4.3. *Let* $0 \leq y \in L^1(\widetilde{\mathcal{M}}) \cap \mathcal{M}$ *be simple, i.e.,*

$$y = \sum_{i=1}^n \alpha_i e_i$$

where e_i, $i = 1, \cdots, n$, *are mutually disjoint projections in* $\mathcal{M}$ *with finite trace and* $0 \leq \alpha_i$, $i = 1, \cdots, n$. *Let* $0 \leq x \in L^p(\widetilde{\mathcal{M}}) + \mathcal{M}$. *If* $y \prec\prec^p x$, *then there exists a bounded positive operator* T *from the couple* $(L^p(\widetilde{\mathcal{M}}), \mathcal{M})$ *to itself of norm at most one such that* $y \leq Tx$.

The proof of Lemma 4.3 may be reduced to its commutative version given by [LS] Lemma 4 by a simple modification of the techniques used to prove Theorem 2.3.

Theorem 4.4. *Let* $1 < p < \infty$.
(i) If E *is intermediate for the Banach couple* $(L^p(\widetilde{\mathcal{M}}), \mathcal{M})$ *and if* E *satisfies the condition:*

(A) $y \in L^p(\widetilde{\mathcal{M}}) + \mathcal{M}$, $x \in E$ *and* $y \prec\prec^p x \quad \Longrightarrow \quad y \in E$,
then E *is an interpolation space for* $(L^p(\widetilde{\mathcal{M}}), \mathcal{M})$.
(ii) If E *is an interpolation space for* $(L^p(\widetilde{\mathcal{M}}), \mathcal{M})$ *and if* E *has the following property:*

$0 \leq x_\alpha \uparrow_\alpha \subseteq E$, $\sup \|x_\alpha\|_E < \infty \Longrightarrow \sup x_\alpha$ *exists in* E *and* $\|\sup x_\alpha\|_E = \sup \|x_\alpha\|_E$,

then E *satisfies the above condition* (A) .

The proof of part (i) via Lemmas 4.1, 4.2 preceding is now exactly like the proof of the corresponding assertion of [LS] Theorem 2. For the proof of part (ii), the proof given in [LS] Theorem 2 again carries over, via Lemma 4.3.

5. Operators of weak type

Following [Ca2], for $1 \leq p < \infty$, the linear map $T : L_{p1}(\widetilde{\mathcal{M}}) \to \widetilde{\mathcal{N}}$ is said to be of weak type (p, q), $1 \leq q \leq \infty$ if T maps $L_{p1}(\widetilde{\mathcal{M}})$ continuously into $\widetilde{\mathcal{N}}$ and if there exists a constant c such that

$$\mu_t(Tx) \leq ct^{-\frac{1}{q}} \|x\|_{p1}, \quad t > 0$$

for all $x \in L_{p1}(\widetilde{\mathcal{M}})$.

Throughout this section we denote by ω a closed segment in the unit square with end points $(\frac{1}{p_1}, \frac{1}{q_1})$, $(\frac{1}{p_2}, \frac{1}{q_2})$ with $p_1 \neq p_2$, $q_1 \neq q_2$. We denote by $W(\omega, \widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ the class of linear maps on $L_{p_11}(\widetilde{\mathcal{M}}) + L_{p_21}(\widetilde{\mathcal{M}})$ into $\widetilde{\mathcal{N}}$ which are simultaneously of weak types (p_1, q_1), (p_2, q_2).

Following [Ca2], the operator $S(\omega)$ is defined on functions on $\mathbb{R}^+$ by setting

$$S(\omega)f(t) = \int_0^\infty f(s) d\Psi_\omega(s,t), \qquad t > 0,$$

where

$$\Psi_\omega(s,t) = \min\{\frac{s^{1/p_1}}{t^{1/q_1}}, \frac{s^{1/p_2}}{t^{1/q_2}}\}, \qquad s,t > 0.$$

We now indicate how to extend Theorem 8 of [Ca2] by a simple modification of the proof given in [Ca2].

Proposition 5.1. *If* $T : L_{p_1 1}(\widetilde{\mathcal{M}}) + L_{p_2 1}(\widetilde{\mathcal{M}}) \longrightarrow \widetilde{\mathcal{N}}$ *is simultaneously of weak types* (p_1, q_1), (p_2, q_2) *then there exists a constant* $c = c(\omega, p, q)$ *such that*

$$\mu(Tx) \leq cS(\omega)\mu(x), \qquad x \in L_{p_1 1}(\widetilde{\mathcal{M}}) + L_{p_2 1}(\widetilde{\mathcal{M}})$$

where ω *is the segment with end points* $(\frac{1}{p_1}, \frac{1}{q_1})$, $(\frac{1}{p_2}, \frac{1}{q_2})$.

