FORMATION OF SINGULARITIES IN SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION WITH CRITICAL POWER NONLINEARITY

HAYATO NAWA

1. INTRODUCTION AND RESULTS

In this paper, the author would like to report his results of recent papers [23–28] concerning the nonlinear Schrödinger equation with critical power nonlinearity (NSC). Our main results are Theorem A ("L^2 concentration" phenomena [23, 24 , 26]), Theorem B (Asymptotics of blow-up solutions [27, 28]) and Theorem C (existence of "blow-up" solutions in the energy space $H^1(\mathbb{R}^N)$ [27, 28]). Moreover, in Sect. 4, he shall briefly mention the further results.

We start with a review of the Cauchy problem for the nonlinear Schrödinger equation:

$$\text{C(p)} \qquad \begin{cases} \text{(NS)} \quad 2i\dfrac{\partial u}{\partial t} + \Delta u + |u|^{p-1}u = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, \\ \text{(IV)} \quad u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

Here $i = \sqrt{-1}$, $u_0 \in H^1(\mathbb{R}^N)$ and Δ is the Laplace operator on $\mathbb{R}^N$. The unique local existence of solutions of C(p) is well known for $1 < p < 2^* - 1$ ($2^* = \frac{2N}{N-2}$ if $N \geqq 3$, $= \infty$ if $N = 1, 2$): For any $u_0 \in H^1(\mathbb{R}^N)$, there exist a unique solution $u(t,x)$ of C(p) in $C([0,T_m); H^1(\mathbb{R}^N))$ for some $T_m \in (0, \infty]$ (maximal existence time), and $u(t)$ satisfies the following three conservation laws of L^2, the energy and the momentum:

$$\|u(t)\| = \|u_0\|, \tag{1.1}$$

$$E_{p+1}(u(t)) \equiv \|\nabla u(t)\|^2 - \frac{2}{p+1}\|u(t)\|_{p+1}^{p+1} = E_{p+1}(u_0), \tag{1.2}$$

$$\Im \int_{\mathbb{R}^N} \nabla u(t,x)\overline{u(t,x)}dx = \Im \int_{\mathbb{R}^N} \nabla u_0(x)\overline{u_0(x)}dx \tag{1.3}$$

for $t \in [0, T_m)$, where $\|\cdot\|$ and $\|\cdot\|_{p+1}$ denotes the L^2 norm and L^{p+1} norm respectively. Furthermore $T_m = \infty$ or $T_m < \infty$ and $\lim_{t \to T_m} \|\nabla u(t)\| = \infty$. For details, see, *e.g.*, [11,12,14].

As for the existence and non-existence of global solutions of C(p), the following is well known.

(i) If $1 < p < 1 + \frac{4}{N}$, there exists a global solution $u \in C_b(\mathbb{R}; H^1(\mathbb{R}^N))$, for any $u_0 \in H^1(\mathbb{R}^N)$, where $C_b(\mathbb{R}; H^1(\mathbb{R}^N)) = C(\mathbb{R}; H^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}; H^1(\mathbb{R}^N))$. See [11,12,14].

(ii) If $1 + \frac{4}{N} \leqq p < 2^* - 1$, there is a subset $\mathcal{B} \in H^1(\mathbb{R}^N)$ such that for any $u_0 \in \mathcal{B}$ the solution of C(p) blows up, *i.e.* the L^2 norm of its gradient explodes in finite time T_m. See [13,29,30,31,36].

As we have seen above, the number $p = 1 + \frac{4}{N}$ is the critical number for the existence of blow-up solutions of C(p). In what follows, we refer to (NS) with $p = 1 + \frac{4}{N}$ as (NSC), *i.e.*,

$$2i\frac{\partial u}{\partial t} + \Delta u + |u|^{\frac{4}{N}}u = 0, \tag{NSC}$$

and we shall use the notations

$$E(v) = E_\sigma(v), \qquad \sigma = 2 + \frac{4}{N}.$$

The nonlinear Schrödinger equation of the form (NSC) is of physical interest, because (NSC) with $N = 2$ arises in a theory of the stationary self-focusig of a laser beam propagating along the t-axis in a nonlinear medium (see *i.e.* [1,2,15,40]). We may say that the blow-up of solution corresponds to the focusing of the laser beam.

Recently, many mathematicians have studied the formation of singularities in blow-up solutions of (NSC) near blow-up time (*e.g.*, [7, 15, 19, 20, 21, 22, 23–29, 35, 37, 38]). Here, it is worth while to note that (NSC) has a remarkable property that it is invariant under the pseudo-conformal transformations (see *e.g.*,[7,29] and (1.5) below). This symmetry seems to be closely related to the structure of solutions of (NSC) (see *e.g.*, Weinstein [37,38], Nawa and M. Tsutsumi [29] and Cazenave and Weisler [7]): in the super critical case ($p > 1 + \frac{4}{N}$), Merle [19] suggested that every blow-up solution of (NS) has a strong limit in L^2 at blow-up time; in the critical case ($p = 1 + \frac{4}{N}$), Nawa [23,24] and Weinstein [38] showed that every blow-up solution of (NSC) loses its L^2 continuity at blow-up time because of the concentration of its L^2 mass (see also Merle [20], Merle and Y. Tsutsumi [21], and Y. Tsutsumi [35]). Moreover we know how amount the blow-up solution of (NSC) concentrate their L^2 mass. Precisely, we can prove [23, 24, 26] (see also [38]):

Theorem A. *Let Q be a nontrivial solution of the elliptic equation*

$$\Delta Q - Q + |Q|^{\frac{4}{N}}Q = 0 \tag{A.1}$$

such that

$$\begin{aligned} \frac{2}{\sigma}\|Q\|^{\frac{4}{N}} &= \inf_{\substack{v \in H^1(\mathbb{R}^N) \\ v \neq 0}} \frac{\|v\|^{\frac{4}{N}}\|\nabla v\|^2}{\|v\|_\sigma^\sigma} \\ &= \inf_{\substack{v \in H^1(\mathbb{R}^N) \\ v \neq 0}} \left\{ \frac{2}{\sigma}\|v\|^{\frac{4}{N}} \; ; \; \|\nabla v\|^2 - \frac{2}{\sigma}\|v\|_\sigma^\sigma \leqq 0 \right\}. \end{aligned} \tag{A.2}$$

Let $u(t)$ be the singular solution of (NSC) such that

$$\lim_{t \to T_m} \|\nabla u(t)\| = \lim_{t \to T_m} \|u(t)\|_\sigma = \infty \tag{A.3}$$

for some $T_m \in (0,\infty]$. We put

$$\lambda(t) = \frac{1}{\|u(t)\|_\sigma^{\sigma/2}}, \tag{A.4}$$

$$u_\lambda(t,x) = \lambda(t)^{\frac{N}{2}} u(t,\lambda(t)x), \tag{A.5}$$

$$A = \sup_{R>0}\left(\liminf_{t\uparrow T_m}\left(\sup_{y\in\mathbb{R}^N}\int_{|x-y|\leqq R}|u_\lambda(t,x)|^2dx\right)\right). \tag{A.6}$$

(1) Then we have

$$A \geqq \|Q\|^2 \tag{A.7}$$

and we have that, for any $\varepsilon \in (0,1)$, there are constans $K>0$ and $T_0>0$, and there is a right continuous function $\gamma(\cdot)\in L^\infty_{loc}([T_0,T_m);\mathbb{R}^N)$ such that

$$\int_{|x-y|\leqq R\lambda(t)} |u(t,x+\lambda(t)\gamma(t))|^2dx \geqq (1-\varepsilon)A, \quad t\in[T_0,T_m), \tag{A.8}$$

for any $R\geqq K$. Moreover if $A>\frac{1}{2}\|u(0)\|^2$, then we have $\gamma(\cdot)\in C([T_0,T_m);\mathbb{R}^N)$.
(2) Suppose in addition that $\|u(0)\|^2 < 2\|Q\|^2$. Then by (1), we have $\gamma(\cdot)\in C([T_0,T_m);\mathbb{R}^N)$.

Remark 1.1. (1) The equation (A.1) is a time-independent version of (NSC) and arises in various domain of physics. See [3,6,27,28,32,36] for the existence of positive solutions of (A.2) and for the associated minimization problems. The standard argument shows that $Q\in\mathcal{S}$ (the space of C^∞ functions of rapid decreasing). We can also prove that $E(Q)=0$.
(2) By the first equality in (A.2) and the conservation law (1.2), we see that if $\|u_0\|<\|Q\|$, the corresponding solution exists globally in time. For this, see Weinstein [36, 37]. Furthermore the initail datum $Q(x)e^{-i|x|^2/2}$ leads to the blow-up solution which blows up at time $t=1$ (see [29,37]). In this sense, the estimate (A.7) is optimal.
(3) The formula (A.8) tells us that the focusing of a laser beam could be understood mathematically as "L^2 concentration" phenomena of blow-up solutions of (NSC). The quantity A measures the "size" of the "largest" singularity, since the blow-up solution, in genaral, has several L^2-concentration points (see Merle [20] and Theorem B of this paper).

However the profiles of blow-up solutions had not been investigated so well.

