COMPENSATED COMPACTNESS AND THE BITING LEMMA

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In this talk I want to discuss some recent developments in the theory of compensated compactness, especially weak continuity of Jacobians of vector-valued functions and weak lower semicontinuity of variational integrals with quasiconvex integrands in the sense of Chacon's Biting Lemma. The characterization of weak continuous functionals has been an important tool for studying nonlinear partial differential equations and the calculus of variations. In particular, the "div-curl" lemma was instrumental in the work of Tartar [22] and DiPerna [14] on conservation laws, and the study of null Lagrangians was central to the work of Ball [4,5] on polyconvex functions and nonlinear elasticity.

The following result, known as Chacon's Biting Lemma, will be used to study weak continuity and weak lower semicontinuity of functionals. For proofs and more general statements of the lemma, see Brooks and Chacon [13], Balder [3], Slaby [24], Ball and Murat [10].

Theorem A. *(Biting Lemma) Let $\Omega \subset R^n$ be bounded and measurable and let $f^{(j)}$ be a bounded sequence in $L^1(\Omega)$. Then there exist a function $f \in L^1(\Omega)$, a subsequence $f^{(\nu)}$ of $f^{(j)}$ and a non-increasing sequence of measurable subsets $E_k \subset \Omega$ with*

$$\lim_{k \to \infty} \operatorname{meas}(E_k) = 0,$$

such that

$$f^{(\nu)} \rightharpoonup f \qquad \text{in } L^1(\Omega \setminus E_k)$$

as $\nu \to \infty$, for each fixed k.

The results E_k, which are removed ("bitten from") Ω, are associated with possible concentrations of the sequence $f^{(j)}$. Here and in the rest of the talk, $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ denote weak convergence and weak-$*$ convergence.

The following is the main weak continuity result of this talk.

Theorem 1. *Let $n \geq 2$, $\Omega \subset R^n$ be open and bounded and $u^{(j)} \rightharpoonup u$ in $W^{1,n}(\Omega, R^n)$. Then there exists a subsequence $u^{(\nu)}$ of $u^{(j)}$ such that*

$$\det Du^{(\nu)} \rightharpoonup \det Du \qquad \text{in the sense of the biting lemma in } \Omega.$$

Theorem 1 is almost optimal, since in general we can not expect that $\det Du_\nu \rightharpoonup \det Du$ in $L^1(\Omega)$ (see Ball and Murat [8, Counterexample 7.3]).

The weak continuity of Jacobians was studied by Reshetnyak [19] (also see Ball, Currie and Olver [7]), who showed that under the same assumptions as Theorem 1,

$$\det Du_j \overset{*}{\rightharpoonup} \det Du \qquad \text{in the sense of measures.}$$

In a more general framework of compensated compactness, Murat [18] and Tartar [21] have proved the "div-curl" lemma, that if $u_j \rightharpoonup u$, $v_j \rightharpoonup v$ in $L^2(\Omega; R^n)$, and $\operatorname{div} u_j$, $\operatorname{curl} v_j$ belong to compact sets of H^{-1}_{loc}. Then

$$u_j \cdot v_j \overset{*}{\rightharpoonup} u \cdot v \qquad \text{in the sense of measures.}$$

An advantage of the weak continuity results for Jacobians and those in the theory of compensated compactness in the sense of the biting lemma, is that they can be used to study systems with measurable coefficients. For example, to prove existence results in nonlinear elasticity when the stored energy function is a Carathéodory function (see [17,20]).

The biting lemma can also be used to study lower semicontinuity problems for variational integrals. To state the result, we need the following well-known weak lower semicontinuity theorem in the multi-dimensional calculus of variations:

Theorem B. *(Acerbi and Fusco [1]) Let $\Omega \subset R^n$ be bounded and open, and let*

$$I(u) = \int_\Omega f(x, u, Du)\, dx, \qquad u \in W^{1,p}(\Omega; R^N),$$

where $1 \le p < \infty$, and where $f : \Omega \times R^N \times R^{Nn} \to R$ satisfies

(i) *f is a Carathéodory function,*

(ii) $0 \le f(x, u, P) \le a(x) + C(|u|^p + |P|^p)$
for every $x \in R^n$, $u \in R^N$ and $P \in R^{Nn}$, where $C > 0$ and $a(\cdot) \in L^1(\Omega)$,

