MOVING BOUNDARY PROBLEMS WITH NONLINEAR DIFFUSION

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1. INTRODUCTION

In keeping with the central topic of this mini–conference, I'll be discussing two moving boundary problems that originate in practical nonlinear diffusion models. In each case, the results have direct application to environmental hydrology.

The first moving boundary problem represents the absorption of water by the soil under a pond. In this case, the moving boundary is the free interface between the saturated and unsaturated zones. My exact solution to this problem is yet to appear in the literature [1].

In the second problem, the moving boundary arises indirectly. Here, we consider infiltration of water at constant rate into a porous stratum of finite depth, underlain by an impermeable barrier. The appropriate boundary value problem is highly nonlinear, in both the governing equation and the boundary conditions. For a particular choice of soil hydraulic properties, this nonlinear problem on a rigid domain may be transformed to a linear boundary value problem on a domain that shrinks linearly in time. This is how the moving boundary problem originates. Our exact solution to this problem appeared two years ago [2]. However, our method of solution has not yet been justified rigorously. We are able to throw some light on this problem here. Also, we are now able to solve the model of water infiltration into a finite soil column that is already partially wetted. Previously we assumed zero initial water content and avoided the more difficult situation of gravity–induced redistribution occurring in the early stages of infiltration.

In order to construct exact solutions, we have chosen special governing nonlinear parabolic transport equations that can be transformed to the linear diffusion equation. These linearisable models can be selected systematically by searching for Lie–Bäcklund symmetries. In the next section, I will list the known linearisable parabolic transport equations that may apply to hydrology.

2. GOVERNING FLOW EQUATIONS FOR UNSATURATED POROUS MEDIA

The macroscopic modelling approach is reviewed in the science citation classic by Philip [3]. In this approach, one neglects the microstructure of the medium and conceptually averages over regions which are large compared to a single grain or pore. Darcy's law [4] was observed experimentally before the Navier–Stokes equations were developed. The version that applies to unsaturated media is

$$\underset{\sim}{v} = -K(\theta)\underset{\sim}{\nabla}\Phi, \tag{2.1}$$

where $\underset{\sim}{v}$ is the volumetric water flux, K is the hydraulic conductivity, θ is the volumetric water content (volume of H_2O per total volume of soil) and Φ is the hydraulic pressure head. Early this century, Buckingham [5] began to regard Φ as the total potential energy per unit weight of water. Then

$$\Phi = \Psi(\theta) - z \tag{2.2}$$

where Ψ represents the soil–water interaction potential and $-z$ is the gravitational potential when z is positive downwards. Substituting (2.2) in (2.1), we obtain a constitutive law

$$\underset{\sim}{v} = K(\theta)\underset{\sim}{\hat{e}}_z - K(\theta)\frac{d\Psi(\theta)}{d\theta}\underset{\sim}{\nabla}\theta \tag{2.3}$$

(here, $\underset{\sim}{\hat{e}}_z$ is the downward unit vector),

which combined with the Equation of Continuity

$$\frac{\partial\theta}{\partial t} + \underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0 \tag{2.4}$$

gives the Richards equation for unsaturated flow

$$\frac{\partial\theta}{\partial t} = \underset{\sim}{\nabla} \cdot [D(\theta)\underset{\sim}{\nabla}\theta] - \frac{dK}{d\theta}\frac{\partial\theta}{\partial z}. \tag{2.5}$$

The nonlinear diffusivity is $D(\theta) = K(\theta)\dfrac{d\Psi}{d\theta}$.

In practice, Equation 2.5 is a highly nonlinear convection–diffusion equation. The diffusion term may be traced back to the soil–water interaction in (2.2) and the convection term may be traced back to gravity.

The Lie–Bäcklund symmetry method for selecting linearisable evolution equations has been described elsewhere [6]. With regard to Equation 2.5, we aim to find soil hydraulic models, represented by a pair of functions $(K(\theta), D(\theta))$, such that (2.5) can be transformed to the linear diffusion equation,

$$\frac{\partial\Theta}{\partial T} = \frac{\partial^2\Theta}{\partial X^2}. \tag{2.6}$$

The canonical Equation 2.6 has a hierarchy of n'th order one–parameter symmetry groups

$$\Theta^* = \Theta + s\Theta_n(X, T),$$

where Θ_n represents $\left(\dfrac{\partial}{\partial X}\right)^n \Theta(X, T)$.

This is the simplest example of a Lie–Bäcklund or generalized symmetry group, in which the new dependent variable depends on derivatives of order higher than one. Possession of a symmetry is an invariant property. Therefore, partial differential equations which can be transformed to a linear equation with constant coefficients, must themselves have Lie–Bäcklund symmetries of arbitrarily high order. Consequently, we search for equations of the type (2.5) that possess a continuous one–parameter symmetry, which in infinitesimal form is

$$\theta^* = \theta + \varepsilon L(t, \underset{\sim}{x}, \theta, \underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \ldots, \underset{\sim}{\theta}_n) + O(\varepsilon^2) \tag{2.7}$$

Here, $\underset{\sim}{\theta}_j$ represents the collection of j'th order spatial derivatives of Θ. Equation 2.7 is extended to higher order variables θ_j^* by prolongation [7]. However, unlike the first order contact transformations, this prolonged transformation cannot operate on a finite dimensional manifold [8].

Invariance of the governing Equation 2.5 under (2.7) implies a set of determining relations for L. These are linear partial differential equations. They have no solution unless the free hydraulic functions $(D(\theta), K(\theta))$ obey certain consistency relations. The first result in this area was provided by Bluman and Kumei [9] who showed that a nonlinear diffusion equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x}\right] \tag{2.8}$$

has a finite order Lie–Bäcklund symmetry if and only if $D^{-1/2}$ is a linear function of θ,

$$D = \frac{a}{(b-\theta)^2}. \tag{2.9}$$

Transformation of this nonlinear diffusion equation to the linear canonical form had already been achieved by Knight [10], by modifying a transformation of Storm [11].