It is worth while to note here that the nonlinear Schrödinger equation (NS) also derived from a field equation for a quantum mechanical nonrelativistic *many body system* in the semi classical limit (*e.g.* [11]). Hence it is expected that the solution of nonlinear Schrödinger equation involves a *many body* character in its behavior. Therefore, in general, it seems difficult to trace directly the dynamics of blow-up solutions. In other words, we may not expect the clear "trajectories" of singularities as in Theorem A (2) generally.

Concerning the asymptotic behavior of blow-up solution of (NSC), the following results were known.

(I) Let $u(t)$ be a solution of (NSC) such that $\|u(t)\| = \|Q\|$ and $\|\nabla u(t)\| \to \infty$ as $t \to T_m$ for some $T_m \in (0, \infty]$. Then we have, for $\tilde{\lambda}(t) = \|\nabla u(t)\|^{-1}$,

$$\left\| \tilde{\lambda}(t)^{\frac{N}{2}} u(t, \tilde{\lambda}(t)(\cdot - \tilde{\gamma}(t))) e^{i\theta(t)} - Q(\cdot) \right\| \to 0 \quad \text{as} \quad t \to T_m \tag{1.4}$$

for some $\tilde{\gamma}(t) \in \mathbb{R}^N$ and $\theta(t) \in \mathbb{R}$ (Weinstein [37]).

(II) Let $u(t)$ be a solution of (NSC) such that $xu(t) \in L^2(\mathbb{R}^N)$ and $\|\nabla u(t)\| \to \infty$ as $t \to T_m$ for some $T_m \in (0, \infty)$. If $u(t)$ satisfies $\|(x - a)u(t)\| \to 0$ $(t \to T_m)$, then $u(t)$ must be of the form:

$$(T_m - t)^{-N/2} \exp\left(\frac{-i|x_t(a, v)|^2}{2(T_m - t)} \right) V\left(\frac{t}{T_m(T_m - t)}, \frac{x_t(a, v)}{T_m - t} \right) e^{\frac{iv}{T_m}(x - \frac{v}{2T_m}t)}, \tag{1.5}$$

where $V(t, x)$ is also a solution of (NSC) in $C(\mathbb{R}_+; H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^2 dx))$ such that $E(V(t)) = 0$, and where

$$x_t(a, v) = x - a + v - \frac{v}{T_m} t$$

for an appropriate $v \in \mathbb{R}^N$ (Nawa and M.Tsutsumi [29]).

(III) For given L points $\{a^1, a^2, \cdots, a^L\} \subset \mathbb{R}^N$, there exist a blow-up solution $u(t)$ of (NSC) such that

$$\left\| u(t) - \sum_{j=1}^{L} Q^j(t) \right\|_\sigma \to 0 \quad \text{as} \quad t \to T_m, \tag{1.6}$$

where

$$Q^j(t, x) = (T_m - t)^{-N/2} \exp\left(\frac{-i|x - a^j|^2}{2(T_m - t)} \right) Q\left(\frac{x - a^j}{T_m - t} \right) e^{it/2T_m(T_m - t)} \tag{1.7}$$

for $T_m \in (0, \infty)$ (Merle [20]).

Remark 1.2. (1) $Q(x)e^{it/2}$, which is a standing wave solution of (NSC), is transformed into Q^j by the space-time transformation appearing in the left hand side of (2.2) with $a = a^j$ and $v = 0$. We call this transformation pseudo-conformal transformation. Since we have $E(Q(\cdot)e^{it/2}) = 0$, Q^j is a blow-up solution of (NSC) such that $\|(x - a^j)u(t)\| \to 0$ $(t \to T_m)$ by virtue of (II).

(2) These results require additional conditions on initial data (or solutions): $\|u_0\| = \|Q\|$ for (I); $|x|u_0 \in L^2(\mathbb{R}^N)$ for (II) and (III).

In [27, 28], the author investigated the asymptotic profile of generic H^1 blow-up soluton of (NSC) to obtain:

Theorem B. *Let $u(t)$ be the singular solution of (NSC) such that*

(B.1) $$\limsup_{t\to T_m}\|\nabla u(t)\| = \limsup_{t\to T_m}\|u(t)\|_\sigma = \infty$$

for some $T_m \in (0,\infty]$. Let $\{t_n\}$ be any sequence such that

(B.2) $$\sup_{t\in[0,t_n]}\|u(t)\|_\sigma = \|u(t_n)\|_\sigma.$$

For this $\{t_n\}$, we put

(B.3) $$\lambda_n = \frac{1}{\|u(t_n)\|_\sigma^{\sigma/2}}$$

and, we consider the scaled functions

(B.4) $$u_n(t,x) = \lambda_n^{\frac{N}{2}}\overline{u(t_n-\lambda_n^2 t, \lambda_n x)}$$

for $t\in[0,t_n/\lambda_n^2)$. Then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), which satisfies the following properties: there exist

(i) *a finite number of nontrivial solutions $u^1, u^2, \cdots, u^L$ of (NSC) in $C_b(\mathbb{R}_+; H^1(\mathbb{R}^N))$ with $E(u^j)=0$ and $\Im\int_{\mathbb{R}^N}\nabla u^j(t,x)\overline{u^j(t,x)}dx = 0$ $(j=1,2,\cdots,L)$, and*

(ii) *sequences $\{\gamma_n^1\}$, $\{\gamma_n^2\}$, $\cdots$, $\{\gamma_n^L\}$ in $\mathbb{R}^N$ with $\lim_{n\to\infty}|\gamma_n^j-\gamma_n^k| = \infty$ $(j\neq k)$,*

such that, for any $T>0$,

(B.5) $$\lim_{n\to\infty}\sup_{t\in[0,T]}\left\|u_n(t,\cdot)-\sum_{j=1}^L u^j(t,\cdot-\gamma_n^j)\right\|_\sigma = 0,$$

(B.6) $$\lim_{n\to\infty}\sup_{t\in[0,T]}\left\|\nabla u_n(t,\cdot)-\sum_{j=1}^L \nabla u^j(t,\cdot-\gamma_n^j)\right\| = 0,$$

(B.7) $$\lim_{n\to\infty}\sup_{t\in[0,T]}\left\|u_n(t,\cdot)-\sum_{j=1}^L u^j(t,\cdot-\gamma_n^j) - \phi_n(t,\cdot)\right\| = 0,$$

where

(B.8) $$\phi_n(t,\cdot) = \exp(\frac{it}{2}\triangle) * \left(u_n(0,\cdot)-\sum_{j=1}^L u^j(0,\cdot-\gamma_n^j)\right).$$

Furthermore we have

(B.9) $$\|u_0\|^2 \geqq \sum_{j=1}^L \|u^j(t)\|^2 \geqq L\|Q\|^2.$$

where Q is a nontrivial solution of (A.1) and (A.2).

Remark 1.3. (1) If the solution satisfies $\limsup_{t\to T_m} \|\nabla u(t)\| = \infty$, then we have, by the energy conservation law $\limsup_{t\to T_m} \|u(t)\|_\sigma = \infty$. So, (B.1) is always assured. If $T_m < \infty$, we have (B.1) with lim sup replaced by lim. We can choose a sequence as in (B.2), since we have (B.1).
(2) The scaled function u_n in (B.4) also solves (NSC), and satisfies $\|u_n(t)\| = \|u(t)\|$ and $E(u_n(t)) = \lambda_n^2 E(u(t))$. This is a special feature of (NSC).
(3) The estimate (B.9) is a consequence of $E(u^j) = 0$ and the characterization of the ground state of Q (see (A.2)).
(4) A typical examle of *zero energy* and *zero momentum* solution of (NSC) in $L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^N))$ is $Q(x)e^{it/2}$.
(5) If $u(0,x)$ is radially symmetric, *i.e.* $u(0,x) = u(0,|x|)$, we have $L = 1$ and $\gamma_n^1 \equiv 0$ in this theorem. If $u(0,x)$ has the same L^2 norm of the ground state $Q(x)$, *i.e.* $\|u(0)\| = \|Q\|$, then we have (B.7) with $L = 1$ and $\phi_n \equiv 0$.

Each u^j can be considered to correspond to the "strong" singularity in blow-up solution, since one has, by (B.7),

$$\lim_{n\to\infty} \sup_{t\in[t_n-\lambda_n^2 T, t_n]} \left\| u(t,\cdot) - \sum_{j=1}^{L} u_n^j(t,\cdot) - \widetilde{\phi}_n(t,\cdot) \right\| = 0, \tag{1.8}$$

where

$$u_n^j(t,x) = \frac{1}{\lambda_n^{N/2}} u^j\left(\frac{t_n - t}{\lambda_n^2}, \frac{x - \gamma_n^j \lambda_n}{\lambda_n}\right), \tag{1.9}$$

$$\widetilde{\phi}_n(t,x) = \frac{1}{\lambda_n^{N/2}} \phi_n\left(\frac{t_n - t}{\lambda_n^2}, \frac{x}{\lambda_n}\right). \tag{1.10}$$

We note that there is a possibility that $\widetilde{\phi}_n$ produce "weak" singularities, around which the rate of blow-up is lower than $\|u(t)\|_\sigma$. Thus we may say that the term $\sum_{j=1}^L u_n^j(t,\cdot)$ is the top term of the asymptotic expansion of blow-up solution near the blow-up time T_m. If $N \geqq 2$, "weak" singularities may form a $N-1$ dimensional manifold as in the case of semilinear heat equations (Giga and Kohn [10]). If $\widetilde{\phi}_n$ produce no singularity, we can safely say that the blow-up set consists of finite number of points as in the case of one-dimensional semilinear heat equations (Chen and Matano [8]).

