SCHAUDER ESTIMATES ON LIE GROUPS

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1. INTRODUCTION

Let $\partial_1, \ldots, \partial_d$ denote the partial differentiation operators on the usual L_p-spaces $L_p(\mathbb{R}^d; dx)$ and $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d}$ their products. Then the subspaces

$$L_{p;n} = \bigcap_{\alpha; |\alpha| \leq n} D(\partial^\alpha) ,$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_d$, are Banach spaces with respect to the norms

$$\varphi \in L_{p;n} \mapsto \|\varphi\|_{p;n} = \sup_{\alpha; |\alpha| \leq n} \|\partial^\alpha \varphi\|_p .$$

Moreover, if $\Delta = -\sum_{i=1}^d \partial_i^2$ is the Laplacian then $L_{p;n} = D(\Delta^{n/2})$ and for each $p \in \langle 1, \infty \rangle$, each $n = 1, 2, \ldots$, and each $\lambda > 0$ there is a $C_{p,n,\lambda} > 0$ such that

$$C_{p,n,\lambda}^{-1} \|\varphi\|_{p;n} \leq \|(\lambda I + \Delta)^{n/2} \varphi\|_p \leq C_{p,n,\lambda} \|\varphi\|_{p;n} \tag{1.1}$$

for all $\varphi \in L_{p;n}$ (see, for example, [Tri] Section 2.2.3).

Thus the differential structure of the L_p-spaces coincides with the structure given by the Laplacian. Our primary aim is to prove similar properties for the differential structures associated with left, or right, translations on the L_p-spaces over a Lie group G.

It should be emphasized that the equivalence of the norms $\varphi \mapsto \|\varphi\|_{p;n}$ and $\varphi \mapsto \|(\lambda I + \Delta)^{n/2} \varphi\|_p$ is not valid if $p = 1$, or $p = \infty$. Nor does it hold on $C_0(\mathbb{R}^d)$. The domination properties

$$\|\varphi\|_{p;n} \leq C_{p,n,\lambda} \|(\lambda I + \Delta)^{n/2} \varphi\|_p ,$$

which are often referred to as à priori inequalities, or Schauder estimates, in the context of elliptic differential operators, are quite delicate and it is perhaps surprising that they extend to all Lie groups. Despite this delicacy the bounds (1.1) can be substantially improved in the classical setting, e.g. for each $p \in \langle 1, \infty\rangle$ and $n = 1, 2, \ldots$ there is a $C_{p;n}$ such that

$$C_{p,n}^{-1} \sup_{\alpha;|\alpha|=n} \|\partial^\alpha \varphi\|_p \leq \|\Delta^{n/2} \varphi\|_p \leq C_{p,n} \sup_{\alpha;|\alpha|=n} \|\partial \varphi\|_p \,. \tag{1.2}$$

These inequalities reflect the fact that the Riesz transforms $\partial_i \Delta^{-1/2}$ extend to bounded operators (see, for example [Ste1] Chapter III). Our secondary aim is to establish bounds analogous to (1.2) for all compact Lie groups. Stein [Ste2] proved a similar result for the bi-invariant Laplacian by a version of Littlewood-Paley theory. Our proof is quite different and applies to all possible Laplacians.

It would be of interest to establish the Lie group version of (1.2) for general groups but this seems to require global estimates which go beyond the methods of this paper. The proof of (1.1) is based on global bounds on L_2, local bounds on L_1, and interpolation. Then (1.2) follows for compact groups with the aid of a spectral bound for Δ.