A symmetry analysis of the Equation 2.5 with its convective term was carried out by Fokas and Yortsos [12]. This uncovered the linearisable model with $D(\theta)$ as in (2.9) and with

$$K(\theta) = \frac{\alpha}{2}(b-\theta)^{-1} + \beta(b-\theta) + \gamma \tag{2.10}$$

α, β and γ being arbitrary constants. Reduction of this model to its linear canonical form was achieved not only by Fokas and Yortsos but also by Rosen [13] and by Kingston and Rogers [14]. Burgers' equation is a better–known linearisable example of (2.5) but this, in fact, may be viewed as a limiting case of the Fokas–Yortsos–Rosen equation [15].

In scale–heterogeneous porous media, gravity–free diffusion is governed by an equation of the type

$$\frac{\partial\theta}{\partial t} = \lambda(x)\frac{\partial}{\partial x}\left[D(\theta)\frac{\partial\theta}{\partial x}\right] - \lambda'(x)E(\theta)\frac{\partial\theta}{\partial x} - \lambda''(x)\int D(\theta) + E(\theta)\ d\theta \tag{2.11}$$

Here, $\lambda(x)$ is the dilation or scale factor representing variable soil texture. A symmetry analysis [16] reveals that Equation 2.11 is linearisable when $D(\theta)$ is of the form 2.9, and

$$\begin{aligned} \Big(\lambda(x), E(\theta)\Big) &= \left([1+mx]^{\alpha}, \left[\frac{1}{\alpha} - \frac{3}{2}\right] D(\theta)\right) \quad (\alpha, m \text{ constant }) \\ \text{or } \Big(\lambda(x), E(\theta)\Big) &= \left(e^{mx}, -\frac{3}{2}D(\theta)\right). \end{aligned}$$

In order to represent distributed plant roots, we may incorporate a water sink term. In unpublished work still in progress, I have shown that a reaction–diffusion equation

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial\theta}{\partial x}\right] - R(\theta, x, t) \tag{2.12}$$

is linearisable if and only if $D(\theta)$ is of the form (2.9) and

$$R = (b-\theta)\frac{d}{dt}\ell n|f(t)| + a_1 f(t) \tag{2.13}$$

with a_1 constant and f a general function. A simple change of variables

$$\begin{aligned} x &= \frac{2}{m}\left[1 - e^{-my/2}\right] \\ \theta &= b + [\rho - b]e^{my/2} \end{aligned}$$

produces a linearisable equation

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial y}\left[\frac{a}{(b-\rho)^2}\frac{\partial\rho}{\partial y}\right] - \frac{m}{2}\frac{a}{(b-\rho)^2}\frac{\partial\rho}{\partial y} - Re^{-my/2} \tag{2.14}$$

which has a sink term decreasing with depth and a nonlinear convection term. So far, no practical use has been made of Equation 2.14.

All of the above results apply to one spatial dimension. The outlook for higher dimensions is grim. In two dimensions, no equation of the form (2.5) has a Lie–Bäcklund symmetry [17].

3. APPLICATION OF THE FOKAS–YORTSOS–ROSEN EQUATION TO UNSATURATED FLOW

The relations 2.9 and 2.10 are very useful for modelling the hydraulic functions of real unsaturated soils [15]. Constant–rate rainfall infiltration into a semi–infinite soil, can be modelled by the exactly solvable nonlinear boundary value problem

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[\frac{a}{(b-\theta)^2}\frac{\partial \theta}{\partial z}\right] - \left[\frac{\lambda/2}{(b-\theta)^2} - \gamma\right]\frac{\partial \theta}{\partial z} \tag{3.1}$$

$$\theta = \theta_i \qquad t = 0 \tag{3.2}$$

$$\frac{\lambda}{2}(b-\theta)^{-1} + \gamma(b-\theta) + \beta - \frac{a}{(b-\theta)^2}\frac{\partial \theta}{\partial z} = R \text{ at } z = 0 \tag{3.3}$$

$$\theta \to \theta_i \text{ as } z \to \infty, \tag{3.4}$$

as shown by Broadbridge and White [15] and independently by Sander et al [18]. In the rest of this paper, I will adopt the scaling conventions of Broadbridge and White [15]. The above governing flow equation is recast in the dimensionless form

$$\frac{\partial \Theta}{\partial t_*} = \frac{\partial}{\partial z_*}\left[\frac{c(c-1)}{(c-\Theta)^2}\frac{\partial \Theta}{\partial z_*}\right] - \frac{d}{d\Theta}\left[\frac{(c-1)\Theta^2}{(c-\Theta)}\right]\frac{\partial \Theta}{\partial z_*} \tag{3.5}$$

where Θ is the normalized water content,

$$\Theta = \frac{\theta - \theta_n}{\theta_s - \theta_n}$$
$$c = \frac{b - \theta_n}{\theta_s - \theta_n}$$
$$z_* = z/\lambda_s \text{ and } t_* = t/t_s.$$

Here, θ_s is the water content at saturation and θ_n is the water content at which the model conductivity (2.10) is a minimum, $K'(\theta_n) = 0$. The length scale λ_s is a sorptive length scale, a natural scale for capillary rise, and the time scale is Philip's gravity time scale [3]. Although λ_s and t_s were not originally defined with this in mind, they can be determined from the simple relations

$$\begin{aligned} \lambda_s^2 t_s^{-1} \text{ (diffusivity scale) } &= \bar{D} \text{ (mean diffusivity)} \\ \text{and} \quad \lambda_s t_s^{-1} \text{ (velocity scale) } &= K_s/(\theta_s - \theta_n) \end{aligned}$$

where $K_s = K(\theta_s)$, the conductivity at saturation. This velocity scale U is the speed of the large–time travelling wave $\theta = f(z - Ut)$ when the surface is kept saturated [19]. Equation 3.5 is highly nonlinear when c is close to 1 and weakly nonlinear when c is large, reducing to Burgers' equation in the limit $c \to \infty$.