However, there still remains a possibility of "ergodic" or "chaotic" behavior of singularities, since (as is mentioned after Theorem A) the solution of nonlinear Schrödinger equation involves a *many body* character in its behavior. Each "singularity" $\{u_n^j\}$ considered to be "a cluster" or "a particle" in the analogy of many body Schrödinger equation. If the remainder term $\widetilde{\phi}_n(t,x)$ plays a role of long range potentials, the motion of "clusters" will be very complicated. Thus the expression of Theorem B using the sequence of scaled solutions (B.4) may be a considerable one.

Remark 1.4. As in the proof of [Theroem C;24], we can show that if $u(0) \in H^1(\mathbb{R}^N) \cap L^2(|x|^2dx)$, then

$$\sup_{n\in\mathrm{N}} |\gamma_n^j \lambda_n| < \infty \quad (j = 1, 2, \cdots, L) \tag{1.11}$$

in Theorem B.

In the next section, we shall sketch the proof of Theorem B, in which we shall need the following Theorem C [28].

Theorem C. *Let $p \geqq 1 + \frac{4}{N}$. If the initial datum $u_0(x) = u(0,x)$ satisfies*

$$\text{(C.1)} \qquad E_{p+1}(u_0) < \frac{\left(\Im \int_{\mathbb{R}^N} \nabla u_0(x)\overline{u_0(x)}dx\right)^2}{\|u_0\|^2},$$

then the corresponding solution $u(t)$ of $C(p)$ satisfies

$$\text{(C.2)} \qquad \sup_{t\in[0,T_m)} \|\nabla u(t)\| = \infty,$$

where T_m is the maximal existence time, i.e., $u(t)$ blows up in finite time:

$$\lim_{t\to T_m} \|\nabla u(t)\| = \lim_{t\to T_m} \|u(t)\|_\sigma = \infty$$

for some $T_m < \infty$; or grows up at infinity:

$$\limsup_{t\to\infty} \|\nabla u(t)\| = \limsup_{t\to\infty} \|u(t)\|_\sigma = \infty.$$

This theorem will play a crucial role to prove the finiteness of "singularities" in Theorem B (see Sect.2), and Theorem C itself is also an improvement of previous results concerning the existence of blow-up solution of (NS) in the sence that we do not require additional conditions on initial data except (C.1). We note here that the condition (C.1) can be reduced to the one that

$$E_{p+1}(u_0) < 0, \tag{1.12}$$

since (NS) is invariant under the Galilei transformations:

$$u(t,x) \longmapsto u(t, x - vt) \exp\left(iv \cdot x - \frac{i}{2}v^2 t\right), \quad v \in \mathbb{R}^N. \tag{1.13}$$

The blow-up of negative energy solutions had been proved under some conditions: $|x|u_0 \in L^2(\mathbb{R}^N)$ (Glassey [13]) while this is an important class of initial data and quite reasonable physically;

$N \geqq 2$ and u_0 is radially symmetric (Ogawa-Y.Tsutsumi[30]); $N = 1$ and $p = 1 + \frac{4}{N}$ (Ogawa-Y.Tsutsumi[31]). Hence the results of Ogawa-Tsutsumi [30,31] ensure that Theorem C with $T_m < \infty$ holds true for the case of $N = 1$, and of $N \geqq 2$ and $u(0)$ being radially symmetric. We may say that Theorem C is a weak version of a theorem which has long been speculated. Here "weak" means that Theorem C does not assert that every negative energy initial datum leads to the blow-up solution of $C(p)$. There remains a possibility that $T_m = \infty$.

We shall give the sketch of proof of Theorem C in Sect.3.

Remark 1.5. Of corse, for the proof of Theorem B, we need Theorem C in the case of $p = 1 + \frac{4}{N}$ only. In [28], one can find the proof of Theorem C for $p = 1 + \frac{4}{N}$ under the assumption (1.12) instead of (C.1). However the argument also works for $p > 1 + \frac{4}{N}$ with slight modifications. The properties in Theorem B that $E(u^j) = 0$ and $\Im \int_{\mathbb{R}^N} \nabla u^j(t,x)\overline{u^j(t,x)}dx = 0$ $(j = 1, 2, \cdots, L)$ are the byproduct of Theorem C.

2. SKETCH OF PROOF OF THEOREM B

In this section, we shall give the sketch of Theorem B under the assumption that Theorem C holds true.

Theorem B seems to be closely related to a phenomenon which has been observed in various nonlinear problems by the name of bubble theorem or concentrated compactness theorem (for example, see [4,16,17,18,22,33,34]). In fact, the proof of this theorem is inspired by Brézis and Coron [4]. One may find that the underlying idea being the method of concentrated compactness due to Lions [17,18]. However, we do not use the general method of it. Our basic tool is the compactness device as in Lieb [16] (see also Brézis and Lieb [6] and Fröhlich, Lieb and Loss [9]). We extend Lieb's compactness lemma to space-time one, with which the Ascoli-Arzela theorem plays a crucial role in our analysis working with the scaled solutions of (NSC) defined by (B.4). The use of the general method of concentrated compactness in the study of blow-up problem for the nonlinear Schrödinger equation can be traced back to Weinstein [37].

The following proposition is the heart of the matter.

Proposition 2.1. *Let $\{v_n\}$ be an equibounded family in $C([0,T]; H^1(\mathbb{R}^N))$ such that*

$$2i\frac{\partial v_n}{\partial t} + \triangle v_n + |v_n|^{\frac{4}{N}} v_n = g_n, \tag{2.1}$$

$$\sup_{t\in[0,T]} \|v_n(t)\|_\sigma \neq 0. \tag{2.2}$$

Here $\{g_n\}$ is an equibounded family in $C([0,T]; L^{\sigma'}(\mathbb{R}^N))$ such that, for any $R > 0$,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|g_n(t,\cdot)\|_{\sigma'} = 0, \tag{2.3}$$

where $\frac{1}{\sigma}+\frac{1}{\sigma'}=1$. *Then there exists*

(i) *a nontrivial solution* v *of (NSC) in* $C([0,T];H^1(\mathbb{R}^N))$ *and*

(ii) *a sequence* $\{\gamma_n\}\subset\mathbb{R}^N$

such that for $\Omega \Subset \mathbb{R}^N$ *and for some subsequence (still denoted by the same letter),*

$$\widetilde{v}_n \equiv v_n(\,\cdot\,,\,\cdot+\gamma_n) \overset{*}{\rightharpoonup} v \quad \text{weakly* in } L^\infty([0,T];H^1(\mathbb{R}^N)), \tag{2.4}$$

$$\widetilde{v}_n \to v \quad \text{strongly in } C([0,T];L^\alpha(\Omega)) \text{ for } \alpha\in[2,\,2^*) \tag{2.5}$$

as $n\to\infty$.

Furthermore we have

$$|\widetilde{v}_n|^{\frac{4}{N}}\widetilde{v}_n - |\widetilde{v}_n - v|^{\frac{4}{N}}(\widetilde{v}_n - v) - |v|^{\frac{4}{N}}v \to 0 \quad \text{strongly in} \quad C([0,T];L^{\sigma'}(\mathbb{R}^N)), \tag{2.6}$$

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\int_{\mathbb{R}^N}\big||\widetilde{v}_n|^\alpha - |\widetilde{v}_n - v|^\alpha - |v|^\alpha\big|dx = 0, \quad \alpha\in[2,2^*), \tag{2.7}$$

$$\lim_{n\to\infty}\int_0^T\{E(\widetilde{v}_n) - E(\widetilde{v}_n - v) - E(v)\}\,dt = 0, \tag{2.8}$$

and for any $t\in[0,T]$

$$\lim_{n\to\infty}\{E(\widetilde{v}_n(t)) - E((\widetilde{v}_n - v)(t)) - E(v(t))\} = 0. \tag{2.9}$$

We can safely say that our analysis investigates, by means of Proposition 2.1, how the "dichotomy" (in the terminology of concentrated compactness) occurs in the sequence $\{u_n\}$. Theorem B asserts that u_n behaves like a finite superposition of *zero energy* time global solutions of (NSC) (see (B.5) – (B.7)).