2. NOTATION AND PRELIMINARIES

Let G be a d-dimensional Lie group, dg left-invariant Haar measure, $d\hat{g}$ right-invariant Haar measure, and m the modular function. Thus $d\hat{g} = dg\, m(g)^{-1}$. Further let $L_p = L_p(G; dg)$ and $L_{\hat{p}} = L_p(G; d\hat{g})$ denote the corresponding scales of L_p-spaces with norms $\|\cdot\|_p$, and $\|\cdot\|_{\hat{p}}$, respectively. The action of left translations L on L_p, or $\hat{L}$ on $L_{\hat{p}}$, is given by

$$(L(g)\,\varphi)(h) = \varphi(g^{-1}\,h) \;,\; (\hat{L}(g)\,\hat{\varphi})(h) = \hat{\varphi}(g^{-1}\,h) \;,$$

for $\varphi \in L_p$, or $\hat{\varphi} \in L_{\hat{p}}$. The action of L is isometric but the operator norm of $\hat{L}$ is given by

$$\|\hat{L}(g)\|_{\hat{p}\to\hat{p}} = m(g)^{-1/p} \;.$$

Next let $a_1, \ldots, a_d$ be a basis of the Lie algebra g of G and $A_i = dL(a_i)$, $\hat{A}_i = d\hat{L}(a_i)$ the generators of the one-parameter subgroups $t \mapsto L(e^{-ta_i})$, $t \mapsto \hat{L}(e^{-ta_i})$, of translation on L_p, and $L_{\hat{p}}$, respectively. We define the C^n-elements $L_{p;n}$ for left translations on L_p as the common domain for all n-th order monomials in the A_i. The C^n-elements $L_{\hat{p};n}$ for $\hat{L}$ on $L_{\hat{p}}$ are defined similarly. The C^∞-elements $L_{p;\infty}$, or $L_{\hat{p};\infty}$, are then introduced on the intersection of the family of subspaces $L_{p;n}$, or $L_{\hat{p};n}$. It is also useful to introduce the C^n-norms $\|\cdot\|_{p;n}$ by the recursive definition $\|\cdot\|_{p;0} = \|\cdot\|_p$ and

$$\|\varphi\|_{p;n} = \|\varphi\|_{p;n-1} + \sup_{1\leq i<d} \|A_i\varphi\|_{p;n-1} \; .$$

Norms $\|\cdot\|_{\hat{p};n}$ on $L_{\hat{p};n}$ are defined analogously.

The Laplacians Δ, and $\hat{\Delta}$, of the basis $a_1, \ldots, a_d$ are then defined by

$$\Delta = -\sum_{i=1}^{d} A_i^2 \; , \; \hat{\Delta} = -\sum_{i=1}^{d} \hat{A}_i^2 \; .$$

These operators are closable on each of the L_p-, or $L_{\hat{p}}$-, spaces, and it is an implication of our estimates (see Theorem 3.1) that they are already closed if $p \in \langle 1, \infty\rangle$. Therefore, at a slight risk of confusion, we will not make a notational distinction between the operators and their closures.

Now with these definitions we aim to establish the inequalities (1.1), and similar inequalities for the $L_{\hat{p}}$-spaces. Thus the classical results transfer directly to the Lie group setting for left translations on either scale of spaces and hence, by symmetry, they also hold for right translations.

The key tools in this investigation are the families of continuous semigroups S, and $\hat{S}$ generated by the (closed) operators Δ, and $\hat{\Delta}$. (If $p \in [1, \infty\rangle$ all statements are with respect to the strong topology, and if $p = \infty$ the weak* topology.) These semigroups were originally constructed by Nelson and Stinespring [NeS], and independently by Langlands [Lan]. The two families interpolate between the spaces, e.g. $S_t(L_p \cap L_r) \subseteq L_p \cap L_r$, and they are consistent, i.e. $S_t(L_p \cap L_{\hat{p}}) = \hat{S}_t(L_p \cap L_{\hat{p}})$. The semigroups S are contractive but

$$\|\hat{S}_t\|_{\hat{p}\to\hat{p}} = e^{t\beta^2/p}$$

where $\beta^2 = \sum_{i=1}^d \beta_i^2$ and the β_i are the left derivatives of m at the identity e of G, i.e. $\beta_i = (A_i\, m)(e)$.