Proof. There exists a constant c' such that

$$\mu_t(Ty) \leq c' t^{-\frac{1}{q_i}} \|\mu(y)\|_{p_i 1}, \qquad t > 0$$

for all $y \in L_{p_i 1}(\widetilde{\mathcal{M}})$, $i = 1, 2$. Let $x \in L_{p_1 1}(\widetilde{\mathcal{M}}) + L_{p_2 1}(\widetilde{\mathcal{M}})$, fix $t > 0$ and let $\alpha = \mu_{t^m}(x)$ where m is the slope of the segment ω. Let $x = u|x|$ be the polar decomposition of x and write

$$|x| = (|x| - \alpha 1)^+ + |x| \wedge n\alpha.$$

Note that it follows from [FK] Lemma 2.5 that

$$\mu((|x| - \alpha 1)^+) = (\mu(x) - \alpha)^+.$$

Suppose now that $p_1 < p_2$. If $t > 0$, then

$$\begin{aligned}
\mu_t(Tx) &\leq \mu_{\frac{t}{2}}(Tu(|x| - \alpha 1)^+) + \mu_{\frac{t}{2}}(Tu(|x| \wedge \alpha 1)) \\
&\leq c'(t^{-\frac{1}{q_1}} \|\mu(u(|x| - \alpha 1)^+)\|_{p_1 1} + t^{-\frac{1}{q_2}} \|\mu(u(|x| \wedge \alpha 1))\|_{p_2 1}) \\
&\leq c'(t^{-\frac{1}{q_1}} \|(\mu(x) - \alpha 1)^+\|_{p_1 1} + t^{-\frac{1}{q_2}} \|\mu(x) \wedge \alpha)\|_{p_2 1}) \\
&\leq c\left(\int_0^{t^m} (\mu(x) - \alpha 1) d(\frac{s^{1/p_1}}{t^{1/q_1}}) \; + \; \int_{t^m}^\infty \mu(x) d(\frac{s^{1/p_2}}{t^{1/q_2}}) \; + \; \alpha \int_0^{t^m} d(\frac{s^{1/p_2}}{t^{1/q_2}})\right) \\
&= c \int_0^\infty \mu(x) d(\Psi_\omega(s,t)) \\
&= c(\sigma, p, q) \; S(\omega) \; \mu(x).
\end{aligned}$$

Via Theorem 1.1 and Proposition 5.1 preceding, the proofs of the following results reduce to those given in [Ca2] Theorem 9 and Theorem 10.

Proposition 5.2. (cf. [Ca2] Theorem 9) *If* $q_1, q_2 > 1$, *if* $x \in L_{p_1 1}(\widetilde{\mathcal{M}}) + L_{p_2 1}(\widetilde{\mathcal{M}})$ *and if* $y \in \widetilde{\mathcal{N}}$ *then the following statements are equivalent.*
(i) There exists $T \in W(\omega, \widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ *such that* $Tx = y$.
(ii) There exists $c > 0$ *such that*

$$\mu(y) \leq cS(\omega)\mu(x).$$

Proposition 5.3. (cf. [Ca2] Theorem 10) *If* q_1, $q_2 > 1$ *and if* A_1, A_2 *are linear subspaces of* $L_{p_1 1}(\widetilde{\mathcal{M}}) + L_{p_2 1}(\widetilde{\mathcal{M}})$, $\widetilde{\mathcal{N}}$ *respectively, then the following statements are equivalent:*
(i) $W(\omega, \widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})(A_1) \subseteq A_2$.
(ii) $x \in A_1$, $y \in \widetilde{\mathcal{N}}$, $\mu(y) \leq S(\omega)\mu(x)$ *implies* $Y \in A_2$.

Finally, let us note that in the commutative setting, the result of Proposition 5.2 is essentially the starting point of the work of Boyd [B1] and we leave to the interested reader the task of extending the results of [B1] to the non-commutative setting. For trace ideals, this task has been carried out in [Ar].

References

[Ar] J. Arazy, *Some remarks on interpolation theorems and the boundness of the triangular projection in unitary matrix spaces*, Integral Equations and Operator Theory, 1 (1978), 453-495.

[BL] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer, Berlin-Heidelberg-New York, **223** (1976).

