RELAXED VARIATIONAL PRINCIPLES AND ALGORITHMS FOR THE EQUILIBRIA OF ROTATING SELF-GRAVITATING FLUIDS

*Giles Auchmuty**

1. Introduction.

In this paper, we shall analyze some aspects of the variational principle for the axisymmetric equilibria of a rotating self-gravitating fluid. The analysis is developed to justify certain relaxed Lagrangean formulations of the problem and to describe an iterative algorithm for computing the solutions.

These equilibrium solutions provide simple, hydrodynamical models of stars and planets. The physical basis of these theories is described in Tassoul [7]. Here our interest is in developing convergent algorithms for finding these solutions.

In sections 2 to 5 we describe a variational principle for these equilibria, analyze the functional involved, derive the extremality conditions and prove that the local minimizers of this problem are, in fact, classical solutions of the hydrodynamical equations. The existence of such minimizers is proven. Much of this analysis is a variation of that of [1] but there are a number of new results, including convexity theorems, that are important for our purposes.

In section 6 we introduce, and analyze, a relaxed Lagrangean for this problem which converts the variational principle into one of minimizing a functional which is convex in each of two variables separately. This formulation is then used in section 7 to develop an algorithm which generates strict descent sequences for the problem and the convergence of this algorithm is studied.

I would like to thank the organizers of this conference and the C.M.A. for their hospitality.

*This research was partially supported by NSF Grant DMS 8901477

2. The Variational Principle.

The axisymmetric equilibrium figures of a rotating self-graviting fluid obeying a given barotropic equation of state may be characterized by a variational principle for the density of the fluid. A description, and analysis, of this problem is given in Auchmuty and Beals [1]. There it is shown that there are minimizers of this variational problem and that the minimizing densities are continuous functions with compact support in space.

Here we shall look at the problem of developing algorithms for constructing minima of this variational principle. To do this we shall pose the problem in the closed ball B_R of finite radius R in space, and consider the case of the polytropic equation of state

$$p = K\rho^{\gamma} \tag{2.1}$$

with $\gamma > 1, K > 0$. Here p is the pressure and ρ is the density of the fluid. Let $L^{\gamma}(B_R)$ be the set of all (equivalence classes of) measurable real-valued functions defined on B_R, whose absolute value is γth power integrable. It is a Banach space under the norm

$$||u||_{\gamma}^{\gamma} = \int_{B_R} |u(x)|^{\gamma} dx.$$

Here, and henceforth, integration will be over the set B_R unless otherwise indicated.

The admissible set D of density distributions is the set of functions ρ in $L^{\gamma}(B_R)$ obeying

(D1): $\rho(x) \geq 0$ a.e. on B_R,

(D2): $\int \rho(x)\, dx = M > 0$, and (2.2)

(D3): ρ is axisymmetric about the z-axis and symmetric about the plane $z = 0$.

It is an (interesting) exercise in measure theory to prove that D is strongly closed in $L^{\gamma}(B_R)$. D is a convex set in $L(B_R)$, so it is also weakly closed.

The rotation law is specified by giving the distribution of angular momentum per unit mass j. This should be a function of a Lagrangean variable of the system. Since we are restricting attention to axisymmetric figures, this Lagrangean variable will be

$$m_\rho(r) = M^{-1} \int_{r(x)<r} \rho(x)\, dx \tag{2.3}$$

This integral is over the region of cylindrical radius r about the z axis. Throughout this paper, (r, θ, z) will be cylindrical polar coordinates of a point x in space, (x_1, x_2, x_3) will be its Euclidean coordinates while $r(x)$ and $|x|$ will be the cylindrical and spherical radii of x respectively.

$m_\rho(r)$ defined by (2.3) is the proportion of mass inside a cylinder of radius r. We shall require

(D4): $j : [0,1] \to [0,\infty)$ is a continuously differentiable, non-decreasing function with $j(0) = 0$.

The condition that j be nondecreasing is the Solberg stability criterian (see Tassoul [7], Section 7.3).