One can easily see that the scaled function u_n satisfies

$$2i\frac{\partial u_n}{\partial t} + \Delta u_n + |u_n|^{\frac{4}{N}}u_n = 0, \tag{2.10}$$

$$\|u_n(t)\| = \|u_0\|, \tag{2.11}$$

$$\sup_{t\in[0,T]}\|u_n(t)\|_\sigma = 1 \quad \text{for any} \quad T>0, \tag{2.12}$$

$$E(u_n(t)) = \lambda_n^2 E(u_0)\,(\to 0) \quad \text{as}\, n\to\infty. \tag{2.13}$$

From (2.10), (2.11) and (2.12), it follows that $\{u_n\}$ is an equi-bounded family in $L^\infty(\mathbb{R}_+;H^1(\mathbb{R}^N))$. Thus, by Proposition 2.1, there exist

(i) a nontrivial solution u^1 of (NSC) in $C([0,\infty);H^1(\mathbb{R}^N))$

(ii) a sequence $\{y_n^1\}\subset\mathbb{R}^N$

such that for $\Omega \Subset \mathbb{R}^N$ and for some subsequence (still denoted by the same letter),

$$u_n^1 \equiv u_n(\,\cdot\,,\,\cdot + y_n^1) \overset{*}{\rightharpoonup} u^1 \text{ weakly* in } L^\infty([0,\infty); H^1(\mathbb{R}^N)), \tag{2.14}$$

$$u_n^1 \to u^1 \quad \text{strongly in } C([0,T]; L^\alpha(\Omega)) \tag{2.15}$$

for $\alpha \in [2, 2^*)$ as $n \to \infty$. Furthermore we have

$$|u_n^1|^{\frac{4}{N}} u_n^1 - |u_n^1 - u^1|^{\frac{4}{N}}(u_n^1 - u^1) - |u^1|^{\frac{4}{N}} u^1 \to 0 \tag{2.16}$$

strongly in $C([0,T]; L^{\sigma'}(\mathbb{R}^N))$,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \int_{\mathbb{R}^N} \big| |u_n^1|^\alpha - |u_n^1 - u^1|^\alpha - |u^1|^\alpha \big| dx = 0, \quad \alpha \in [2, 2^*), \tag{2.17}$$

$$\lim_{n\to\infty} \int_0^T \{E(u_n^1) - E(u_n^1 - u^1) - E(u^1)\}\, dt = 0, \tag{2.18}$$

and, for any $t \in \mathbb{R}_+$

$$\lim_{n\to\infty} \{E(u_n^1(t)) - E((u_n^1 - u^1)(t)) - E(u^1(t))\} = 0. \tag{2.19}$$

Suppose that $\limsup_{n\to\infty} \|u_n^1 - u^1\|_\sigma \neq 0$. $u_n^1 - u^1$ satisfies

$$2i\frac{\partial(u_n^1 - u^1)}{\partial t} + \triangle(u_n^1 - u^1) + |u_n^1 - u^1|^{\frac{4}{N}}(u_n^1 - u^1) = g_n^1, \tag{2.20}$$

where

$$g_n^1(t,x) = -\big(|u_n^1|^\sigma - |u_n^1 - u^1|^\sigma - |u^1|^\sigma\big)(t,x). \tag{2.21}$$

We note here that (2.16) implies that for any $\{x_n\} \subset \mathbb{R}^N$ and for any $T > 0$,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|g_n^1(t, \cdot + x_n)\|_\sigma = 0. \tag{2.22}$$

Clearly $\{u_n^1 - u^1\}$ is an equi-bounded family in $L^\infty([0,\infty); H^1(\mathbb{R}^N))$. At this stage we apply Proposition 2.1 to $\{u_n^1 - u^1\}$.

We iteratively use Proposition 2.1 to construct u^j's, and we have

$$\lim_{n\to\infty} \{E(u_n(t)) - E((u_n^j - u^j)(t))\} = \sum_{k=1}^{j} E(u^k(t)). \tag{2.23}$$

Here the important thing is the finiteness of u^j's. If the iteration were not terminated at some finite index, we would have by the construction of u^j's that

$$\lim_{j\to\infty} \lim_{n\to\infty} \sup_{t\in[0,T]} \|(u_n^j - u^j)(t)\|_\sigma = 0. \tag{2.24}$$

The formula (2.23) together with (2.13) and (2.24) yields that

$$\lim_{j\to\infty} \sum_{k=1}^{j} E(u^k(t)) \leqq \lim_{j\to\infty} \lim_{n\to\infty} \sup_{t\in[0,T]} \|(u_n^j - u^j)(t)\|_\sigma^\sigma = 0. \tag{2.25}$$

On the other hand, since each u^j is in $L^\infty([0,\infty); H^1(\mathbb{R}^N))$ and solves(NSC), Theorem C tells us that $E(u^j) \geqq 0$ for any j. From this fact and (2.25), we have $E(u^j) = 0$ for any j. Thus we have $\|u^j\| \geqq \|Q\|$ for any j (see the definition of Q in Theorem A), which lead to a contradiction. Therefoe we obtain $L < \infty$.

3. SKETCH OF PROOF OF THEOREM C

In this section, we shall sketch the proof of Theorem C for $p = 1 + \frac{4}{N}$. We can prove the other cases ($p > 1 + \frac{4}{N}$) analogously.

We begin with

Lemma 3.1. *Let $\Phi \in W^{4,\infty}(\mathbb{R}^N)$, and let $2\Psi = \nabla\Phi = 2(\Psi^1, \Psi^2, \cdots, \Psi^N)$. Then H^1 solution $u(t)$ of C(p) ($p \in (1, 2^*)$) satisfies, for any t, $t_0 \in \mathbb{R}$,*

$$\langle \Psi^k, |u(t)|^2\rangle = \langle \Psi^k, |u(t_0)|^2\rangle + \Im \int_{t_0}^{t} \langle \nabla\Psi^k(s) \cdot \nabla u(s)\, ,\, u(s)\rangle ds. \tag{3.1}$$

and

$$\langle \Phi(t), |u(t)|^2\rangle = \langle \Phi(t_0), |u(t_0)|^2\rangle + 2\Im \int_{t_0}^{t} \langle u(s)\, ,\, \Psi(s)\nabla u(s)\rangle ds. \tag{3.2}$$

Furthermore we have the genegalized dilation identity:

$$\begin{aligned} &-2\Im\langle u(t), \Psi(t)\cdot\nabla u(t)\rangle + 2\Im\langle u(t_0), \Psi(t_0)\cdot\nabla u(t_0)\rangle \\ &= \frac{p-1}{p+1}\int_{t_0}^{t} ds\,\langle \nabla\cdot\Psi(s)|u(s)|^{p+1}\rangle - 2\Re\int_{t_0}^{t} ds\,\langle \partial_j u(s)\, ,\, \partial_k\Psi^j(s)\partial_k u(s)\rangle \\ &\quad + \frac{1}{2}\int_{t_0}^{t} ds\,\langle \triangle(\nabla\cdot\Psi)\, ,\, |u(s)|^2\rangle\,; \end{aligned} \tag{3.3}$$

and the generalized "variance" (or pseudo-conformal) identity:

$$\begin{aligned} \langle \Phi(t), |u(t)|^2\rangle &= \langle \Phi(t_0), |u(t_0)|^2\rangle + 2\Im t\langle u(t_0), \Psi(t_0)\cdot\nabla u(t_0)\rangle + t^2 E_{p+1}(u_0) \\ &\quad + \int_{t_0}^{t} ds \int_{t_0}^{s} d\tau \left\langle \frac{2}{p+1}(\nabla\cdot\Psi(\tau)+2) - \nabla\cdot\Psi(\tau)\, ,\, |u(\tau)|^{p+1}\right\rangle \\ &\quad - 2\int_{t_0}^{t} ds \int_{t_0}^{s} d\tau\,\langle (\delta_{ij} - \Re\partial_i\Psi^j)\partial_i u(\tau)\, ,\, \partial_j u(\tau)\rangle \\ &\quad + \frac{1}{2}\int_{t_0}^{t} ds \int_{t_0}^{s} d\tau\,\langle \triangle(\nabla\cdot\Psi)\, ,\, |u(\tau)|^2\rangle\,. \end{aligned} \tag{3.4}$$

Proof. This lemma is an analogue of Ehrenfest's law in quantum mechanics. We give a formal calculation below. It can be easily justified by approximating the initial datum u_0 with a sequence in $H^2(\mathbb{R}^N)$ (see Kato [14]). We use the following notations: $\mathcal{L}u = (2i\partial_t + \triangle)u; F = F(u) = -|u|^{p-1}u; [A, B] = AB - BA; \mathfrak{d} = a(t,x)\partial_t + b(t,x)\cdot\nabla + c(t.x); \langle zf, g\rangle = \overline{z}\langle f, g\rangle, z \in \mathbb{C}$. Let $u(t)$ satisfy $\mathcal{L}u = F(u)$. Then we have (Ehrenfest's law)

$$2i\frac{d}{dt}\langle u, \mathfrak{d}u\rangle = \langle u, \mathfrak{d}F\rangle - \langle F, \mathfrak{d}u\rangle + \langle u, [\mathcal{L}, \mathfrak{d}]u\rangle. \tag{3.5}$$

Putting $\mathfrak{d} = \Psi + it\nabla$ (the generalized generator of Galilean transformations), we have (3.1). One can easily obtain (3.2) from (3.3) with $\mathfrak{d} = \Phi$. In order to obtain (3.3), we put $\mathfrak{d} = 2t\partial_t + \Psi\cdot\nabla + \frac{1}{2}\nabla\Phi$

(the generalized generator of space-time dilations). Combining (3.2) and (3.3), we obtain (3.4) with the help of the energy conservation law (1.2).