[B1] D.W. Boyd, *Indices of function spaces and their relationship to interpolation*, Canad. J. Math. **21** (1969), 1245-1254.

[Ca1] A.P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 133-190.

[Ca2] A.P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. **26** (1966), 273-299.

[DDP1] P.G. Dodds, T.K. Dodds and B. de Pagter, *Non-commutative Banach function spaces*, Math. Z. **201** (1989), 583-597.

[DDP2] P.G. Dodds, T.K. Dodds and B. de Pagter, *Non-commutative Köthe duality*, in preparation.

[DDP3] P.G. Dodds, T.K. Dodds and B. de Pagter, *A general Markus inequality*, this volume.

[DDP4] P.G. Dodds, T.K. Dodds and B. de Pagter, *Fully symmetric operator spaces*, in preparation.

[FK] T. Fack and H. Kosaki, *Generalized s-numbers of τ-measurable operators*, Pacific J. Math. **123** (1986), 269-300.

[Fr] D.H. Fremlin, *Stable subspaces of $L^1 + L^\infty$*, Math. Proc. Cambridge Philos. Soc. **64** (1968), 625-643.

[Ho] J.R. Holub, *A characterization of the norm ideals of compact operators on Hilbert space*, J. Math. Anal. Appl. **50** (1975), 596-606.

[Ko] H. Kosaki, *Non-commutative Lorentz spaces associated with a semi-finite von Neumann algebra and applications*, Proc. Jap. Acad., Ser. A **57** (1981), 303-306.

[KPS] S.G. Krein, Ju.I. Petunin and E.M. Semenov, *Interpolation of linear operators*, Translations of Mathematical Monographs, Amer. Math. Soc. **54** (1982).

[Ku] R. Kunze, *L^p-Fourier transforms in locally compact unimodular groups*, Trans. Amer. Math. Soc. **89** (1958), 519-540.

[LS1] G.G. Lorentz and T. Shimogaki, *Interpolation theorems for operators in function spaces*, J. Functional Anal. **2** (1968), 31-51.

[LS2] G.G. Lorentz and T. Shimogaki, *Interpolation theorems for the pairs of spaces* (L^p, L^∞) *and* (L^1, L^q), Trans. Amer. Math. Soc. **159** (1971), 207-221.

[Lu] W.A.J. Luxemburg, *Rearrangement invariant Banach function spaces*, Queen's Papers in Pure and Applied Mathematics, No. **10** (1967), 83-144.

[Ne] E. Nelson, *Notes on non-commutative integration*, J. Functional Anal. **15** (1974), 103-116.

[Ov1] V.I. Ovčinnikov, *s-numbers of measurable operators*, Funktsional'nyi Analiz i Ego Prilozheniya **4** (1970), 78-85 (Russian).

[Ov2] V.I. Ovčinnikov, *Symmetric spaces of measurable operators*, Dokl. Nauk SSSR **191** (1970), 769-771 (Russian), English Translation: Soviet Math. Dokl. **11** (1970), 448-451.

[Ov3] V.I. Ovčinnikov, *Symmetric spaces of measurable operators*, Trudy inst. matem. VGU **3** (1971), 88-107.

[PS] J. Peetre and G. Sparr, *Interpolation and non-commutative integration*, Ann. Mat. Pura Appl.. **104** (1975), 187-207.

[Ru] G.I. Russu, *On intermediate symmetrically normed ideals*, Funktsional'nyi Analiz i Ego Prilozheniya, **3** (1969), 94-95 (Russian).

[Ry] J.V. Ryff, *Orbits of* L^1*-functions under doubly stochastic transformations*, Trans. Amer. Math. Soc. **117** (1965), 92-100.

[Su1] F.A. Sukochev, *Construction of non-commutative symmetric spaces*, Dokl. Akad. Nauk UzSSR **8** (1986), 4-6.

[Su2] F.A. Sukochev, *Symmetric spaces on finite von Neumann algebras*, Thesis, Tashkent, (1987).

[Su3] F.A. Sukochev, *(en)-invariant properties of non-commutative spaces of measurable operators*, Dokl. Akad. Nauk UzSSR **7** (1985), 6-8.

[Ye1] F.J. Yeadon, *Non-commutative* L^p*-spaces*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 91-102.

[Ye2] F.J. Yeadon, *Ergodic theorems for semifinite von Neumann algebras: II*, Math. Proc. Cambridge Philos. Soc. **88** (1980), 135-147.

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