The initial and boundary conditions (3.2) to (3.4) become

$$\Theta = \Theta_i \qquad t_* = 0. \tag{3.6}$$

$$\begin{aligned} \frac{(c-1)\Theta^2}{(c-\Theta)} - \frac{c(c-1)}{(c-\Theta)^2}\frac{\partial\Theta}{\partial z_*} &= R_* &\qquad z_* = 0 \\ &= \frac{-K_n}{K_s - K_n} &\qquad z_* = \ell_* \end{aligned} \tag{3.7}$$

where $R_* = (R - K_n)/(K_s - K_n)$, and $K_n = K(\theta_n)$.

For the remainder of this paper, I assume $K_n = 0$. This is not a bad approximation since in practice $K(\theta)$ is strongly increasing so that if θ_n is much less than θ_s, then K_n will be negligible compared to K_s. Note that I am allowing the initial moisture content θ_i to exceed θ_n. The initial conductivity $K(\theta_i)$ may be greater than zero and the zero value of $K(\theta_n)$ is of little consequence.

4. ABSORPTION OF WATER UNDER A POND

Consider water being supplied, under positive pressure head Ψ_0, at the surface of an initially dry semi-infinite soil. The positive pressure may be due to a pond of depth Ψ_0. The zone of positive pressure extends to some depth $\ell(t)$ within the saturated zone. In the deeper unsaturated zone, the water content is less than its maximum value, $\theta < \theta_s$ and Ψ is negative. In some hydrological models, there is a tension- saturated zone, in which $\Psi < 0$ and $\theta = \theta_s$. This will be ignored here but it would be a simple matter to incorporate it in the following analysis, by assigning some negative value Ψ_s, to the saturated/unsaturated interface.

From the general argument of Philip [20], the main effect of the positive pressure head Ψ_0 is to be seen at early times, when gravity is negligible. Therefore, in order to investigate this main effect, we may neglect gravity in the governing flow equations. In terms of dimensionless variables, the appropriate free boundary problem is

$$-\frac{\partial \Psi_*}{\partial z_*} = \frac{\Psi_{0*}}{\ell_*(t_*)} \qquad 0 < z_* < \ell_*(t_*) \tag{4.1}$$

$$\Theta = 1 \qquad z_* = \ell_*(t_*) \tag{4.2}$$

$$\frac{-c(c-1)}{(c-\Theta)^2}\frac{\partial \Theta}{\partial z_*} \longrightarrow \frac{\Psi_{0*}}{\ell_*(t_*)} \quad as \;\; z_* \to \ell_*(t_*)^+ \tag{4.3}$$

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial z_*}\left[\frac{c(c-1)}{(c-\Theta)^2}\frac{\partial \Theta}{\partial z_*}\right] \qquad z_* > \ell_* \tag{4.4}$$

$$\Theta = \Theta_i \qquad t_* = 0 \tag{4.5}$$

$$\Theta \longrightarrow \Theta_i \qquad z_* \to \infty$$

$$\ell_*(0) = 0 \tag{4.6}$$

Here, $\Psi_* = \Psi/\lambda_s$.

Equation 4.1 expresses the fact that the one-dimensional flux must be uniform throughout the saturated zone, where concentration is constant. Statement 4.3 is an expression of the continuity of flux at the free interface $z_* = \ell_*(t_*)$.

Now it is convenient to define a new coordinate $y_* = z_* - \ell_*(t_*)$, whose origin is at the moving interface. The unknown function $\ell_*(t_*)$ will then appear in the governing partial differential equation rather than in a boundary condition. This results in a boundary value problem

$$\Theta = 1 \qquad y_* = 0 \tag{4.7}$$

$$\frac{-c(c-1)}{(c-\Theta)^2}\frac{\partial\Theta}{\partial y_*} \to \frac{\Psi_{0*}}{\ell_*(t_*)} \text{ as } y_* \to 0^+ \tag{4.8}$$

$$\frac{\partial\Theta}{\partial t_*} = \frac{\partial}{\partial y_*}\left[\frac{c(c-1)}{(c-\Theta)^2}\frac{\partial\Theta}{\partial y_*}\right] + \dot{\ell}_*(t_*)\frac{\partial\Theta}{\partial y_*} \tag{4.9}$$

$$\Theta = \Theta_i \qquad t_* = 0 \tag{4.10}$$

$$\Theta \to \Theta_i \qquad y_* \to \infty \tag{4.11}$$

$$\ell_*(0) = 0 \tag{4.12}$$

The above problem is amenable to a symmetry analysis. Provided

$$\ell_*(t_*) = m_* t_*^{\frac{1}{2}} \tag{4.13}$$

with $m_* = m/\lambda_s$ (constant), there is a consistent similarity reduction $\Theta = f(\phi)$, where ϕ is the Boltzmann similarity invariant, $\phi = y_* t_*^{-\frac{1}{2}}$.

From Equations 4.1 and 4.13, the flux of water entering through the saturated zone is

$$v(0,t) = \frac{K_s \Psi_0}{m} t^{-\frac{1}{2}}$$

Hence, the cumulative amount of water having been absorbed is

$$\begin{aligned} i(t) &= \int_0^t v(0,t)dt \\ &= St^{\frac{1}{2}} \end{aligned}$$

$$\text{where} \quad S = \frac{2K_s \Psi_0}{m} \tag{4.14}$$

S is known as the *sorptivity* [21] a quantity that can be directly measured.