Define the functional $\mathcal{E} : D \to \bar{\mathbb{R}} = [-\infty, \infty]$ by

$$\mathcal{E}(\rho) = \int \left[K_0 \rho(x)^\gamma + \frac{1}{2}\rho(x) j^2(m_\rho(r)) r^{-2} - \frac{1}{2}\rho(x) V\rho(x)\right] dx \tag{2.4}$$

$$= E_1(\rho) + T(\rho) - \mathcal{V}(\rho). \tag{2.5}$$

Here $K_o = K(\gamma-1)^{-1}$ and $V\rho(x) = \displaystyle\int \frac{\rho(y)dy}{|x-y|}$. (2.6)

Units are chosen here so that the gravitational constant $G = 1$. $T(\rho)$ and $\mathcal{V}(\rho)$ are, respectively, the kinetic energy and the gravitational potential energy associated with the density distribution ρ.

The variational principle is find extremals (or critical points) of $\mathcal{E}$ on D. In section 4 it will be shown that these extremals will be the densities of possible equilibrium figures for a self-gravitating fluid obeying the equation of state (2.1) and rotating with this given angular momentum per unit mass $j(m)$.

3. Analysis of the Functional.

In this section, various functional analytic results about the quantities arising in this variational principle will be obtained. We need to develop a calculus for functionals defined on the domain $D \subseteq L^\gamma(B_R)$. All integrals and measures will be Lebesgue integrals and measures. When terms are used here without definition, they should be taken as in Zeidler [8].

When ρ is in D, we define the cone of allowable directions at ρ to be

$$P_\rho = \{t(u-\rho): \quad t \geq 0, u \in D\}. \tag{3.1}$$

P_ρ is a closed, convex cone in $L^\gamma(B_R)$ as D is a closed convex set. For each h in P_ρ, there is a $\delta \geq 1$ such that $\rho + th$ is in D for $0 \leq t < \delta$. The support, supp ρ, of a function ρ in D is defined to be the set of all x in B_R such that for each $r > 0, |\{B_r(x) \cap E_o\}| \neq 0$ where $B_r(x) = \{y \in B_R : |x-y| < r\}, E_o = \{x \in B_R : \rho(x) > 0\}$ and $|G|$ is the Lebesgue measure of G.

<u>Lemma 3.1</u>. *Suppose ρ is in D and h is in P_ρ, then h obeys (D3),*

(H1): $\quad h(x) \geq 0$ a.e. *on* $B_R -$ *supp* ρ, *and* *(3.2)*

(H2): $\quad \int h \, dx = 0.$ *(3.3)*

Proof: Each h in P_ρ obeys (D3) since (3.1) holds with u, ρ obeying (D3). Since $\rho + th$ is in D for $0 \leq t < \delta$, (2.2) implies that

$$\int (\rho + th) dx = M \qquad \text{for } 0 \leq t < \delta.$$

Hence (3.3) holds as ρ obeys (D2).

If x is not in supp ρ, then there exists $r > 0$ such that $|B_r(x) \cap E_o| = 0$. Therefore $\rho(x) = 0$ a.e. on $B_r(x)$.

From (D1), we must have $(\rho + th)(x) \geq 0$ a.e. on $B_r(x)$ for $0 \leq t < \delta$, so $h(x) \geq 0$ a.e. on $B_r(x)$.

Since B_R- supp u is an open set, (H1) follows. □

The essential domain of a functional $\mathfrak{F} : D \to (-\infty, \infty]$ is $D_o = \{u \in D : |\mathfrak{F}(u)| < \infty\}$. $\mathfrak{F}$ is convex on D if

$$\mathfrak{F}((1-t)u + tv) \leq (1-t)\mathfrak{F}(u) + t\mathfrak{F}(v),$$

for all u, v in D and $0 \leq t \leq 1$.