Now we turn to the proof of Theorem C. We argue by contradiction: We assume that the solution $u(t)$ of C$(1+\frac{4}{N})$ exists globaly in time in the space $C_b([0,\infty);H^1(\mathbb{R}^N))$, and $u(t)$ satisfies

$$M \equiv \sup_{t\in\mathbb{R}_+} \|u(t)\|_{H^1(\mathbb{R}^N)} < \infty, \tag{3.6}$$

$$E(u(t)) = E(u_0) \equiv -E_0 < 0. \tag{3.7}$$

We take any sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$, and put

$$u_n(t,x) \equiv u(t+t_n,x). \tag{3.8}$$

In what follows, we shall often extract subsequences without mentioning this fact. We note that $\{u_n\}$ is a bounded sequece in $L^\infty(\mathbb{R}_+;H^1(\mathbb{R}^N))$, and satisfies

$$\|u_n(t)\|_\sigma^\sigma > \frac{\sigma}{2}E_0, \tag{3.9}$$

$$E(u_n(t)) = -E_0 \tag{3.10}$$

for any $t \in [0,\infty)$ by (3.6). In the same way as in the proof of Theorem B, we iteratively use Proposition 2.1 to obtain

Lemma 3.2. *There exists*

(i) *a family of solutions of (NSC) in* $C_b(\mathbb{R}_+;H^1(\mathbb{R}^N))$: $\mathfrak{A}_1 = \{w_1^1, w_1^2, \cdots\}$, *and*

(ii) *a family of sequences in* $\mathbb{R}^N$: $\mathfrak{B}_1 = \{\{y_{1,n}^1\},\{y_{1,n}^2\},\cdots\}$

such that we have

$$\lim_{n\to\infty}\left|\sum_{k=2}^{j} y_{1,n}^k\right| = \infty \qquad (j \geqq 2), \tag{3.11}$$

and, for some subsequence (still denoted by the same letter), we have

$$w_{1,n}^1 \equiv u_n(t\,,\,\cdot + y_{1,n}^1) \to w_1^1 \not\equiv 0, \tag{3.12}$$

$$w_{1,n}^j \equiv \left(w_{1,n}^{j-1} - w_1^{j-1}\right)(t\,,\,\cdot + y_{1,n}^j) \to w_1^j \not\equiv 0 \quad (j \geqq 2), \tag{3.13}$$

weakly in* $L^\infty(\mathbb{R}_+;H^1(\mathbb{R}^N))$ *and strongly in* $C([0,T];L^q(\Omega))$ *for any* $T>0$, $\Omega \Subset \mathbb{R}^N$ *and* $\alpha \in [2,2^*)$, *and*

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\int_{\mathbb{R}^N}\left||w_{1,n}^j|^\alpha - |w_{1,n}^j - w_1^j|^\alpha - |w_1^j|^\alpha\right|dx = 0, \tag{3.14}$$

$$\lim_{n\to\infty}\int_0^T\left\{E(w_{1,n}^j) - E(w_{1,n}^j - w_1^j) - E(w_1^j)\right\}dt = 0, \tag{3.15}$$

and, for any $t \in \mathbb{R}_+$,

$$\lim_{n\to\infty} \left\{ E(w_{1,n}^j(t)) - E((w_{1,n}^j - w_1^j)(t)) - E(w_1^j(t)) \right\} = 0, \tag{3.16}$$

$$\lim_{n\to\infty} (\|w_{1,n}^1\|_\alpha^\alpha - \|(w_{1,n}^j - w_1^j)(t)\|_\alpha^\alpha) = \sum_{k=1}^{j} \|w_1^k(t)\|_\alpha^\alpha, \tag{3.17}$$

$$\lim_{n\to\infty} \left\{ E(w_{1,n}^1(t)) - E((w_{1,n}^j - w_1^j)(t)) \right\} = \sum_{k=1}^{j} E(w_1^k(t)). \tag{3.18}$$

Furthermore, we have: If $L \equiv \#\mathfrak{A}_1 < \infty$,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|(w_{1,n}^L - w_1^L)(t)\|_{L^\sigma(\mathbb{R}^N)} = 0, \tag{3.19}$$

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\{ \sup_{y\in\mathbb{R}^N} \int_{|x-y|<R} |(w_{1,n}^L - w_1^L)(t,x)|^2 dx \right\} = 0; \tag{3.20}$$

if $\#\mathfrak{A}_1 = \infty$,

$$\lim_{j\to\infty} \lim_{n\to\infty} \sup_{t\in[0,T]} \|(w_{1,n}^j - w_1^j)(t)\|_{L^\sigma(\mathbb{R}^N)} = 0 \tag{3.21}$$

$$\lim_{j\to\infty} \lim_{n\to\infty} \sup_{t\in[0,T]} \left\{ \sup_{y\in\mathbb{R}^N} \int_{|x-y|<R} |(w_{1,n}^j - w_1^j)(t,x)|^2 dx \right\} = 0, \tag{3.22}$$

for any $R > 0$.

From (3.18), there exists a number $k(1) \in \mathbb{N}$ such that $E(w_1^{k(1)}) < 0$. We put

$$u^1(t,x) \equiv w_1^{k(1)}(t,x), \tag{3.23}$$

so that we have

$$u^1 \in C_b([0,\infty); H^1(\mathbb{R}^N)) \tag{3.24}$$

$$2i\frac{\partial u^1}{\partial t} + \Delta u^1 + |u^1|^{\frac{4}{N}} u^1 = 0, \tag{3.25}$$

$$\|u^1(t)\| = \|u^1(0)\|, \quad -E_1 \equiv E(u^1(t)) < 0. \tag{3.26}$$

Next stage we consider

$$u_n^1(t\,,\,x) \equiv u^1(t + t_n\,,\,x). \tag{3.27}$$

Just as in the case of $\{u_n\}$, we obtain a family of solutions of (NSC) $\mathfrak{A}_2 = \{w_2^1, w_2^2, \cdots\}$. Among them we can find a function

$$u^2(t,x) \equiv w_2^{k(2)}(t,x) \quad \text{for some } k(2) \in \mathbb{N} \tag{3.28}$$

such that

$$u^2 \in C_b([0,\infty); H^1(\mathbb{R}^N)) \tag{3.29}$$

$$2i\frac{\partial u^2}{\partial t} + \triangle u^2 + |u^2|^{\frac{4}{N}} u^2 = 0, \tag{3.30}$$

$$\|u^2(t)\| = \|u^2(0)\|, \quad -E_2 \equiv E(u^2(t)) < 0. \tag{3.31}$$

Repeating the arguements above, we obtain a family $\{u^j\}$ of solutions of (NSC) such that

$$u^j \in C_b([0,\infty); H^1(\mathbb{R}^N)) \tag{3.32}$$

$$2i\frac{\partial u^j}{\partial t} + \triangle u^j + |u^j|^{\frac{4}{N}} u^j = 0, \tag{3.33}$$

$$\|u^j(t)\| = \|u^j(0)\|, \quad -E_j \equiv E(u^j(t)) < 0. \tag{3.34}$$

For this family of solutions of (NSC), we have

Lemma 3.3. *For each j, we introduce*

$$D_j = \sup_{R>0}\left(\liminf_{t\uparrow\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{|x-y|\leqq R}|u^j(t,x)|^2dx\right)\right). \tag{3.35}$$

Then we have

$$\|Q\| \leqq \|u^{j+1}(0)\|^2 \leqq D_j, \tag{3.36}$$

$$\|u^j(0)\|^2 - D_j \downarrow 0 \quad (j\uparrow L). \tag{3.37}$$

$$E(u^j) < -E^* \tag{3.38}$$

for some positive constant E^* for sufficiently large j.
Furthermore, if $\|u^l(0)\|^2 = D_l$ for some finite $l\in\mathbb{N}$, we have $\#\{u^j\} = l < \infty$.

One can prove this lemma by the construction of u^j's.

As in the proof of Theorem A (2) (see [24, 27]), we have, by the definition of D_j, that

Lemma 3.4. *We take j large enough to obtain*

$$\|u^j(0)\|^2 < 2D_j.$$

(This is possible by Lemma 3.3.) For any $\varepsilon\in(0,1)$, there is a constant $R_j>0$ such that there exists a Lipshitz continuous function $\gamma_j(\cdot)\in C([T_j,\infty);\mathbb{R}^N)$ such that

$$\int_{|x|\leqq R}|u^j(t,x+\gamma_j(t))|^2dx \geqq (1-\varepsilon/2)D_j, \quad t\in[T_j,\infty), \tag{3.36}$$

for any $R \geqq R_j$ and for some $T_j > 0$.