(iii) *f is quasiconvex in P.*

Let $u^{(j)} \rightharpoonup u$ in $W^{1,p}(\Omega; R^N)$. Then

$$I(u) \le \liminf_{j \to \infty} I(u^{(j)}).$$

A function $f : R^{Nn} \to R$ is quasiconvex (see Morrey [16], Ball [4, 5], Ball, Currie and Olver [7]) if

$$\int_U f(P + D\phi(x))\, dx \ge f(P)\, \text{meas}(U)$$

for every $P \in R^{Nn}$, $\phi \in C^1_0(U; R^N)$, and every open bounded subset $U \subset R^n$. A Carathéodory function $f : \Omega \times R^N \times R^{Nn} \to R$ is **quasiconvex in** P if there exists a subset $I \subset \Omega$ with $\text{meas}(I) = 0$ such that $f(x, u, \cdot)$ is quasiconvex for all $x \in \Omega \setminus I$, $u \in R^N$.

A function $f : \Omega \times R^N \times R^s \to \bar{R}$ is a **Carathéodory function** if

(i) $f(\cdot, u, a)$ is measurable for every $u \in R^N$, $a \in R^s$,

(ii) $f(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$.

We also use the following theorem concerning the existence and properties of Young measures. For results in a more general context and proofs the reader is referred to Berliocchi and Lasry [12], Balder [2] and Ball [6].

Theorem C. *Let $z^{(j)}$ be a bounded sequence in $L^1(\Omega; R^s)$. Then there exist a subsequence $z^{(\nu)}$ of $z^{(j)}$ and a family $(\nu_x)_{x\in\Omega}$ of probability measures on R^s, depending measurably on $x \in \Omega$, such that for any measurable subset $A \subset \Omega$,*

$$f(\cdot, z^{(\nu)}) \rightharpoonup \langle \nu_x, f(x, \cdot)\rangle \qquad \text{in } L^1(A),$$

for every Carathéodory function $f : \Omega \times R^s \to R$ such that $f(\cdot, z^{(\nu)})$ is sequentially weakly relatively compact in $L^1(A)$.

The following is the main lower semicontinuity result of this talk.

Theorem 2. *Let $1 \le p < \infty$ and let $f : R^{nN} \to R$ be a continuous quasiconvex function satisfing*

$$| f(P) | \le C(1+ | P |^p)$$

where C is a nonnegative constant. Then given any sequence $u^{(j)} \rightharpoonup u$ in $W^{1,p}(\Omega; R^N)$, there exist a subsequence $u^{(\nu)}$ and a family $(\nu_x)_{x\in\Omega}$ of probability measures on R^{Nn}, depending measurably on x, such that

$$f(, Du^{(\nu)}) \rightharpoonup l(x) := \langle \nu_x, f(\cdot)\rangle \qquad \text{in the sense of the biting lemma in } \Omega \tag{1}$$

and

$$\langle \nu_x, f(\cdot)\rangle \ge f(Du(x)) \qquad \text{a.e. in } \Omega. \tag{2}$$

Moreover, if E_k denotes the sequence of measurable subsets of Ω in the biting lemma corresponding to (1) and satisfying $\lim_{k\to\infty} \operatorname{meas}(E_k) = 0$, then for each fixed k,

$$\int_{\Omega\setminus E_k} f(Du(x))\, dx \le \liminf_{\nu\to\infty} \int_{\Omega\setminus E_k} f(Du^{(\nu)}(x))\, dx. \tag{3}$$

The proofs given in this talk are essentially the ones given in [10]. Although the result here can be extended to the case in which f is a function depending on (x, u, P), for simplicity we shall restrict to the case in which f does not depend on (x, u).

Proof of Theorem 2. For $m = 1, 2, \ldots$, let $g_m(t) = \max(t, -m)$, $t \in R$. Then g_m is convex and monotone, so that $g_m \circ f$ is still quasiconvex in P. Also

$$-m \le g_m(f(P)) \le C(1+ | P |^p) \tag{4}$$

for all $P \in R^{Nn}$. Therefore by Theorem A applied to f, $| f |$ and $g_m \circ f$, there exists a subsequence $u^{(\nu)}$ of $u^{(j)}$ such that

$$f(Du^{(\nu)}) \rightharpoonup \langle \nu_x, f(\cdot) \rangle \qquad \text{in } \Omega, \tag{5}$$

$$| f(Du^{(\nu)}) | \rightharpoonup \langle \nu_x, | f(\cdot) | \rangle \qquad \text{in } \Omega, \tag{6}$$

$$g_m(f(Du^{(\nu)})) \stackrel{b}{\rightharpoonup} \langle \nu_x, g_m(f(\cdot)) \rangle \qquad \text{in } \Omega, \tag{7}$$