When ρ is in D_o, h is in P_ρ, then $\mathfrak{F}$ is said to be differentiable in the direction h if

$$\lim_{t \to 0+} t^{-1}\{\mathfrak{F}(\rho + th) - \mathfrak{F}(\rho)\} \qquad \text{exists .}$$

Let $D_\rho = \{h \in P_\rho : \mathfrak{F}$ is differentiable in the direction h at $\rho\}$. Then D_ρ will be a cone. $\mathfrak{F}$ is said to be G-differentiable at ρ if $D_\rho = P_\rho$ and there exists g in $L^{\gamma'}(B_R)$ such that

$$\lim_{t \to 0+} t^{-1}[\mathfrak{F}(\rho + th) - \mathfrak{F}(\rho)] = \int gh \, dx$$

Here $\gamma' = \gamma(\gamma - 1)^{-1}$ is the conjugate index to γ and when this holds we shall write $D\mathfrak{F}(\rho) = g$.

To simplify the notation, and some analysis, we shall define $X = \{\rho \in L^\gamma(B_R) : \rho$ obeys $(D3)\}$. X is a closed subspace of $L^\gamma(B_R)$; hence is regarded as a Banach space in its own right with

$$||\rho||_\gamma^\gamma = \int |\rho(x)|^\gamma dx = 4\pi \int_0^R \int_0^{\sqrt{R^2-r^2}} r|\rho(r,z)|^\gamma dr \, dz \tag{3.4}$$

upon using (D3) and cylindrical polar coordinates.

D is a closed convex subset of X. The dual space of X with respect to the usual pairing

$$f(\rho) = < f, \rho > = \int \rho(x) f(x)\ dx \tag{3.5}$$

is $X^* = \{u \in L^{\gamma'}(B_R) : u \text{ obeys } (D3)\}$ with γ' being the conjugate index to γ.

The first term in (2.4) is $E_1 : D \to [0, \infty)$ defined by

$$\begin{aligned} E_1(\rho) &= K_o \int \rho^\gamma dx \\ &= K_o ||\rho||_\gamma^\gamma \quad \text{for } \rho \text{ in } D. \end{aligned} \tag{3.6}$$

<u>RESULT 3.2</u>. *When $1 < \gamma < \infty$, E_1 is convex, strongly continuous and weakly lower semi-continuous (l.s.c.) on D. It is G-differentiable with*

$$DE_1(\rho)(x) = \gamma K_o \rho(x)^{\gamma - 1}. \tag{3.7}$$

Proof: The extension $\tilde{E}_1$ of E_1 to X defined by (3.6) is convex and strongly continuous from the definition (3.4) of the norm on X. Hence it is weakly l.s.c. on X and thus on D.

Then (3.7) follows by a straightforward computation. □

Next consider the gravitational potential energy functional

$$\mathcal{V}(\rho) = \frac{1}{2} \int \rho\ V \rho\ dx. \tag{3.8}$$

<u>RESULT 3.3</u>. *The integral operator $V : X \to X^*$ defined by (2.6) is compact when $\frac{6}{5} < \gamma < \infty$. For such γ, the functional $\mathcal{V}$ is weakly continuous and G-differentiable on X. It is strictly convex on D and there is a constant C_1 such that, for all ρ in D,*

$$\mathcal{V}(\rho) \le C_1 M^{2-\beta} ||\rho||_\gamma^\beta \tag{3.9 }$$

with $\beta = \frac{\gamma'}{3}$.

Proof: The integral operator $V : L^{\gamma}(B_R) \to L^{\gamma'}(B_R)$ defined by (2.6) is compact from the potential theoretic version of the Sobolev imbedding theorem when $\frac{1}{\gamma} - \frac{2}{3} < \frac{1}{\gamma'}$. See Sobolev [6] Chapter 1. This inequality yields $\gamma > \frac{6}{5}$. From the symmetry properties of the kernel, a direct analysis shows that when ρ obeys (D3) so does $V\rho$. Hence V maps X into X^* compactly when $\gamma > \frac{6}{5}$.