As in the unponded ($m = 0$) case [10], Equation 4.9 may be linearised by applying consecutively the Kirchhoff transformation [22]

$$\mu = \frac{c(c-1)}{c - \Theta} \tag{4.15}$$

and the Storm transformation [23]

$$\chi = \; [c(c-1)]^{\frac{1}{2}} \int_0^{y_*} \mu^{-1} dy_* \;\; ; \tau = t. \tag{4.16}$$

In terms of the new variables, the left hand side of the governing Equation 4.9 is

$$\begin{aligned} \frac{\partial \Theta}{\partial t_*} &= c(c-1)\mu^{-2}\frac{\partial \mu}{\partial \tau} - c(c-1)\mu^{-3}\left[\frac{\partial \mu}{\partial \chi}\right]^2 \\ &- \frac{1}{2}[c(c-1)]^{\frac{1}{2}}(c-1)m_*\tau^{-\frac{1}{2}}\mu^{-2}\frac{\partial \mu}{\partial \chi} \\ &+ \frac{1}{2}[c(c-1)]^{\frac{3}{2}}m_*\tau^{-\frac{1}{2}}\mu^{-3}\frac{\partial \mu}{\partial \chi} \\ &- [c(c-1)]^{\frac{1}{2}}\mu^{-2}\frac{\partial \mu}{\partial \chi}\frac{\Psi_{0*}}{\ell_*} \end{aligned} \tag{4.17}$$

whereas the right hand side is

$$\begin{aligned} &\frac{\partial}{\partial y_*}\left[\frac{c(c-1)}{(c-\Theta)^2}\frac{\partial \Theta}{\partial y_*}\right] + \frac{1}{2}m_* t_*^{-\frac{1}{2}}\frac{\partial \Theta}{\partial y_*} \\ &= c(c-1)\mu^{-2}\frac{\partial^2 \mu}{\partial \chi^2} - c(c-1)\mu^{-3}\left(\frac{\partial \mu}{\partial \chi}\right)^2 \\ &+ \frac{1}{2}m_*\tau^{-\frac{1}{2}}[c(c-1)]^{\frac{3}{2}}\mu^{-3}\frac{\partial \mu}{\partial \chi} \end{aligned} \tag{4.18}$$

After equating 4.17 and 4.18, it is the cencellation of the nonlinear $\mu^{-3}\left(\frac{\partial\mu}{\partial\chi}\right)^2$ terms that enables the governing flow equation to be transformed to a linear canonical form when $m_* = 0$ [10]. Even for ponded absorption ($m_* \neq 0$), the extra nonlinear term $\frac{1}{2}m_*\tau^{-\frac{1}{2}}[c(c-1)]^{\frac{3}{2}}\mu^{-3}\frac{\partial\mu}{\partial\chi}$ cancels when 4.17 and 4.18 are equated. By equating 4.17 and 4.18 and then multiplying throughout by $[c(c-1)]^{-1}\mu^2$, we are left with a linear equation

$$\frac{\partial\mu}{\partial\tau} - \frac{1}{2}\gamma\tau^{-\frac{1}{2}}\frac{\partial\mu}{\partial\chi} - \frac{\partial^2\mu}{\partial\chi^2} = 0 \tag{4.19}$$

where

$$\gamma = [c(c-1)]^{-\frac{1}{2}}[(c-1)m_* + 2\frac{\Psi_{0*}}{m_*}]$$

The initial and boundary conditions are

$$\mu = c-1, \quad \tau = 0 \tag{4.20}$$

$$\mu \to c-1, \qquad \chi \to \infty \tag{4.21}$$

$$-[c(c-1)]^{\frac{1}{2}}\mu^{-1}\frac{\partial\mu}{\partial\chi} \to \frac{1}{2}S_*\tau^{-\frac{1}{2}}, \quad x \to 0 \tag{4.22}$$

$$\mu \to c, \quad x \to 0 \tag{4.23}$$

In (4.22), the dimensionless sorptivity is

$$S_* = \left[\frac{c(c-1)}{a}\right]^{\frac{1}{2}} S$$

The similarity solution, a function of $\phi = \chi\tau^{-\frac{1}{2}}$, is

$$\mu = g(\phi)$$

$$= (c-1) - A\,\mathrm{erfc}\left(\frac{\phi+\gamma}{2}\right) \tag{4.24}$$

$$\text{where } A = \frac{\frac{1}{2}(c-1)\pi^{\frac{1}{2}}h^{-\frac{1}{2}}}{\frac{1}{2}\pi^{\frac{1}{2}}h^{-\frac{1}{2}}\mathrm{erfc}(\frac{1}{2}\gamma) - \exp(-\frac{1}{4}\gamma^2)} \tag{4.25}$$

with h the solution of

$$\frac{1}{c} = \frac{1}{2}\pi^{\frac{1}{2}}h^{-\frac{1}{2}}\exp([\frac{1}{2}h^{-\frac{1}{2}} + h^{\frac{1}{2}}\frac{\Psi_{0*}}{c}]^2)\mathrm{erfc}(\frac{1}{2}h^{-\frac{1}{2}} + h^{\frac{1}{2}}\frac{\Psi_{0*}}{c}) \tag{4.26}$$

Equation 4.26 is very close to the trascendental equation that determines the position of the phase boundary in the classical Stefan problem (see e.g. [24]). Given the values of c and Ψ_{0*}, Equation 4.26 may be readily inverted numerically to obtain h. With h viewed as a function of Ψ_{0*} and c, we have

$$S = a^{\frac{1}{2}}/\sqrt{h(\Psi_{0*}, c)} \tag{4.27}$$

The length and time scales, introduced in Section 3, may be expressed [15]

$$\lambda_s = h(0,c)S_0^2/c(c-1)(\theta_s - \theta_n)K_s \tag{4.28}$$

and

$$t_s = h(0,c)S_0^2/c(c-1)K_s^2 \tag{4.29}$$

Therefore, from Equations 4.14 and 4.27, we obtain

$$m_* = 2\Psi_{0*}\sqrt{h(\Psi_{0*}, c)}/c(c-1) \tag{4.30}$$

Given μ from Equation 4.24, the rescaled water content is

$$\Theta = c[1 - (c-1)/\mu(\chi, \tau)] \tag{4.31}$$

Inverting the Storm transformation 4.16, the depth is

$$\begin{aligned} y_* &= [c(c-1)]^{-\frac{1}{2}}\int_0^x \mu(\chi,\tau)d\chi \\ &= [c(c-1)]^{-\frac{1}{2}}\{(c-1)\phi + 2A\pi^{-\frac{1}{2}}\exp(-\frac{1}{4}[\phi+\gamma]^2) \\ &- 2A\pi^{-\frac{1}{2}}\exp(-\frac{1}{4}\gamma^2) - A(\phi+\gamma)\mathrm{erfc}\left(\frac{\phi+\gamma}{2}\right) \\ &+ A\gamma\mathrm{erfc}(\frac{1}{2}\gamma)\} \end{aligned} \tag{4.32}$$

Equations 4.31 and 4.32 constitute an exact parametric solution. In Figure 1, we plot normalized water content $(\theta - \theta_i)/(\theta_s - \theta_i)$ against reduced depth $-x_* t_*^{-\frac{1}{2}}$ for a highly nonlinear soil $(c = 1.05)$ and for dimensionless pond depths $\Psi_{0*} = 0, 2$ and 8.