Langlands [Lan] established that the action of S, and $\hat{S}$, is determined by a universal integral kernel K,

$$(S_t\,\varphi)(g) = \int_G dh\, K_t(h)\,\varphi(h^{-1}\,g) = (\hat{S}_t\,\varphi)(g)$$

for all $\varphi \in L_p \cap L_{\hat{p}}$. Properties of this kernel, and in particular upper bounds on the kernel and its derivatives, are fundamental for the sequel. Such bounds have been given by many authors and the information we use can be found in [Dav] [Var] [Rob1] or references cited therein.

The kernel K is pointwise positive and jointly analytic in t and g. Moreover one has upper bounds

$$K_t(g) \le a(1 \wedge t)^{-d/2}\, e^{\beta^2 t/4}\, e^{-b|g|^2/t} \tag{2.1}$$

where $a, b > 0$ and $g \mapsto |g|$ is the modulus associated with a right-invariant Riemannian metric on G. In particular $|g| = \infty$ and $K_t(g) = 0$ if $g \notin G_0$, the connected component of the identity. These bounds can be improved for special classes of groups, or for special choices of the modulus, but we will not need these improvements.

The resolvents $(\lambda I + \Delta)^{-1}$ of the (closed) Laplacian Δ are defined for $\lambda > 0$ as bounded operators on the L_p-spaces by Laplace transformation of S,

$$(\lambda I + \Delta)^{-1}\,\varphi = \int_0^\infty dt\, e^{-\lambda t} S_t\,\varphi\,.$$

It then follows that the resolvents have positive kernels R such that

$$\left((\lambda I + \Delta)^{-1}\,\varphi\right)(g) = \int_G dh\, R_\lambda(h)\,\varphi(h^{-1}\,g)\ .$$

The resolvents associated with $\hat{\Delta}$ on $L_{\hat{p}}$ can also be defined for $\lambda > \beta^2/p$ and their action is determined by the same kernel. The bounds on K then translate into bounds on R. One finds that there are b, c, $\lambda_0 > 0$ and for each $\lambda > \lambda_0$ an a_λ such that

$$R_\lambda(g) \le a_\lambda\, f(g)\, e^{-(b\lambda^{1/2} - c)|g|}$$

for all $g \in G\backslash\{e\}$ where

$$f(g) = |g|^{-(d-2)} \qquad \text{if } d > 2$$

$$= 1 + \big|\log|g|\,\big| \qquad \text{if } d = 2\;.$$

The semigroups S, and $\hat{S}$, map the spaces L_p, and $L_{\hat{p}}$, into the C^∞-elements $L_{p;\infty}$, and $L_{\hat{p};\infty}$, for left translations. Therefore the operators $A_i\,S_t$, $A_i\,A_j\,S_t$, etc. are bounded, for all $t > 0$ and $i, j = 1, \ldots, d$, and have kernels $A_i\,K_t$, $A_i\,A_j\,K_t$ etc. where the derivatives A_i, A_j or K_t are defined by pointwise limits. These latter kernels have bounds [Rob1] similar to (2.1);

$$\text{(2.3)} \qquad \begin{cases} |(A_i\,K_t)(g)| \le a'\,(1 \wedge t)^{-d/2}\,t^{-1/2}\,e^{\beta^2 t/4}\,e^{-b|g|^2/t}\;, \\ |(A_i\,A_j\,K_t)(g)| \le a''\,(1 \wedge t)^{-d/2}\,t^{-1}\,e^{\beta^2 t/4}\,e^{-b|g|^2/t}\;, \end{cases}$$

for all $g \in G$ and $t > 0$ etc. Thus each additional derivative introduces an extra factor $t^{-1/2}$.