If ρ_n converges weakly to ρ in X, then $V\rho_n$ converges strongly to $V\rho$ in X^* and

$$2[\mathcal{V}(\rho) - \mathcal{V}(\rho_n)] = \int \{(\rho - \rho_n)V\rho + \rho_n(V\rho - V\rho_n)\}dx$$

so this converges to 0 as $n \to \infty$. Hence $\mathcal{V}$ is weakly continuous. From the symmetry of V, one finds that

$$D\mathcal{V}(\rho)h = \int h(x)V\rho(x)\,dx \tag{3.10}$$

for any ρ, h in X, so $\mathcal{V}(\rho)$ is $G-$differentiable on X.

(3.8) may be rewritten as

$$\mathcal{V}(\rho) = \frac{1}{2}\int\int \frac{\rho(x)\rho(y)}{|x-y|}dx\,dy$$

for all ρ in X, so $\mathcal{V}(\rho) > 0$ as ρ obeys $(D1)$ and thus $\mathcal{V}$ is strictly convex on D, as it is quadratic in ρ.

Applying Hölder's inequality to (3.8),

$$|\mathcal{V}(\rho)| \leq \frac{1}{2}\|\rho\|_q \|V\rho\|_{q'} \leq \frac{C}{2}\|\rho\|_q^2 \tag{3.11}$$

provided $q \geq \frac{6}{5}$ upon using the continuity of V as above.

When ρ is in D and $1 < q < \gamma$, then from the standard Hölder type interpolation inequality

$$\| \rho \|_q \leq M^{1-\alpha} \|\rho\|_\gamma^\alpha \tag{3.12}$$

as $\| \rho \|_1 = M$ and with $\alpha = \frac{\gamma'}{q'}$. Substituting this in (3.11)

$$|\mathcal{V}(\rho)| \leq \frac{1}{2} C M^{2(1-\alpha)} \| \rho \|_\gamma^{2\alpha}.$$

Choose $q = \frac{6}{5}$ in (3.12), then $\alpha = \frac{\gamma'}{6}$ and (3.9) follows □

The remaining term is the rotational kinetic energy functional

$$T(\rho) = \frac{1}{2} \int \rho(x) j^2(m_\rho(r)) r^{-2}\, dx. \tag{3.13}$$

T is only defined on D. This integral is always non-negative when ρ is in D, but when $\gamma < \frac{3}{2}$, this integral may be infinite. Let $D_o = \{\rho \in D : T(\rho) < \infty\}$, the essential domain of $\mathcal{E}$ will be D_o whenever $\gamma \geq \frac{6}{5}$.

<u>Result 3.4</u>. *When (D4) holds and $\gamma > 1$, T is weakly l.s.c. on D.*

Proof: Suppose $\{\rho_n : n \geq 1\} \subset D$ converges weakly to a density ρ in D. Then there exists a constant C such that $\|\rho_n\|_\gamma \leq C$. Let $m_n(r) = m_{\rho_n}(r)$ be defined by (2.3). Hölder's inequality yields

$$|m_n(r_1) - m_n(r_2)| \leq C \|\chi_1 - \chi_2\|_{\gamma'}$$

$$\text{where} \qquad \chi_j(x) = \begin{cases} 1 & \text{if } r(x) \leq r_j \\ 0 & \text{otherwise} \end{cases}$$

Here $j = 1$ or 2 and x is in B_r. Then a straightforward computation yields

$$\|\chi_1 - \chi_2\|_q^q = \frac{4\pi}{3} [\phi(r_2) - \phi(r_1)]$$

for $1 \leq q < \infty$ and $0 \leq r_1 \leq r_2 < R$ and with $\phi(r) = [R^2 - r^2]^{\frac{3}{2}}$. Hence there is a constant C_2, such that

$$|m_n(r_2) - m_n(r_1)| \leq C_2 |r_2 - r_1|^\alpha$$

with $\alpha = (\gamma')^{-1}$.

This implies that $\{m_n : n \geq 1\}$ is a bounded equicontinuous family of continuous functions defined on [0,R]. Since ρ_n converges weakly to ρ, $m_n(r)$ will converge pointwise to $m(r)$ for each r in [0,R] and $j(m_n(r))$ will converge uniformly to $j\ (m(r))$ on [0,R].