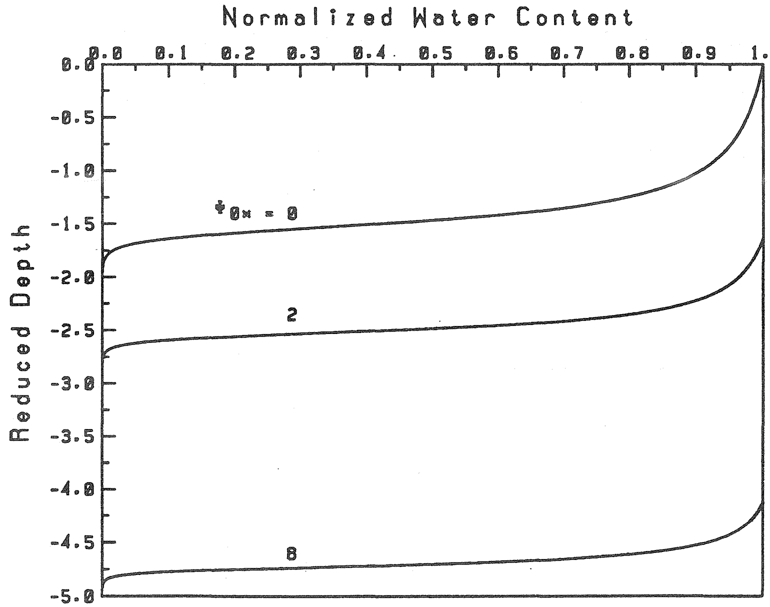

Figure 1.

Analytic solution of soil water absorption under a pond, for the nonlinear soil with $c = 1.05$ and with dimensionless pond depths $0, 2$ and 8.

5. INFILTRATION INTO A FINITE SOIL COLUMN

We consider one-dimensional unsaturated flow into a finite layer of soil, underlain by an impermeable rock layer. The boundary value problem to be solved is

$$\frac{\partial \Theta}{\partial t_*} = \frac{\partial}{\partial z_*}\left[D_*(\Theta)\frac{\partial \Theta}{\partial z_*}\right] - \frac{d}{d\Theta}K_*(\Theta)\frac{\partial \Theta}{\partial z_*} \tag{5.1}$$

$$\Theta = \Theta_i, \ t_* = 0 \tag{5.2}$$

$$K_*(\Theta) - D_*(\Theta)\frac{\partial \Theta}{\partial z_*} = R_*, \quad z_* = 0 \tag{5.3}$$

$$K_*(\Theta) - D_*(\Theta)\frac{\partial \Theta}{\partial z_*} = 0, \quad z_* = \ell_* \tag{5.4}$$

In this particular problem, we are interested in the sudden change in the water content profile when a wetting front reaches the lower boundary. By the time it takes a wetting front to traverse a macroscopic later, gravity should become a significant driving force, relative to capillarity. Hence, the governing Equation (5.1) includes a nonlinear convection term, due to gravity. The uniform initial water content Θ_i is not necessarily zero, so that there may be an initial non-zero flux $K_*(\Theta)$, due to gravity. The boundary condition 5.3 denotes uniform flux $v = R$ at the surface $z = 0$. The dimensionless flux R_* is R/K_s, where K_s is the soil hydraulic conductivity at saturation. The boundary condition 5.4 specifies zero flux at the impervious basement $z_* = \ell_*$.

As in Section 4, we apply the Kirchhoff transformation 4.15 and the Storm transformation 4.16. The governing Equation 5.1 becomes

$$\frac{\partial \mu}{\partial \tau} = \frac{\partial^2 \mu}{\partial \chi^2} - m^{\frac{1}{2}}[1 - 2\rho(1 + 1/\rho) + \mu/(c-1)]\frac{\partial \mu}{\partial \chi} \tag{5.5}$$

where $m = 4c(c-1)$ and $\rho = R_*/m$. The extra nonlinearity in Equation 5.5 originates from the nonlinear convection term in 5.1. However, this model convection term has been chosen so that the transformed Equation 5.5 is essentially Burgers' equation, which can be linearised by a further transformation. Now we apply the Hopf-Cole transformation [25, 26]

$$1 - 2\rho(1 + 1/\rho) + \mu/(c-1) = -2u^{-1}\frac{\partial u}{\partial \zeta} \tag{5.6}$$

with $\zeta = m^{\frac{1}{2}}\chi$. In order for Equation 5.5 to be satisfied, it is sufficient that $u(\zeta, \tau)$ satisfies the linear diffusion equation,

$$\frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial \zeta^2} \tag{5.7}$$

with $T = m\tau$.

The initial and boundary conditions transform to

$$u = \exp(Q\zeta), \quad T = 0 \tag{5.8}$$

$$u = \exp(\rho[\rho+1]T), \quad \zeta = 0 \tag{5.9}$$

$$u = \exp(2Q[c-\Theta_i]\ell_*)\exp(-\rho^2 T), \quad \zeta = 2(c-\Theta_i)\ell_* - 2\rho T, \tag{5.10}$$

where $Q = \frac{1}{2} + \rho - \frac{1}{2}\frac{c}{c-\Theta_i}$.