in the sense of the biting lemma, as $\nu \to \infty$, for each fixed m. Let E_k, $E_{m,k}$ be the sequences of measurable subsets of Ω in the biting lemma corresponding to (5), (7) respectively, and denote by $\chi_{m,k}$ the characteristic function of $\Omega \setminus E_{m,k}$. Let $\phi \in L^\infty(\Omega)$, $0 \leq \phi(x) \leq 1$ a.e. $x \in \Omega$. Let

$$g_{m,k}(x, P) = \phi(x)\chi_{m,k}(x)g_m(f(P)).$$

Then $g_{m,k} + m$ is quasiconvex in P and so by (4) and Theorem B we have

$$\liminf_{\nu \to \infty} \int_\Omega g_{m,k}(Du^{(\nu)})\, dx \geq \int_\Omega g_{m,k}(Du)\, dx,$$

that is

$$\liminf_{\nu \to \infty} \int_{\Omega \setminus E_{m,k}} \phi(x) g_m(f(Du^{(\nu)}))\, dx \geq \int_{\Omega \setminus E_{m,k}} \phi(x) g_m(f(Du))\, dx. \tag{8}$$

By (7) the left-hand side of (8) equals

$$\int_{\Omega \setminus E_{m,k}} \phi(x) \langle \nu_x, g_m(f(\cdot)) \rangle\, dx. \tag{9}$$

Since ϕ is arbitrary with $0 \leq \phi \leq 1$, and since $\lim_{k\to\infty} \text{meas}(E_{m,k}) = 0$, it follows from (8), (9) that

$$\langle \nu_x, g_m(f(\cdot)) \rangle \geq g_m(f(Du(x)) \qquad \text{a.e. in } \Omega. \tag{10}$$

But by (6),

$$\langle \nu_x, | f(\cdot) | \rangle < \infty \qquad \text{a.e. in } \Omega,$$

and since $| g_m(t) | \leq | t |$ for all t,

$$| g_m(f(P)) | \quad \leq \quad | f(P) |$$

for all $x \in \Omega$, $P \in R^{Nn}$. Passing to the limit $m \to \infty$ in (10) using Lebesgue's dominated convergence theorem we therefore obtain (2), from which (3) follows immediately.

□

Proof of Theorem 1. Since $\det P$ is a null Lagrangian (cf. Ball [4, 5]), $\pm \det P$ are quasiconvex, so that the hypotheses of Theorem 2.1 are satisfied with $p = n$. Hence

$$\langle \nu_x, \det P \rangle = \det Du(x) \qquad \text{a.e. } x \in \Omega.$$

□

Theorem 1 can be used to study the existence problem in nonlinear elasticity to obtain an improved version of an existence result due to Ball and Murat [8], when the stored energy density is a Carathéodory function (see [23]).

Using the maximal function method, a "div-curl" lemma in the sense of the biting can be obtained (see [23]). Partly motivated by Theorem 1, Müller [17] proved the following result in the case the Jacobian of the mapping is non-negative:

Theorem D. *(Müller). Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be bounded and open, let $u : \Omega \to \mathbb{R}^n$ be in $W^{1,n}(\Omega;\mathbb{R}^n)$ and assume that* $\det Du \geq 0$ *a.e.. then for every compact set $K \subset \Omega$, $|\det Du| \log(2+|\det Du|) \in L^1(K)$ and*

$$\| |\det Du| \log(2+|\det Du|)\|_{L^1(K)} \leq C(K, n, \|u\|_{W^{1,n}}).$$

One implication of Müller's result is that if $u_j \rightharpoonup u$ weakly in $W^{1,n}(\Omega;\mathbb{R}^n)$ and $\det Du_j \geq 0$, a.e. in Ω, then up to a subsequence, $\det Du_j \rightharpoonup \det Du$ in $L^1_{loc}(\Omega)$. Therefore, in this case the sets E_k to be removed in the biting lemma are close to the boundary of Ω.

Motivated by Müller's result, Coifman, Lions, Meyer and Semmes [13] proved that in fact, if $u \in W^{1,n}(R^n, R^n)$, then $\det Du \in \mathcal{H}^1(R^n)$, the Hardy space $\mathcal{H}^1$, based entirely on "hard" harmonic analysis. A recent result of Jones and Journé [15] compares the weak convergence in the sense of the biting lemma and the weak-$*$ convergence in $\mathcal{H}^1$, i.e. if $f_j \overset{*}{\rightharpoonup} f$ in $\mathcal{H}^1$ and $f_j \rightharpoonup \tilde{f}$ in the sense of the biting lemma, then $f = \tilde{f}$ a.e..

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