Thus $\dfrac{j^2(m_n(r))}{r^2}$ converges uniformly to $\dfrac{j^2(m_\rho(r))}{r^2}$ on $[\varepsilon, R]$ for any $0 < \varepsilon < R$ and

$$\lim_{n\to\infty} \int\limits_{r(x)\geq\varepsilon} \rho_n(x) \frac{j^2(m_n(r))}{r^2} dx = \int\limits_{r(x)\geq\varepsilon} \rho(x) \frac{j^2(m_\rho(r))}{r^2} dx$$

as ρ_n converges weakly to ρ in X.

Call this last integral $T_\varepsilon(\rho)$, then it is a nondecreasing function of ε as ε goes to zero. Also

$$T_\varepsilon(\rho) = \lim_{n\to\infty} T_\varepsilon(\rho_n) \leq \lim_{n\to\infty}\ \inf T(\rho_n).$$

for any $\varepsilon\ > 0$. Let ε go to zero now, then T is weakly *l.s.c.* at ρ in D as claimed. □

Finally we shall describe the derivative of $T(\rho)$ and show that T is convex. To do this we shall use the fact that

$$T(\rho) = \frac{M}{2} \int\limits_0^R m'_\rho(r) \frac{j^2(m_\rho(r))}{r^2} dr \tag{3.14}$$

where $m'_\rho(r)$ is the Radon-Nikodym derivative of $m_\rho(r)$.

Also, we shall sometimes require that allowable perturbations h obey

(H3): there exists $\delta > 0$ such that $m_h(r) = 0$ for $0 \leq r \leq \delta$.

RESULT 3.5. *Suppose $\gamma > 1$, ρ is in D_o and (D4) holds. If h is in P_ρ and obeys (H3), then*

$$\lim_{t\to 0+} t^{-1}[T(\rho+th) - T(\rho)] = \int h(x) W_\rho(r)dx \tag{3.15}$$

$$\text{where} \qquad W_\rho(r) = \int_r^R s^{-3} j^2(m_\rho(s))ds, \tag{3.16}$$

and T is convex on D_o and D.

Proof: Assume ρ, h obey these conditions, then $\rho + th$ is in D_o for $0 \le t \le 1$ and

$$t^{-1}[T(\rho+th) - T(\rho)] = \frac{M}{2t} \int_\delta^R [tm_h'(r)L_t(r) + m_\rho'(r)(L_t(r) - L_o(r))]r^{-2}dr \tag{3.17}$$

where $L_t(r) = j^2(m_{\rho+th}(r))$ and we have used (3.14).

The first integral on this right hand side converges, as t goes to 0, to

$$\frac{M}{2} \int_\delta^R m_h'(r) \frac{L_o(r)}{r^2} dr = \frac{-M}{2} \int_\delta^R m_h(r) \frac{d}{dr}(\frac{L_o(r)}{r^2})dr. \tag{3.18}$$

Also

$$\lim_{t\to 0+} \frac{L_t(r) - L_o(r)}{t} = 2j(m_\rho(r))j'(m_\rho(r))m_h(r)$$

as j is continuously differentiable and this function is continuous on $[\delta, R]$. Thus the dominated convergence theorem implies

$$\lim_{t\to 0+} \frac{1}{2t} \int_\delta^R \frac{m_\rho'(r)}{r^2}[L_t(r) - L_o(r)]dr = \int_\delta^R \frac{m_\rho'(r)}{r^2} j(m_\rho(r))j'(m_\rho(r))m_h(r)dr. \tag{3.19}$$

Combining (3.17) - (3.19), one obtains

$$\lim_{t\to 0+} t^{-1}[T(\rho+th) - T(\rho)] = M \int_\delta^R m_h(r) j^2(m_\rho(r)) r^{-3} dr$$

or

$$DT(\rho)h = \int hW_\rho(r)dx$$

where W_ρ is defined by (3.16). W_ρ is the centrifugal potential in this problem.