Now we face a linear boundary value problem on a domain that shrinks linearly in time. Curiously, King [27] found an efficient method for solving a similar problem using Laplace transforms in the T variable, even though T is bounded by the time at which the spatial domain shrinks to nothing. Applying the Laplace transform to Equations 5.7 to 5.9, we obtain

$$u(\zeta, T) \to \tilde{u}(\zeta, p)$$

$$\tilde{u} = Ae^{-p^{\frac{1}{2}}\zeta} + Be^{p^{\frac{1}{2}}\zeta} + \frac{e^{Q\zeta}}{p - Q^2} \tag{5.11}$$

with

$$B = -A + \frac{1}{p - \rho(\rho+1)} - \frac{1}{p - Q^2} \tag{5.12}$$

Following King [27], in order to implement the lower boundary condition 5.10, we define a moving coordinate

$$\xi = \zeta + 2\rho T \tag{5.13}$$

$$v(\xi, T) = u(\zeta, T)$$

so that ξ is fixed at the lower boundary. The lower boundary condition is

$$v = \exp(2Q[c - \Theta_i]\ell_*)\exp(-\rho^2 T), \ \xi = 2[c - \Theta_i]\ell_* \tag{5.14}$$

Equation 5.7 transforms to

$$\frac{\partial v}{\partial T} + 2\rho\frac{\partial v}{\partial \xi} - \frac{\partial^2 v}{\partial \xi^2} = 0.$$

After applying the Laplace transform $v(\xi, T) \to \tilde{v}(\xi, p)$, the general solution is

$$\begin{aligned}\tilde{v} &= E\exp(\rho + \sqrt{\rho^2 + p})\xi + F\exp(\rho - \sqrt{\rho^2 + p})\xi \\ &+ \frac{e^{Q\xi}}{p - Q(Q - 2\rho)}\end{aligned} \tag{5.15}$$

King's theorem on Laplace transform boosts [27] allows us to compare Laplace transforms $\tilde{u}$ and $\tilde{v}$ obtained in different reference frames and to incorporate all boundary conditions in the same transform. The Laplace transform boost yields

$$\begin{aligned}\tilde{v}(\xi, p) &= \exp(2\rho\frac{\partial}{\partial \xi}\frac{\partial}{\partial p})\Big\{A(\sqrt{p})\exp(-\xi\sqrt{p}) + B(\sqrt{p})\exp(\xi\sqrt{p}) + \frac{e^{Q\xi}}{P - Q^2}\Big\} \\ &= \left[1 - \frac{\rho}{\sqrt{\rho^2 + p}}\right]A\Big(-\rho + \sqrt{\rho^2 + p})\exp(-\xi[-\rho + \sqrt{\rho^2 + p}]\Big) \\ &+ \left[1 + \frac{\rho}{\sqrt{\rho^2 + p}}\right]B\Big(\rho + \sqrt{\rho^2 + p})\exp(\xi[\rho + \sqrt{\rho^2 + p}]\Big) \\ &+ \frac{e^{Q\xi}}{p + 2Q\rho - Q^2}\end{aligned} \tag{5.16}$$

By comparing Equations 5.15 and 5.16, the coefficients E and F may be related to $A(\sqrt{p})$ and $B(\sqrt{p})$ (here viewed as functions of $\sqrt{p}$).

After defining $\nu = \sqrt{p_1} = \rho + \sqrt{\rho^2 + p}$ and $H(\nu) = \sqrt{p_1}A(\sqrt{p_1})$, the full set of boundary conditions implies a difference equation

$$\begin{aligned}&H(\nu) - H(\nu - s\rho)\exp(-4[c - \Theta_i]\ell_*[\nu - \rho]) \\ &= \sum_{j=1}^{4} S_j\frac{1}{\nu - b_j} + \sum_{i=1}^{3} w_i\frac{\exp(-2[c - \Theta_i]\ell_*\nu)}{\nu - a_i},\end{aligned} \tag{5.17}$$

where
$$s_j = \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$$
$$b_j = \sqrt{\rho(\rho+1)}, -\sqrt{\rho(\rho+1)}, Q, -Q \text{ for } j = 1,2,3,4$$

and
$$\omega_1 = -\exp(2[c-\Theta_i]\ell_* Q)$$
$$\omega_2 = \omega_3 = -\frac{1}{2}\omega_1$$
$$a_i = \rho, 2\rho - Q, Q \text{ for } i = 1,2,3$$

The exponential terms on the right hand side of (5.17) are due entirely to the non-zero initial water content Θ_i, ignored in the earlier analysis of Broadbridge *et al.* [2]. To find a particular solution to Equation (5.17), we first recast it in the form

$$LH(\nu) = \sum_{j=1}^{4} s_j \frac{1}{\nu - b_j} + \sum_{i=1}^{3} \omega_i \frac{\exp(-2[c-\Theta_i]\ell_*\nu)}{\nu - a_i} \tag{5.18}$$

where

$$L = 1 - \exp(-4[c-\Theta_i]\ell_*[\nu - \mathcal{R}])\exp(-2\mathcal{R}\frac{\partial)}{\partial\nu}$$

The operator L is not invertible, as it has an infinite-dimensional kernel [28]. However, a formal geometric series inversion gives a large-ν asymptotic expansion for a useful solution $H(\nu)$ [M. King, personal communication]. The result is

$$\begin{aligned}\tilde{u}(\zeta,p) = p^{-\frac{1}{2}}\left\{e^{-p^{\frac{1}{2}}\zeta} - e^{p^{\frac{1}{2}}\zeta}\right\}&\left\{\sum_{j=1}^{4} s_j \sum_{n=0}^{\infty} \frac{\exp(-4n[c-\Theta_i]\ell_*[p^{\frac{1}{2}} - n\rho])}{p^{\frac{1}{2}} - 2n\rho - b_j}\right.\\ &\left. + \sum_{i=1}^{3}\omega_i \sum_{n=0}^{\infty} \frac{\exp(-2[c-\Theta_i]\ell_*[(2n+1)p^{\frac{1}{2}} - 2n(n+1)\rho])}{p^{\frac{1}{2}} - 2n\rho - a_i}\right\}\\ &+ \frac{1}{p-\rho(\rho+1)}e^{p^{\frac{1}{2}}\zeta} - \frac{1}{p-Q^2}e^{p^{\frac{1}{2}}\zeta} + \frac{1}{p-Q^2}e^{Q\zeta}\end{aligned} \tag{5.19}$$