Now the resolvents leave the C^∞-elements invariant and hence $A_i(\lambda I + \hat{\Delta})^{-1}$, $A_i\,A_j(\lambda I + \hat{\Delta})^{-1}$, etc. are defined on these subspaces. But for $\varphi \in L_{p;\infty}$ it follows from the dominated convergence theorem that

$$(A_i(\lambda I + \hat{\Delta})^{-1}\,\varphi)(g) = \int_G dh\,\lim_{t\to 0} t^{-1}(R_\lambda(h) - R_\lambda(e^{ta_i}\,h))\,\varphi(h^{-1}\,g)\;.$$

The pointwise derivatives $g \in G\backslash\{e\} \mapsto (A_i\,R_\lambda)(g)$ exist, however, by the bounds (2.2) and another application of the dominated convergence theorem. Therefore the action of $A_i(\lambda I + \hat{\Delta})^{-1}$, $A_i\,A_j(\lambda I + \hat{\Delta})^{-1}$, etc. on $L_{p;\infty}$ is given by the derivatives $A_i\,R_\lambda$, $A_i\,A_j\,R_\lambda$, etc. of the kernel R_λ. The bounds (2.3) on the derivatives of the kernel then lead to bounds

$$|(A_i\,R_\lambda)(g)| \le a'_\lambda\,|g|^{-(d-1)}\,e^{-(b\lambda^{1/2}-c)|g|}\;,$$

$$|(A_i\,A_j\,R_\lambda)(g)| \le a''_\lambda\,|g|^{-d}\,e^{-(b\lambda^{1/2}-c)|g|}\;,$$

for $\lambda > \lambda_0$ and $g \in G\backslash\{e\}$, etc. Thus each additional derivative introduces an extra factor $|g|^{-1}$. These bounds are now valid for all $d \ge 2$.

3. A PRIORI ESTIMATES

For the remainder of this paper it will be assumed that G is a unimodular Lie group. So $dg = \hat{d}g$, $L_p = L_{\hat{p}}$ etc. This makes it easier to describe the results but the assumption is not essential. See [BuR] for a complete exposition.

The principal problem in establishing the Lie group version of the basic inequalities (1.1) on the L_p-spaces is to establish that the operators $A_i(\lambda I + \Delta)^{-1/2}$, and $A_i\, A_j(\lambda I + \Delta)^{-1}$, are bounded. Here, the square root is defined by the algorithm

$$X^{-1/2} = \pi^{-1} \int_0^\infty d\lambda\, \lambda^{-1/2}(\lambda I + X)^{-1}$$

(see, for example, [Paz]).

Theorem 3.1. *If $p \in (1, \infty)$ and $\lambda > 0$ then there exists a $C_p > 0$ such that*

$$\|A_i\, A_j\, \varphi\|_p \le C_p(\|\Delta\, \varphi\|_p + \|\varphi\|_p) \quad \text{for all } \varphi \in L_{p;2} \text{ and } i, j = 1, \ldots, d \tag{3.1}$$

$$\|\varphi\|_{p;1} \le C_p\, \|(\lambda I + \Delta)^{1/2}\, \varphi\|_p \quad \text{for all } \varphi \in L_{p;1} \tag{3.2}$$

Proof. Since the proofs of (3.1) and (3.2) are similar in outline only the proof of (3.1) will be described. There are four steps.

Step 1. Inequality (3.1) is first proved for $p = 2$. A brief algebraic calculation is involved. See [Rob2] for details.

Step 2. Restricted weak L_1 estimate.

From step 1 the operator $A_i\, A_j(\lambda I + \Delta)^{-1}$ extends to a bounded operator on L_2. We now prove that this operator satisfies a restricted form of boundedness as an operator from L_1 into the weak L_1-space.

Let $X = X_\lambda^{i,j}$ denote the operator $A_i\,A_j(\lambda I+\Delta)^{-1}$ defined on $L_{1;\infty}$. Let $\varphi \in L_{1;\infty}$ and for $\gamma > 0$ define φ^γ by $\varphi^\gamma(g) = \varphi(g)$ if $|\varphi(g)| > \gamma$ and $\varphi^\gamma(g) = 0$ if $|\varphi(g)| \leq \gamma$. We need to prove that there is a $C > 0$ such that

$$\int_G dg\,\{g \in G \;:\; |(X\,\varphi^\gamma)(g)| > \alpha\} \leq \alpha^{-1}\,C\,\|\varphi^\gamma\|_1 \tag{3.3}$$

$$\text{for all } \varphi \in L_{1;\infty} \text{ and } \alpha, \gamma > 0 \;.$$

The estimate (3.3) is a consequence of the following proposition which is proved in [BuR].