If ρ_1, ρ_2 are in D, then T is convex on D provided

$$\phi(t) = T((1-t)\rho_1 + t\rho_2) \leq (1-t)T(\rho_1) + tT(\rho_2) = (1-t)\phi(0) + t\phi(1)$$

If ρ_1 and/or ρ_2 is in $D - D_o$, then this holds automatically, so we can assume ρ_1 and ρ_2 are in D_o.

Assume $m_1(r) = m_2(r)$ for $0 \leq r \leq \delta$, then from above one has

$$\phi'(0) = DT(\rho_1)(\rho_2 - \rho_1) = \int hW_o(r)dx$$

where $h = \rho_2 - \rho_1$, and $W_o = W_{\rho_1}$.

Also,

$$\begin{aligned}\phi'(t) - \phi'(o) &= M \int_\delta^R m_h'(r)[W_t(r) - W_o(r)]dr \\ &= -M \int_\delta^R m_h(r)[\frac{dW_t}{dr}(r) - \frac{dW_o}{dr}(r)]dr \qquad (3.20)\end{aligned}$$

upon itegrating by parts. Here $W_t(r) = W_{(1-t)\rho_1 + t\rho_2}(r)$.

Also $\frac{d}{dt}W_t(r) = 2\int_r^R s^{-3}j(m_t(s))j'(m_t(s))m_h(s)ds$, from (3.16) so

$$W_t(r) - W_o(r) = 2\int_o^t \int_r^R s^{-3}j(m_\tau(s))j'(m_\tau(s))m_h(s)ds\ d\tau.$$

Differentiating this with respect to r and substituting in (3.20)

$$\phi'(t) - \phi'(o) = 2M\int_\delta^R r^{-3}m_h(r)^2(\int_o^t j(m_\tau(r)j'(m_\tau(r))d\tau)dr.$$

Divide both sides by t, let t go to zero, then the dominated convergence theorem yields

$$\phi''(0) = 2M \int_{\delta}^{R} r^{-3} m_h(r)^2 j(m_o(r)) j'(m_o(r)) dr.$$

From (D4), $j'(m) \geq 0$ for all m, so $\phi'' \geq 0$ and thus ϕ is convex. Hence T will be convex for all such ρ_1, ρ_2. Letting δ go to zero and approximating if necessary, T will be convex on D_o.

4. Extremality Conditions.

In this section we shall show that the local minimizers of $\mathcal{E}$ on D actually do provide axisymmetric equilibrium models of rotating self-gravitating fluids obeying the polytropic equation of state (2.1) whenever $\gamma > \frac{6}{5}$. That is, they obey Euler's equation for the uniformly rotating motion of an inviscid, self-gravitating compressible fluid obeying (2.1).

A function $\widetilde{\rho}$ in D_o is said to be a local minimizer of $\mathcal{E}$ on D if, for each h in $P_{\widetilde{\rho}}$, /there is a $\delta > 0$ such that

$$\mathcal{E}(\widetilde{\rho} + th) \geq \mathcal{E}(\widetilde{\rho}) \qquad \text{for } 0 \leq t < \delta. \tag{4.1}$$

<u>THEOREM 1</u>. *Assume $\widetilde{\rho}$ is a local minimizer of $\mathcal{E}$ on $D_o, \gamma > \frac{6}{5}$ and (D4) holds. Then there is a real number λ such that $\widetilde{\rho}$ obeys*

$$\gamma K_o \rho(x)^{\gamma-1} + \int_r^R s^{-3} j^2(m_\rho(s)) ds - V\rho(x) \geq \lambda \quad \text{a.e. on } B_R \tag{4.2}$$

with equality holding here on any open set where $\rho(x) > 0$ and $r(x) > 0$.