In what follows, by space-time translations, we always assume that

$$T_j = 0, \quad \text{and} \quad \gamma_j(0) = 0, \tag{3.37}$$

and we can also assume that

$$\Im\langle \nabla u^j(0), u^j(0)\rangle = 0 \tag{3.38}$$

by Galilei transformations: indeed; if not, taking $v = -\Im\langle \nabla u^j(0), u^j(0)\rangle \,/\, \|u^j(0)\|^2$ yields that

$$\Im\langle \nabla(u^j(0)e^{iv\cdot x}), u^j(0)e^{iv\cdot x}\rangle = 0$$

and

$$E\left(u^j(0)e^{iv\cdot x}\right) = E\left(u^j(0)\right) - \frac{\Im\langle \nabla u^j(0), u^j(0)\rangle^2}{\|u^j(0)\|^2} < E\left(u^j(0)\right). \tag{3.39}$$

To go further, we need the fine property of γ_j found in Lemma 3.4. Following Ogawa - Y.Tsutsumi [30,31], here we introduce a $W^{3,\infty}(\mathbb{R})$ odd function

$$\phi(r) = \begin{cases} r, & 0 \leqq r < 1, \\ r - (r-1)^3, & 1 \leqq r < 1 + \frac{1}{\sqrt{3}}, \\ \text{smooth}, \quad (\phi' \leqq 0) & 1 + \frac{1}{\sqrt{3}} \leqq r < 2, \\ 0, & 2 \leqq r, \end{cases} \tag{3.40}$$

and put, for $m > 0$,

$$\Psi_m(x) = \frac{x}{r}\phi_m(r) = \frac{x}{r} m\phi(\frac{r}{m}), \tag{3.41}$$

$$\Phi_m(x) = 2\int_0^r \phi_m(s)ds, \tag{3.42}$$

where $r = |x|$. Here we note that

$$\left|\frac{\partial^l}{\partial x_l}\phi_m(r)\right| \leqq \frac{K_l}{m^{l-1}}, \quad l = 0,1,2,3, \tag{3.43}$$

$$\nabla\Phi_m(x) = 2\frac{x}{r}\phi_m(r) = 2\Psi_m(x), \tag{3.44}$$

The following lemma shows that we can take $\gamma_j(t)$, for an appropriate time interval long enough, as the approximate center of mass $\gamma_j^{av}(t)$ defined by (3.50) in Lemma 3.5 below.

Lemma 3.5. *Let $\varepsilon > 0$ be arbitrary number such that $\varepsilon < 1/520$, and put*

$$\sqrt{\varepsilon_j} \equiv 2N(1 + C_\Psi)M\left(\frac{\varepsilon}{2} + \frac{\|u^j(0)\|^2 - D_j}{\|u^j(0)\|^2}\right)^{1/2}, \tag{3.45}$$

where C_Ψ is a positive constant defined by

$$C_\Psi \equiv \sup_{1\leqq k\leqq N}\sup_{x\in\mathbb{R}^N}|\nabla\Psi^k_{20R_j}(x)|(<\infty) \tag{3.46}$$

independent of j (see (3.41) and (3.43)). We take j large enough to obtain

$$\frac{\|u^j(0)\|^2 - D_j}{\|u^j(0)\|^2} < \frac{\varepsilon}{2}, \tag{3.48}$$

so that we have

$$\sqrt{\varepsilon_j} < 2N(1+C_\Psi)M\sqrt{\varepsilon}. \tag{3.49}$$

(This is possible by Lemma 3.3 and $\|u^j(0)\| \geqq \|Q\|$.) For u^j, we put

$$\gamma_j^{av}(t) \equiv \langle\Psi_{20R_j}, |u^j(t)|^2\rangle / \|u^j(0)\|^2, \tag{3.50}$$

where Ψ_{20R_j} is as in (3.39). Then, we have

$$\gamma_j^{av}(\cdot) \in C^1([0, 3R_j\|Q\|/\sqrt{\varepsilon_j}); \mathbb{R}^N), \tag{3.51}$$

and there are constants $R_j > 0$ such that

$$|\dot{\gamma}_j^{av}(t)| < \sqrt{\varepsilon_j}, \quad t \in [0, 3R_j\|Q\|/\sqrt{\varepsilon_j}), \tag{3.52}$$

$$\int_{|x|\leqq 4R_j} |u^j(t, x+\gamma_j^{av}(t))|^2 dx \geqq (1-\varepsilon/2)D_j, \quad t \in [0, 3R_j\|Q\|/\sqrt{\varepsilon_j}). \tag{3.53}$$

Proof. We shall show that γ_j^{av} to satisfy (3.52) and (3.53) for sufficiently large j such that we have $\varepsilon_j < N(1+C_\Psi)M\varepsilon$. From Lemma 3.4 we know that

$$\int_{|x|\leqq R_j} |u^j(t, x+\gamma_j(t))|^2 dx \geqq (1-\varepsilon/2)D_j, \quad t \in 0, \infty). \tag{3.54}$$

If $|\gamma_j(t)| \leqq 10R_j$, we have

$$\begin{aligned}
&\left|\left\langle\Psi_{R_j}(\cdot - \gamma_j(t)), |u^j(t)|^2\right\rangle + \gamma_j(t)\|u^j(0)\|^2 - \langle\Psi_{20R_j}, |u^j(t)|^2\rangle\right| \\
&= \int_{[|x-\gamma_j(t)|\leqq R_j]} + \int_{[|x-\gamma_j(t)|\geqq R_j]} \left(\Psi_{R_j}(\cdot - \gamma_j(t))|u^j(t)|^2 + \gamma_j(t)|u^j(0)|^2 - \Psi_{20R_j}|u^j(t)|^2\right) dx \\
&= \int_{[|x-\gamma_j(t)|\geqq R_j]} < (2R_j + 10R_j + 40R_j) \int_{[|x-\gamma_j(t)|\geqq R_j]} |u^j(t)|^2 dx \\
&\leqq 52R_j\left(\|u^j(0)\|^2 - \left(1-\frac{\varepsilon}{2}\right)D_j\right) \leqq 52R_j\left(\frac{\varepsilon}{2} + \frac{\|u^j(0)\|^2 - D_j}{\|u^j(0)\|^2}\right)\|u^j(0)\|^2 \\
&< \frac{1}{10}R_j\|u^j(0)\|^2.
\end{aligned} \tag{3.55}$$

We have used the fact that

$$\left\langle \Psi_m(\cdot - y), |u^j(t)|^2 \right\rangle / \|u^j(0)\|^2 < 2m \tag{3.56}$$

for any $y \in \mathbb{R}^N$ and $m > 0$. We also here note that $\gamma_j^{av}(t) \equiv \langle \Psi_{20R_j}, |u^j(t)|^2 \rangle / \|u^j(0)\|^2$ is almost the center of the mass, and $\left\langle \Psi_{R_j}(\cdot - \gamma_j(t)), |u^j(t)|^2 \right\rangle / \|u^j(0)\|^2$ is almost the center of the "cluster". By (3.55) and (3.56), "the center of the mass" $\gamma_j^{av}(t) \equiv \langle \Psi_{20R_j}, |u^j(t)|^2 \rangle / \|u^j(0)\|^2$ satisfies

$$|\gamma_j^{av}(t) - \gamma_j(t)| < 2R_j + \frac{1}{10}R_j < 3R_j, \tag{3.57}$$

$$\int_{|x| \leqq 4R_j} |u^j(t, x + \gamma_j^{av}(t))|^2 dx \geqq (1-\varepsilon)D_j \tag{3.58}$$

as far as $|\gamma_j(t)| \leqq 10R_j$. From (3.1) with $\Psi^k = \Psi^k_{20R_j}$, we have that $\gamma_j^{av} \in C^1([0,\infty); \mathbb{R}^N)$. On the other hand, we have from (1.3), (3.38) and (3.1) with $\Psi^k = \Psi^k_{20R_j}$ that

$$\begin{aligned}
|\dot{\gamma_j^{av}}(t)| &\leqq 2\sum_{k=1}^N \left| \langle \nabla \Psi^k_{20R_j} \cdot \nabla u^j(t)\,,\, u^j(t) \rangle \right| \Big/ \|u^j(0)\|^2 \\
&\leqq 2\sum_{k=1}^N \left| \int_{\mathbb{R}^N} \nabla \overline{u^j(t)} u^j(t) dx - \int_{\mathbb{R}^N} (1 - \nabla \Psi^k_{20R_j}) \cdot \nabla \overline{u^j(t)} u^j(t) dx \right| \Big/ \|u^j(0)\|^2 \\
&< 2(1 + C_\Psi) N \int_{[|x| \geqq 20R_j]} |\nabla u^j(t)| |u^j(t)| dx \Big/ \|u^j(0)\|^2 \\
&< 2N(1 + C_\Psi) M \left(\frac{\varepsilon}{2} + \frac{\|u^j(0)\|^2 - D_j}{\|u^j(0)\|^2} \right)^{1/2} \Big/ \|u^j(0)\| \\
&< \sqrt{\varepsilon_j} \,/\, \|Q\|
\end{aligned} \tag{3.59}$$

as far as $B(\gamma_j(t); R_j) \subset B(0; 20R_j)$. Now we set

$$t_j = \inf\{t > 0; |\gamma_j(t)| = 9R_j\}.$$

If $t \leqq t_j$, then we have

$$|\gamma_j^{av}(t) - \gamma_j(t)| < 3R_j, \quad |\dot{\gamma_j^{av}}(t)| < \sqrt{\varepsilon_j}/\|Q\|.$$