Proposition 3.2. *Let X be a left-convolution operator from $C_c^\infty(G)$ into $L_{\hat{1}} \cap L_{\hat{2}}$ with kernel R, i.e.*

$$(X\,\varphi)(g) = \int_G dh\,R(h)\,\varphi(h^{-1}\,g) \;.$$

Assume that

1. *X extends to a bounded operator from $L_{\hat{2}}$ to $L_{\hat{2}}$,*
2. *R has support in a ball B_r, is once pointwise left-differentiable on $B_r\backslash\{e\}$, and satisfies bounds*

$$|R(g)| \leq a\,|g|^{-d} \;,\; |(A_i\,R)(g)| \leq a\,|g|^{-d-1}$$

for all $g \in B_r\backslash\{e\}$ and $i = 1, \ldots, d$.

It follows that there is a $C > 0$ such that

$$\int_G d\hat{g}\,\{g \in G\;;\; |(X\,\varphi^\gamma)(g)| > \alpha\} \leq \alpha^{-1}\,C\,\|\varphi^\gamma\|_{\hat{1}}$$

for all $\varphi \in C_c^\infty(G)$ and α, $\gamma > 0$.

The proposition is very similar to results developed by Stein for the discussion of singular integrals (see [Ste1] Chapter II). One key idea of the proof is to reduce it to the case $G = \mathbb{R}^d$ by choosing r small and using the exponential map.

Step 3. Interpolation.

The interpolation argument of Stein ([Ste1] p.21–22) now gives (3.1) for $p \in (1, 2]$.

Step 4. Duality.

A duality argument can be used to obtain (3.1) for $p \in (2, \infty)$.

4. DIFFERENTIAL STRUCTURE

The à priori inequalities of Theorem 3.1 provide the principal information for the characterization of the left differential structure on the L_p-spaces by the Laplacian. Using Theorem 3.1, induction, and some algebra we can establish the following theorem.

Theorem 4.1. *For each $p \in (1, \infty)$, $n = 1, 2, \ldots$ and $\lambda > 0$ $L_{p;n} = D((\lambda I + \Delta)^{n/2})$ and there is a $C_{p,n,\lambda} > 0$ such that*

$$C_{p,n,\lambda}^{-1} \, \|\varphi\|_{p;n} \le \|(\lambda I + \Delta)^{n/2} \, \varphi\|_p \le C_{p,n,\lambda} \, \|\varphi\|_{p;n}$$

$$\text{for all } \varphi \in L_{p;n} \; .$$

Remark: The identification $L_{p;n} = D((\lambda I + \Delta)^{n/2})$ implies a certain universality. The definition of $L_{p;n}$ is independent of the choice of basis $a_1, \ldots, a_d$. But the definition of Δ is certainly basis dependent. Nevertheless one concludes $D((\lambda I + \Delta)^{n/2})$ is basis independent. But $D(\Delta^{n/2}) = D((\lambda I + \Delta)^{n/2})$. Hence the domains $D(\Delta^{n/2})$ are the same for all Laplacians.

5. COMPACT GROUPS

In the introduction we indicated that the relations between differential structures revealed by Theorem 4.1 can be considerably strengthened if $G = \mathbb{R}^d$. A similar situation prevails for compact groups.

Theorem 5.1. *Let G be a compact group. For each $p \in (1, \infty)$ and $n = 1, 2, \ldots,$ $L_{p;n} = D(\Delta^{n/2})$ and there is a $C_{p;n}$ such that*

$$C_{p,n}^{-1} \, \sup \, \|M_n \, \varphi\|_p \le \|\Delta^{n/2} \, \varphi\|_p \le C_{p,n} \, \sup \, \|M_n \, \varphi\|_p$$

for all $\varphi \in L_{p;n}$, where the supremum is over all n-th order monomials in the generators $A_1, \ldots, A_d$.

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