Proof. Choose h in $P_{\widetilde{\rho}}$ and assume it obeys (H3). Let $\phi(t) = \mathcal{E}(\widetilde{\rho} + th) - \mathcal{E}(\widetilde{\rho})$, then from results 3.2, 3.3 and 3.5, one obtains $\phi'(0) = \int [\gamma K_o \rho(x)^{\gamma-1} + W_\rho(r) - V\rho(x)] h(x)\, dx.$

Since $\widetilde{\rho}$ is a local minimizer this must be non-negative. There are sufficiently many allowable h's obeying (H1)–(H3), that this implies there is a real λ such that

$$\gamma K_o \widetilde{\rho}(x)^{\gamma-1} + W_{\widetilde{\rho}}(r) - V\widetilde{\rho}(x) - \lambda \geq 0 \quad \text{a.e for } r(x) \geq \delta$$

with equality holding here on any open subset of $\{x : \widetilde{\rho}(x) > 0\}$. δ is an arbitrary positive number so (4.2) follows as claimed. □

When a function ρ in D obeys (4.2), then

$$\gamma K_o \rho(x)^{\gamma-1} = \max\,(0, \lambda + V\rho(x) - W_\rho(r)) \tag{4.3}$$

where λ is chosen so that the solution of this obeys (2.2). Note that the last term in (4.3) is a strictly monotone increasing function of λ, for fixed ρ.

<u>LEMMA</u>. *If ρ in D obeys (4.3) and $\gamma > \frac{6}{5}$, then ρ is a continuous function on B_R. If $G_o = \{x \in B_R : |x| < R \text{ and } \rho(x) > 0\}$, then G_o is open and ρ is continuously differentiable on G_o.*

Proof. First consider the case $\gamma > \dfrac{3}{2}$. When ρ is in $L^\gamma(B_R)$ with $\gamma > \dfrac{3}{2}$, then the Sobolev imbedding of theorem guarantees that $V\rho$ is continuous on B_R.

From the definition (3.16) of $W_\rho(r)$, one sees that W_ρ is an absolutely continuous non-increasing and non-negative function on $(0, R)$.

Thus $\lambda + V\rho(x) - W_\rho(r)$ will be continuous on $[0\ R]$. If $\lim_{r\to 0^+} W_\rho(r) = w_o$ is finite, then this combination is continuous and bounded on B_R. When $w_o = +\infty$, then there exists a $\delta > 0$ such that $|r| < \delta \Rightarrow$

$$\lambda + V\rho(x) - W_\rho(r) < 0$$

as $\lambda + V\rho$ is bounded on B_R. Hence the right hand side of (4.3) is again bounded and continuous on B_R, so ρ will be.

Since ρ is continuous, it is bounded on B_R and G_o is open.

When ρ is continuous and bounded, then $V\rho$ will be continuously differentiable on B_R and W_ρ will be continuously differentiable on (0, R). Thus the right side of (4.3) will be continuously differentiable on G_o or on any open subset of B_R on which $\rho(x) \equiv 0$. Hence the lemma is proved.

When $\frac{6}{5} < \gamma < \frac{3}{2}$, then a bootstrapping argument as in the proof of theorem A in section 4 of [1], shows that a solution of (4.3) will be in L^γ for some $\gamma > \frac{3}{2}$, and then the preceding argument applies. □

This result shows that (4.2) may be interpreted pointwise in the usual manner. On G_o,

$$\gamma K_o \rho(x)^{\gamma-1} = V\rho(x) - \int_r^R \frac{j^2(m_\rho(s))}{s^3} ds + \lambda.$$

Taking the gradient of both sides here,

$$\gamma K_o \text{ grad } \rho(x)^{\gamma-1} = \text{grad } V\rho(x) + \frac{j^2(m_\rho(r))}{r^3}\hat{i}_r \tag{4.4}$$

where $\hat{i}_r$ is the unit vector in the cylindrical radial direction. Multiplying both sides by ρ, one obtains Euler's equation for a rotating, compressible, self-graviting fluid obeying the equation of state (2.1)

$$\text{grad } K\rho(x)^\gamma = \rho(x)[\text{grad } V\rho(x) + r^{-3} j^2(m_\rho(r))\hat{i}_r]. \tag{4.5}$$

This holds as $K = K_o(\gamma - 1)$ from (2.6). It shows that the local minimizers of $\mathcal{E}$ on D actually are classical solutions of the Euler equation (4.5) on the set G_o.