Hence we have from (6.50) and (6.59) that

$$t_j \geqq 3R_j \|Q\| / \sqrt{\varepsilon_j},$$

so that we have

$$|\gamma_j(t)| \leqq 9R_j, \quad t \in [0, 3R_j\|Q\|/\sqrt{\varepsilon_j}). \tag{3.60}$$

Thus we obtain

$$\int_{|x| \leqq 4R_j} |u^j(t, x + \gamma_j^{av}(t))|^2 dx \geqq (1 - \varepsilon/2) D_j, \quad t \in [0, 3R_j\|Q\|/\sqrt{\varepsilon_j}). \tag{3.61}$$

Proof of Theorem B concluded

Let $\varepsilon > 0$ be arbitrary such that

$$\max\left(\varepsilon, \frac{K_3\varepsilon\|u(0)\|^2}{800}, \frac{2}{\sigma}\Big((NK_1+2)+NK_1\Big)2C_N(\varepsilon\|u(0)\|^2)^{2/N}(\varepsilon\|u(0)\|^2+M)\right) < \min\left(\frac{1}{520}, \frac{E^*}{4}\right), \tag{3.62}$$

where $C_N = \frac{\sigma}{2\|Q\|^{4/N}}$ (see Theorem A). We take j large enough to obtain (3.48) and (3.49). By virtue of Lemma 3.5 and its proof, there exists a positive constant $R_j > \max(K_2/10, 1)$ (for convinience) such that we have that for any $R > 10R_j$

$$\int_{|x|\leqq R} |u^j(t,x)|^2 dx \geqq (1-\varepsilon/2)D_j, \quad t \in [0, 3R_j\|Q\|/\sqrt{\varepsilon_j}), \tag{3.63}$$

where K_2 is the positive constant appearing in (3.43).

The rest of the proof is a modification of the argument performed in Glassey [13] and Ogawa – Y. Tsutsumi. [30, 31]. We shall work with a modified "variance identity" (3.4) together with a suitable weight-function Φ. Let $\phi(r)$ be as in (3.40), and put, for $m > 0$,

$$\tilde{\Psi}_m^k(x) = \phi_m(x_k) = m\phi(\frac{x_k}{m}), \quad (k = 1, 2, \cdots, N), \tag{3.64}$$

$$\tilde{\Phi}_m(x) = 2\sum_{k=1}^{N}\int_0^{x_k}\phi_m(s)ds, \tag{3.65}$$

where $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$. Here we note that

$$\left|\frac{\partial^l}{\partial x_l}\phi_m(r)\right| \leqq \frac{K_l}{m^{l-1}}, \quad l = 0,1,2,3, \tag{3.43}$$

$$\nabla\tilde{\Phi}_m(x) = 2\tilde{\Psi}_m(x) = 2(\tilde{\Psi}_m^1(x_1), \tilde{\Psi}_m^2(x_2), \cdots, \tilde{\Psi}_m^N(x_N)). \tag{3.66}$$

we take Ψ and Φ in (3.4) with $p = 1 + \frac{4}{N}$ as follows:

$$\Psi(t,x) = \tilde{\Psi}_{20R_j}(x), \quad \Phi(t,x) = \tilde{\Phi}_{20R_j}(x).$$

Then we have

$$\begin{aligned}\langle\tilde{\Phi}_{20R_j}, |u^j(t)|^2\rangle &= \langle\tilde{\Phi}_{20R_j}, |u^j(0)|^2\rangle + 2\Im t\langle u^j(0), \tilde{\Psi}_{20R_j}\cdot\nabla u^j(0)\rangle + t^2E(u^j(0))\\ &+ \int_0^t ds\int_0^s d\tau\left\langle\frac{2}{\sigma}\Big(\nabla\cdot\tilde{\Psi}_{20R_j}+2\Big) - \nabla\cdot\tilde{\Psi}_{20R_j}, |u^j(t)|^{\sigma}\right\rangle\\ &- 2\Re\int_0^t ds\int_0^s d\tau\left\langle(\delta_{ik} - \partial_i\tilde{\Psi}_{20R_j}^k)\partial_i u^j(t), \partial_k u^j(t)\right\rangle\\ &+ \frac{1}{2}\int_0^t ds\int_0^s d\tau\left\langle\Delta(\nabla\cdot\tilde{\Psi}_{20R_j}), |u^j(t)|^2\right\rangle\\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.\end{aligned} \tag{3.67}$$

One can easily deduce from the first term I that

$$\mathrm{I} \leqq K_0\|u(0)\|^2(20R_j)^2 + 2K_0\|u(0)\|^2 M(20R_j)t - E^* t^2. \tag{3.68}$$

We recall from the proof of Lemma 3.5 that for $R > 10R_j$

$$\begin{aligned}\int_{[|x|>R]} |u^j(t)|^2 dx &\leqq \left(\|u^j(0)\|^2 - \left(1-\frac{\varepsilon}{2}\right) D_j\right) = \left(\frac{\varepsilon D_j}{2} + \|u^j(0)\|^2 - D_j\right) \\ &\leqq \left(\frac{\varepsilon}{2} + \frac{\|u^j(0)\|^2 - D_j}{\|u^j(0)\|^2}\right)\|u^j(0)\|^2 < \varepsilon\|u(0)\|^2 \end{aligned} \tag{3.69}$$

as long as $t \in [0, 3R_j\|Q\|/\sqrt{\varepsilon_j})$. Using this, we get

$$\begin{aligned}&\left\langle \frac{2}{\sigma}\left(\nabla\cdot\tilde{\Psi}_{20R_j} + 2\right) - \nabla\cdot\tilde{\Psi}_{20R_j}\,,\, |u^j(t)|^\sigma \right\rangle \\ &\leqq \int_{[|x|>20R_j]} \frac{2}{\sigma}\left(\left(|\nabla\cdot\tilde{\Psi}_{20R_j}| + 2\right) + |\nabla\cdot\tilde{\Psi}_{20R_j}|\right)|u^j(t,x)|^\sigma dx \\ &\leqq \frac{2}{\sigma}\left((NK_1+2) + NK_1\right)\int_{[|x|>20R_j]} |u^j(t,x)|^\sigma dx \\ &\leqq \frac{2}{\sigma}\left((NK_1+2)+NK_1\right) 2C_N(\varepsilon\|u(0)\|^2)^{2/N}(\varepsilon\|u(0)\|^2 + M). \end{aligned} \tag{3.70}$$

Here we have used that fact that for $v \in H^1(\mathbb{R}^N)$

$$\|v\|^\sigma_{L^\sigma([|x|>20R_j])} \leqq 2C_N\|v\|^{\frac{4}{N}}_{L^2([|x|>10R_j])}\|v\|^2_{H^1([|x|>10R_j])}, \tag{3.71}$$

which follows easily from $10R_j > K_2$ and

$$\|\chi v\|^\sigma_\sigma \leqq C_N\|\chi v\|^{\frac{4}{N}}\|\nabla(\chi v)\|^2,$$

where

$$\chi(x) = 1 - \frac{1}{N}\nabla\cdot\left(\frac{x}{r}\phi_{10R_j}(r)\right), \quad r = |x|.$$

Consequently we obtain

$$\mathrm{II} \leqq \frac{2}{\sigma}\left((NK_1+2)+NK_1\right)2C_N(\varepsilon\|u(0)\|^2)^{2/N}(\varepsilon\|u(0)\|^2+M)t^2 \leqq \frac{1}{4}E^*t^2. \tag{3.72}$$

We shall estimate the third term III. Noting that $1 - \partial_k\tilde{\Psi}^k_{20R_j}(x_k) \geqq 0$, we have

$$\begin{aligned}&-2\Re\left\langle (\delta_{ik} - \partial_i\tilde{\Psi}^k_{20R_j})\partial_i u^j(t)\,,\, \partial_k u^j(t)\right\rangle \\ &= -2\Re\int_{\mathbb{R}^N}\sum_{k=1}^N (1 - \partial_k\tilde{\Psi}^k_{20R_j}(x_k))|\partial_k u^j(t,x)|^2 dx \\ &\leqq 0. \end{aligned} \tag{3.73}$$

Hence we get

$$\mathrm{III} \leqq 0. \tag{3.74}$$

Making use of (3.69), one can easily obtain

$$\mathrm{IV} \leqq \frac{1}{2}\frac{K_3}{(20R_j)^2}\varepsilon\|u(0)\|^2t^2 < \frac{K_3\varepsilon\|u(0)\|^2}{800}t^2 < \frac{1}{4}E^*t^2. \tag{3.75}$$

Collecting the estimates (3.67), (3.68), (3.72), (3.74) and (6,75), we have

$$\langle\tilde{\Phi}_{20R_j}, |u^j(t)|^2\rangle \leqq K_0\|u(0)\|^2(20R_j)^2 + 2K_0\|u(0)\|^2M(20R_j)t - \frac{1}{2}E^*t^2. \tag{3.76}$$

Therefore, we can show that

$$\begin{aligned} &\langle\tilde{\Phi}_{20R_j}, |u^j(2R_j\|Q\|/\sqrt{\varepsilon_j})|^2\rangle \\ &\leqq R_j^2\left(400K_0\|u(0)\|^2 + 800K_0\|u(0)\|^2M\frac{1}{\sqrt{\varepsilon_j}} - 2E^*\frac{1}{\varepsilon_j}\right) \\ &< 0 \end{aligned} \tag{3.77}$$

for sufficiently small ε (and sufficiently large j). Thus, we reach a contradiction.

