Rotating Drops Trapped Between Parallel Planes

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ABSTRACT. We derive the existence of local minimizers of the functional $\mathcal{F}_\Omega(E)$ describing the energy of a liquid drop $E \subset \mathbb{R}^3$, trapped between two parallel hyperplanes and rotating with constant angular velocity $\sqrt{2\Omega}$, for small $\Omega > 0$.

The study of rotating drops is motivated by problems in astrophysics and physical chemistry. Many physicists and mathematicians have worked on related problems, including Newton, MacLaurin [23], Jacobi [20], Plateau [25], Poincaré [26], Darwin [12], Lord Rayleigh [27], Hölder [19], Appell [2], Lichtenstein [21], Lyttleton [22], Chandrasekhar [8,9], Auchmuty [5], Caffarelli and Friedman [7], Friedman and Turkington [17,18], Brown and Scriven [6].

The question investigated here will be the existence of rotating drops with free boundaries; precisely, of a drop situated between two parallel planes and rotating with constant angular velocity. Of particular interest is the stability of connected drops.

The methods used in this paper will be those introduced by De Giorgi [14,15] for the treatment of variational problems (compare also [16,24]), related to the notion of sets of finite perimeter.

A Lebesgue measurable set $E \subset \mathbb{R}^n$, with characteristic function χ_E, is said to have *finite perimeter* in A, $A \subset \mathbb{R}^n$ open, if the total variation of the vector valued measure $D\chi_E$ satisfies

$$\int_A |D\chi_E| = \sup\left\{\int_A \chi_E \operatorname{div} g(x)\,dx : g \in C_0^1(A,\mathbb{R}^n), |g(x)| \leq 1 \text{ for } x \in E\right\} < +\infty .$$

We denote by $\Pi_1 = \{x = (y,z) \in \mathbb{R}^2 \times \mathbb{R} : z = 0\}$, $\Pi_2 = \{x = (y,z) \in \mathbb{R}^2 \times \mathbb{R} : z = d\}$, two parallel planes of distance $d > 0$, and by $G = \{x = (y,z) \in \mathbb{R}^2 \times \mathbb{R} : 0 < z < d\}$ the domain between them. The mathematical model of the rotating drop will be to minimize its energy, which is the sum of surface tension, capillarity and rotational energy, and is described by the functional

$$\mathcal{F}_\Omega(E) = \int_G |D\chi_E| + \nu \sum_{i=1}^{2} \int_{\Pi_i} \chi_E^+ \, d\mathcal{H}^2 - \Omega \int_G |y|^2 \chi_E \, dz\, dy$$

where $\nu, \Omega \in \mathbb{R}$, $0 \le \nu < 1$, $0 \le \Omega$. By χ_E^+ we denote the trace of χ_E for $x \in \Pi_i$, $i = 1, 2$, (compare [15,16,24]). The class of admissible sets is chosen to be

$$\mathcal{C} = \left\{ E \subset G \text{ Lebesgue measurable } : \int_G |D\chi_E| < \infty,\ |E| = 1 \right.$$
$$\left. \text{and } \int_E y_i \, dx = 0,\ i = 1, 2 \right\},$$

that is, the sets E with prescribed volume and barycenter lying on the axis $(0, 0, z)$. The functional $\mathcal{F}_\Omega$ describes the energy of a liquid drop rotating at a constant angular velocity $\sqrt{2\Omega}$ around its own barycenter.

As the energy functional is unbounded from below, we shall only treat the question of the existence of local minimizers for $\mathcal{F}_\Omega$.

Let $G(R) = \{(y, z) \in G : |y| < R\}$ for any $R \in \mathbb{R}$. We call $E \in \mathcal{C}$ a *local minimizer* if there exists $R > 0$ such that

(i) $E \subset\subset G(R)$
(ii) $\mathcal{F}_\Omega(E) \le \mathcal{F}_\Omega(F)$ for all $F \in \mathcal{C}$, $F \subset G(R)$.

We define $\mathcal{C}_R = \{E \in \mathcal{C} : E \subset G(R)\}$.