Although this is a large-p asymptotic expansion rather than a convergent series, the term by term Laplace inversion yields a series that converges at all physically meaningful times:

$$
\begin{aligned}
u = {} & e^{Q\zeta+Q^2T} + \frac{1}{2}e^{-\zeta^2/4T} f\left(\frac{\zeta}{2\sqrt{T}} - \sqrt{\rho(\rho+1)T}\right) \\
& + \frac{1}{2}e^{-\zeta^2/4T} f\left(\frac{\zeta}{2\sqrt{T}} + \sqrt{\rho(\rho+1)T}\right) \\
& - \frac{1}{2}e^{-\zeta^2/4T} f\left(\frac{\zeta}{2\sqrt{T}} - Q\sqrt{T}\right) - \frac{1}{2}e^{-\zeta^2/4T} f\left(\frac{\zeta}{2\sqrt{T}} + Q\sqrt{T}\right) \\
& + \sum_{n=1}^{\infty}\sum_{j=1}^{4} S_j \left\{\exp(4n^2[c-\Theta_i]\ell_*\rho - [4n[c-\Theta_i]\ell_* + \zeta]^2/4T \right. \\
& \times f\left(\frac{4n[c-\Theta_i]\ell_* + \zeta}{2\sqrt{T}} - (2n\rho + b_j)\sqrt{T}\right) \\
& - \exp\left(4n^2[c-\Theta_i]\ell_*\rho - [4n[c-\Theta_i]\ell_* - \zeta]^2/4T\right) \\
& \left. \times f\left(\frac{4n[c-\Theta_i]\ell_* - \zeta}{2\sqrt{T}} - (2n\rho + b_j)\sqrt{T}\right) \right\} \\
& + \sum_{n=0}^{\infty}\sum_{i=1}^{3} w_i \left\{\exp([4n^2+4n][c-\Theta_i]\ell_*\rho - [[4n+2][c-\Theta_i]\ell_* + \zeta]^2/4T \right. \\
& \times f\left(\frac{[4n+2][c-\Theta_i]\ell_* + \zeta}{2\sqrt{T}} - (2n\rho + a_i)\sqrt{T}\right) \\
& - \exp([4n^2+4n][c-\Theta_i]\ell_*\rho - [[4n+2][c-\Theta_i]\ell_* - \zeta]^2/4T \\
& \left. \times f\left(\frac{[4n+2][c-\Theta_i]\ell_* - \zeta}{2\sqrt{T}} - (2n\rho + a_i)\sqrt{T}\right) \right\}
\end{aligned} \tag{5.20}
$$

The function f in Equation 5.20 is defined by

$$f(x) = e^{x^2}\operatorname{erfc}(x)$$

Since this function is bounded, it can be seen, by comparison with $\sum e^{n^{-2}}$, that the above series converge provided T is less than the time taken to fill all initial pore space,

$$T < [1-\theta_i]\ell_*/\rho$$

The function $u(\zeta, T)$ can be differentiated explicitly, and Θ can be obtained from Equations 4.15 and 5.6, as

$$\Theta = c[1-(c-1)/\mu] \tag{5.21}$$

$$\frac{\mu}{c-1} = 1 + 2\rho - 2u^{-1}\frac{\partial u}{\partial \zeta} \tag{5.22}$$

Finally, from Equations 4.16 and 5.6, we deduce

$$z_* = c^{-1}[(\rho + \frac{1}{2})\zeta - \ell_n(u) + \rho(\rho+1)T] \tag{5.23}$$

Equations 5.21 to 5.23 provide an exact parametric solution to the boundary value problem posed at the beginning of this section. An example is shown in Figure 2.

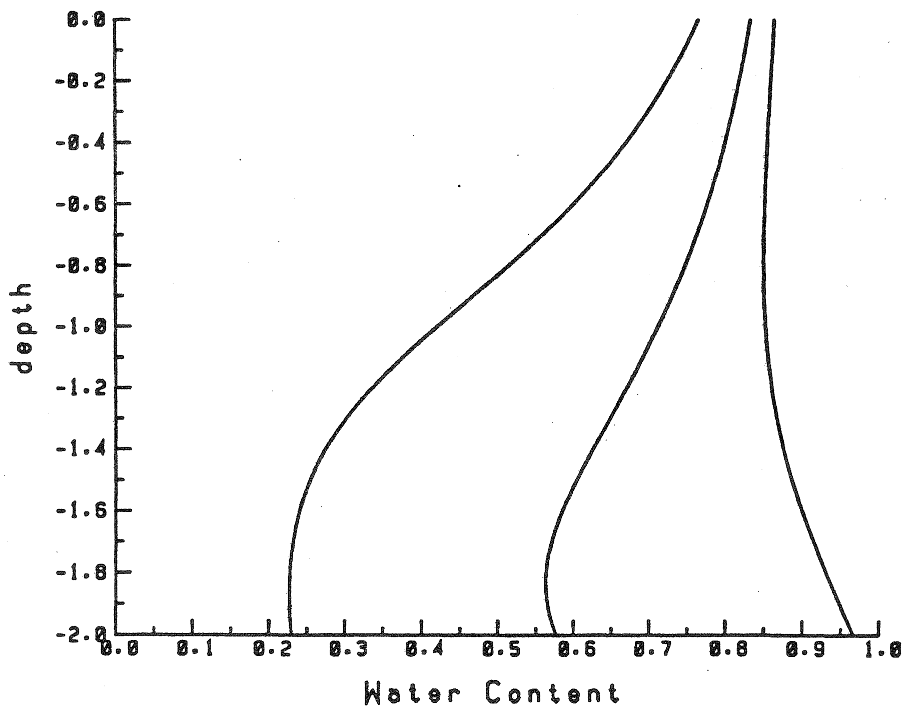

Figure 2. Analytic solution for infiltration into a finite column. $c = 1.2, R_* = 0.5, \ell_* = 2.0, \Theta_i = 0.2$. Output times for the three curves are $t_* = 1.0, 2.0$ and 2.7.