4. FURTHER RESULTS AND REMARKS

Our method can be applied to the following Hartree type equation with $N \geqq 3$:

$$2i\frac{\partial u}{\partial t} + \Delta u + (V * |u|^2)u = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, \tag{4.1}$$

where

$$V(x) = \frac{1}{|x|^2}, \tag{4.2}$$

and $*$ denotes the convolution in $\mathbb{R}^N$. It is worth while to note here that the equation (4.1) – (4.2) is also invariant under the pseudo-conformal transformations as well as (NSC). We have frequently used the symmetric property of (NSC) in both the proofs of Theorems B and C. Moreover our method works for the following system of nonlinear Schrödinger equation:

$$\begin{cases} 2i\dfrac{\partial u}{\partial t} + \Delta u + |v|^2u = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ 2i\dfrac{\partial v}{\partial t} + \Delta v + |u|^2v = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2. \end{cases} \tag{4.3}$$

This equation is closely related to the (BCS) theory of superconductivity (J. Bardeen, L. N. Cooper and J. R. Schrieffer. *Phys. Rev.*, **108** (1957)). The blow-up of solutions of (4.3) may correspond to the formation of "Cooper pairs" on the Fermi surface: note that the "singulaities" in Theorem B are described by *zero energy* and *zero momentum* solutions.

In [41], we discuss the inverse problem of Theorem B.

Acknowledgement. The author would like to thank professors B. Thompson and T. Cranny for their kind hospitality during his stay in Brisbane. The living expense of the author during this miniconference was partially supported by The Unit for Mathematical Analysis of The University of Queensland, The Centre for Mathematics and its Applications of the Australian National University and The Australian Mathematical Society, for which he is greatful. Finaly, it is a pleasure to thank professor N. Trudinger and his family for their kind and warm hospitality he received during his stay in Brisbane.

References

1. S.A. Akhmanov, A.P.Sukhorukov and R.V. Khokhlov, *Self-focusing and self-trapping of intence light beams in a nonlinear medium*, Soviet physics JETP **23** (1966), 1025–1033.
2. ———, *Self-focusing and diffraction of light in a nonlinear medium*, Soviet physics USPEKHI **93** (1968), 609–633.
3. H. Berestycki and P.L. Lions, *Nonlinear scalar field equations.* I, II, Archi. Rat. Mech. Anal. **82** (1983), 313–376.
4. H. Brézis and J.M. Coron, *Convergence of $H-$system or how to blow bubbles*, Archi. Rat. Mech. Anal. **89** (1985), 21–56.
5. H. Brézis and E.H. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Am. Math. Soc. **88** (1983), 485–489.
6. ———, *Minimum action solutions of some vector field equations*, Commun. Math. Phys. **96** (1983), 97–113.
7. T. Cazenave and F.B. Weissler, *The structure of solution to the pseudoconformaly invariaint nonlinear Schrödinger equations*, preprint.
8. X.-Y.Chen and H. Matano, *Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations*, J. Differential Equations **78** (1989), 160–190.
9. J. Fröhlich, E.H. Lieb and M. Loss, *Stability of coulomb systems with magnetic fields I, The one- electron atom*, Commun. Math. Phys. **104** (1986), 251–270.
10. Y. Giga and R.V. Kohn, *Nondegeneracy of blowup for semilinear heat equations*, Commun. Pure Appl. Math. **17** (1989), 845–884.
11. J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equations.* I, II, J. Funct. Anal. **32** (1979), 1–71.
12. ———, *The global Cauchy problem for the nonlinear Schrödinger equation revisited*, Ann. Inst. Henri Poincaré, Analyse Nonlinéaire. **2** (1985), 309–327.
13. R.T. Glassey, *On the blowing up solution to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. **18** (1979), 1794–1797.
14. T. Kato, *On nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré, Physique Theorique **46** (1987), 113–129.
15. B. LeMesurier, C. Papanicolau, C. Sulem and P.L. Sulem, *The focusing singularity of the nonlinear Schrödinger equation*, Direction in Partial Differential Equations (Crandall,M.G., Rabinowitz, P.H. and Turner,R.E., eds.), Academic Press, New York, 1987, pp. 159–201.
16. E.H. Lieb, *On the lowest eigenvalue of Laplacian for the intersection of two domains*, Invent. Math. **74** (1983), 441–448.
17. P.L. Lions, *The concentration compactness principle in the calculus of variations. The locally compact case.* I, II, Ann. Inst. Henri Poincaré, Analyse Nonlinéaire. **1** (1984), 109–145, 223–283.
18. ———, *The concentration compactness principle in the calculas of variations. The limit case.* I, II., Riv. Math. Iberoamericana **1** (1985), 45–121, 145–201.
19. F. Merle, *Limit of the solution of the nonlinear Schrödinger equation at the blow-up time*, J. Funct. Anal. **84** (1989), 201–214.
20. ———, *Construction of solutions with exactly k blow-up points for the Schrödinger equation with the critical power nonlinearity*, Commun. Math. Phys. **129** (1990), 223–240.
21. F. Merle and Y. Tsutsumi, *L^2 concentration of blow-up solutions for the nonlinear Schödinger equation with the critical power nonlinearity*, J. Differential Equations **84** (1990), 205–214.
22. K. Nagasaki and T. Suzuki, *Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities*, Asymptotic Analysis **3** (1990), 173–188.
23. H. Nawa, *"Mass concentration" phenomenon for the nonlinear Schrödinger equation with the critical power nonlinearity*, Funk. Ekva. **35** (1992), 1–18.
24. ———, *"Mass concentration" phenomenon for the nonlinear Schrödinger equation with the critical power nonlinearity.* II, Kodai Math. J. **13** (1990), 333–348.

25. ______, *Convergence theorems for the pseodo-conformally invariant nonlinear Schrödinger equation*, Proc. Japan Acad. **66** (A) (1990), 214–216; RIMS Kokyuroku **755** (1991), 73–92.
26. ______, *Formation of singularities in solutions of the nonlinaer Schrödinger equation*, Proc. Japan Acad. **67** (A) (1991), 29–34.
27. ______, *Asymptotic profiles of blow-up solutions of the nonlinear Schrödinger equation with critical power nonlinearity*, J. Math. Soc. Japan (to appear).
28. ______, *Asymptotic profiles of blow-up solutions of the nonlinear Schrödinger equation*, Singularities in fluids, plasmas and optics (R. E. Caflisch and G. C. Papanicolau, eds.), NATO ASI series, Kluwer, Dordrecht, 1993, pp. 221–253; its revised manuscript, T.I.T. preprint series, 1993.
29. H. Nawa and M. Tsutsumi, *On blow-up for the pseudo-conformally invariant nonlinear Schrödinger equation*, Funk. Ekva. **32** (1989), 417–428.
30. T .Ogawa and Y. Tsutsumi, *Blow-up of H^1-solution for the nonlinear Schrödinger equation*, J. Differential Equations **92** (1991), 317–330.
31. ______, *Blow-up of H^1-solution for the one dimensional nonlinear Schrödinger equation with critical power nonlinearity*, Proc. Amer. Math. Soc. **111** (1991), 487–496.
32. W. A. Strauss, *Existence of solitary waves in higher dimensions*, Commun. Math. Phys. **55** (1977), 149–162.
33. M. Struwe, *A global compactness result for elliptic boundary value problem involving limiting nonlinearities*, Math. Z. **187** (1983), 567–576.
34. S. Takakuwa, *Behavior of minimizing sequences for the Yamabe problem*, Tokyo Metropolitan University Mathematics Preprint Series **7** (1990).
35. Y. Tsutsumi, *Rate of L^2 concentration of blow-up solutions for the nonlinear Schödinger equation with critical power nonlinearity*, Nonlinear Anal. T.M.A. **15** (1990), 719–724.
36. M.I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates.*, Commun. Math. Phys. **87** (1983), 511–517.
37. ______, *On the structure and formation singularities in solutions to nonlinear dispersive evolution equations*, Commun. in Partial Diffrential Equations **11** (1986), 545–565.
38. ______, *The nonlinear Schrödinger equation - Singularity formation, Stability and dispersion*, The Connection between Infinite and Finite Dimensional Dynamical Systems, Contemporary Math.99, 1989, pp. 213–232.
39. K. Yajima, *Existence of solutions for Schrödinger evolution equations*, Commun. Math. Phys. **110** (1987), 415–426.
40. V.E. Zakharov and V.S. Synakh, *The nature of self-focusing singularity*, Sov. Phys. JETP **41** (1976), 441–448.
41. H. Nawa and T. Ozawa, in preparation.

Hayato NAWA

Department of Mathematics, Faculty of Science

Tokyo Institute of Technology

oh-okayama meguro, Tokyo 152

Japan

e–mail address: nawa@math.titech.ac.jp