The techniques are the same as those used by Albano and Gonzalez in [1]. In our case, the special difficulty arises from the "free boundary" of E in Π_i, due to the additional capillarity term in the functional.

Related results for rotating drops with obstacles are also obtained by Congedo, Emmer and Gonzalez [11], and Congedo [10] — here the obstacle is assumed to be a graph with a certain growth at infinity. Sturzenhecker [28] treats the cases of pendent rotating drops.

The main result we present is

Main Theorem. *There exists $\Omega_1 > 0$ such that for $0 < \Omega < \Omega_1$, the energy functional $\mathcal{F}_\Omega$ has a local minimizer.*

The complete proofs being given in [4], we summarise here the main ideas.

1. General existence results.

Using the standard compactness theorem for $BV(G(R))$-functions uniformly bounded in the $BV(G(R))$-norm (see [16]) and the lower semicontinuity of $\mathcal{F}_\Omega$ with respect to L^1-convergence, we obtain the following two results:

Theorem. *Let $R \in \mathbb{R}$, $R > 0$, be such that $|G(R)| > 1$. Then, for each $\Omega \ge 0$ there exists $E_\Omega \in \mathcal{C}_R$ minimizing $\mathcal{F}_\Omega$*

$$\mathcal{F}_\Omega(E_\Omega) = \inf\{\mathcal{F}_\Omega(F) : F \in \mathcal{C}_R\}.$$

Theorem. *For a sequence $\{\Omega_j\}_{j \in \mathbb{N}}$ with $\Omega_j \to 0$, as $j \to \infty$, we obtain*

$$E_{\Omega_j} \to E_0 \quad \text{in } L^1(G(R)),$$

where E_0 is $\mathcal{F}_0$-minimizing.

This allows the use of the author's results in [3], where a detailed discussion of the geometrical properties of E_0 is given.

2. A stability result for E_0.

In the case $\Omega = 0$ the minimizer E_0 is known to be an analytic, rotationally symmetric, periodic surface of constant mean curvature. Furthermore, it intersects Π_i at a constant angle γ, for which $\cos\gamma = \nu$, and consists of at most one period. In $\mathbb{R}^3$ the possible minimizers are classified: they are the Delaunay surfaces [13]. Using this, we know the possible shapes of drops to which E_{Ω_j} would converge for $\Omega_j \to 0$. We also prove that for "small" (related to the distance of the planes) volume $|E_0|$, the minimizer cannot be connected and have non-empty intersections with both Π_1 and Π_2. In this case, E_0 is part of a ball satisfying the volume and boundary (contact angle) conditions. (For more details see [3,4].)

3. Existence of local minimizers.

The proof consists of two steps. First we have:

Theorem. *Choose R large enough that $\frac{R}{2} > \max\left\{ \left(\frac{4}{3}\pi\right)^{\frac{1}{3}}, \left(\frac{8}{\pi d}\right)^{\frac{1}{2}} \right\}$. Then there exists $\Omega_0 > 0$ such that, for $0 < \Omega < \Omega_0$, there exists t, $\frac{R}{2} \le t \le \frac{3R}{4}$, with*

$$\int_{G \cap \{|y| = t\}} \chi_{E_\Omega} \, d\mathcal{H}^2 = 0 \, .$$

Intuitively, the drops E_{Ω_j} concentrate more and more in a neighbourhood of E_0, given the L^1-convergence for $\Omega_j \to 0$. Eventually, there will be some cylinder $\{x \in G : |y| = t\}$, which intersects E_{Ω_j} in at most a set of lower-dimension, for $0 < \Omega < \Omega_0$.

The final step is to show that E_Ω has no component outside this cylinder.

Theorem. *Choose R as large as above. Then there exists $\Omega_1 > 0$ such that, for $0 < \Omega < \Omega_1$ there exists t, $\frac{R}{2} \le t \le \frac{3R}{4}$, with*

$$\int_{G(R) \setminus G(t)} \chi_{E_\Omega}(x) \, dx = 0 \, .$$

The idea is to cut off any part outside $G(t)$, rescale what is left axially so as to restore the prescribed volume, and translate so that the barycentre once again lies on the z-axis. The resulting set is then shown to have lower energy. This completes the proof of the main theorem.

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