6. ALTERNATIVE APPROACH TO THE FINITE COLUMN

In section 5, a nonlinear boundary value problem on a rigid domain was transformed to a linear boundary value problem on a shrinking domain. It is the linear boundary value problem on the shrinking domain that offers the additional challenge.

The solution method used in Section 5 raises many questions. Firstly, the Laplace transform has been applied to a function $u(\zeta, T)$ defined only on a finite time domain. For what boundary value problems can the Laplace transform technique succeed in producing a correct solution $u(\zeta, T)$ even though the Laplace transform $\tilde{u}(\zeta, T)$ cannot exist!? In familiar applications of the Laplace transform, the boundary conditions lead to algebraic equations which uniquely determine the free parameters of a general solution. However, in Section 5, the boundary condition led to a difference Equation 5.17,

$$(1 - L_1)H(\nu) = J(\nu) \tag{6.1}$$

for which there is an infinite dimensional solution space to the homogeneous equation. In this case, a formal operator inversion

$$H(\nu) = \{1 + \sum_{n=1}^{\infty} L_1^n\}J(\nu) \tag{6.2}$$

led to a good solution. The formal sums in Equation 5.19 are large-p asymptotic expansions, rather than convergent series, corresponding to the series in Equation 5.20 having only a finite radius of convergence in T. A formal Laplace inversion could be applied term by term to Equation 5.19. However this is not always possible, following the formal inversion expressed in Equation 6.2. In a related practical problem with different boundary conditions [29], the solution obtained by Equation 6.2 could not be inverted. Before the inverse Laplace transform could be applied, an appropriate solution to the homogeneous difference equation was added. There is at this time no rigorously justified criterion for uniquely selecting an appropriate solution of the difference Equation 5.17 so that the inverse Laplace transform can be applied to the constructed expression 5.19.

King's Laplace transform boost theorem [27] applies to a change of Galilean reference frame. If it cannot be applied to two reference frames which are accelerating relative to each other, then perhaps an alternative special technique should be found for treating domains that expand or shrink *linearly* in time. The difficult questions raised above might then be avoided. The best way to treat a moving boundary problem is to transform it to a fixed boundary problem. If we inspect the group of point symmetries of the linear heat equation [30], we find a transformation that does the job. Let

$$\overline{T} = \frac{T}{1 - T\rho/[c-\Theta_i]\ell_*}$$
$$\overline{\zeta} = \frac{\zeta}{1 - T\rho/[c-\Theta_i]\ell_*}$$
$$\overline{u} = [1 - T\rho/[c-\Theta_i]\ell_*]^{\frac{1}{2}} \exp\left(\frac{-\rho\zeta^2}{4[c-\Theta_i]\ell_*}\frac{1}{1 - T\rho/[c-\Theta_i]\ell_*}\right) u \tag{6.3}$$

Then the boundary value problem given by Equations 5.7 to 5.10 transforms to a linear boundary value problem on a rigid domain defined at all positive times.

$$\frac{\partial \overline{u}}{\partial \overline{T}} = \frac{\partial^2 \overline{u}}{\partial \overline{\zeta}^2} \quad \text{on } [0, 2(c-\Theta_i)\ell_*] \times [0,\infty)$$
$$\overline{U}(\overline{\zeta},0) = f(\overline{\zeta}) = \exp\left(Q\overline{\zeta} - \frac{\rho\overline{\zeta}^2}{4[c-\Theta_i]\ell_*}\right)$$
$$\overline{u}(0,\overline{T}) = \phi_1(\overline{T})$$
$$= [1 + \overline{T}\rho/[c-\Theta_i]\ell_*]^{-\frac{1}{2}} \exp\left[\rho(\rho+1)\frac{\overline{T}}{1+\rho\overline{T}/[c-\Theta_i]\ell_*}\right]$$
$$\overline{u}(2[c-\Theta_i]\ell_*, \overline{T}) = \phi_2(\overline{T})$$
$$= [1 + \rho\overline{T}/[c-\Theta_i]\ell_*]^{-\frac{1}{2}} \exp(-[c-\Theta_i]\ell_*[\rho + 2Q]) \tag{6.4}$$

This reformulation should dispel any doubts about uniqueness of the solution. In terms of Fourier expansions, the solution is

$$\overline{u} = \frac{1}{[c-\Theta_i]\ell_*}\sum_{n=1}^{\infty} e^{-n^2\pi^2\overline{T}/(2[c-\Theta_i]\ell_*)^2} \sin\left(\frac{n\pi\overline{\zeta}}{2[c-\Theta_i]\ell_*}\right)$$
$$\times \left[\int_0^{2[c-\Theta_i]\ell_*} f(x)\sin\left(\frac{n\pi x}{2[c-\Theta_i]\ell_*}\right)dx\right.$$
$$\left. + \frac{n\pi}{2[c-\Theta_i]\ell_*}\int_0^{\overline{T}} e^{n^2\pi^2 s/(2[c-\Theta_i]\ell_*)^2}\{\phi_1(s) - (-1)^n\phi_2(s)\}ds\right] \tag{6.5}$$

The integrals in Equation 6.5 have not been evaluated explicitly and it is doubtful whether any sequence of commonly used approximate integration schemes, with progressively finer discretization, would converge faster than the partial sums in Equation 5.20. Neither approach has yet been adapted to treat a domain that shrinks or expands *nonlinearly* in time.

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