5. Existence of Minima.

Here we shall look at the question of minimizing $\mathcal{E}$ on D and finding

$$\alpha = \inf_{\rho \in D} \quad \mathcal{E}(\rho). \tag{5.1}$$

We shall first show that when $\gamma > \frac{4}{3}, \alpha$ is finite and there is a minimizer of $\mathcal{E}$ on D.

<u>THEOREM 2</u>. *Suppose $\gamma > \frac{4}{3}$ and (D4) holds, then for any $M > 0$, the infimum α of $\mathcal{E}$ on D is finite and there is a $\hat{\rho}$ in D such that*

$$\mathcal{E}(\hat{\rho}) = \inf_{\rho \in D} \mathcal{E}(\rho). \tag{5.2}$$

Proof. From (2.4), (3.9) and the fact that $T(\rho) \geq 0$,

$$\mathcal{E}(\rho) \geq K_0 \; ||\rho||_\gamma^\gamma \; - C_1 \; M^{2-\beta} \; ||\rho||_\gamma^\beta \tag{5.3}$$

with $\beta = \frac{1}{3}\gamma'$. Thus $\gamma > \beta$ provided $\gamma > \frac{4}{3}$ so in this case $\mathcal{E}(\rho) \to \infty$ in D as $|| \; \rho \; ||_\gamma \to \infty$.

Choose ρ_o in D_o, then from (5.3) with $\gamma > \frac{4}{3}$, there is a C_2 such that $\mathcal{E}(\rho) \leq \mathcal{E}(\rho_o)$ and ρ in D implies $|| \; \rho \; ||_\gamma \leq C_2$. The subset D_2 of D obeying $|| \; \rho \; ||_\gamma \leq C_2$ is a weakly compact, convex set as X and $L^\gamma(B_R)$ are reflexive.

E_1 and T are weakly *l.s.c.* on D from results 3.2 and 3.4, while $\mathcal{V}$ is weakly continuous on D from 3.3. Thus $\mathcal{E}$ will be weakly *l.s.c.* on D and hence on D_2. Since D_2 is weakly compact, $\mathcal{E}$ attains its infinum on D_2 and thus on D. So the theorem follows. □

The minimizer $\hat{\rho}$ here will depend on R - when R is small. In [1], however, it was shown that for R large enough, the minimizer of $\mathcal{E}$ on D will not change. In other words, even when $R = \infty$ there is a minimizer of $\mathcal{E}$ on D and this minimizer has compact support. In [4], section 8, some lower bounds on this radius were described.

When $\frac{6}{5} < \gamma < \frac{4}{3}$ and $j(m) \equiv 0$, then by a simple scaling argument, one can construct a sequence of densities in D with $\lim_{n\to\infty} \mathcal{E}(\rho_n) = -\infty$ so $\alpha = -\infty$ in this case.

The critical value $\gamma = \frac{4}{3}$ is a classical stability criterion for non-rotating models and is discussed in Chandrasekhar [5], Section 2.10.

6. Relaxed Variational Principles.

The original variational principle $(\mathcal{P})$ for these models is to minimize the non-convex functional

$$\mathcal{E}(\rho) = E_1(\rho) + T(\rho) - \mathcal{V}(\rho) \tag{6.1}$$

on the closed convex subset D of X. From the results of section 3, we know that each of E_1, T and $\mathcal{V}$, individually, is convex so that $\mathcal{E}$ is the difference of two convex functionals.

This enables the application of non-convex duality theory as described in [2] and [3].

Let $\mathcal{V}^* : X^* \to [0,\infty]$ be the conjugate convex functional of $\mathcal{V}$. That is

$$\mathcal{V}(u) = \sup_{\rho\in X} \int \rho(u - \frac{1}{2}V\rho)dx \tag{6.2}$$

$$= \begin{cases} \frac{1}{2} < u, V^{-1}u > & \text{if } u \in V(X) \\ \infty & \text{otherwise.} \end{cases} \tag{6.3}$$

As will be seen the explicit form of $\mathcal{V}^*$ will not be needed. It is sufficient that $\mathcal{V}^*$ is a convex and weakly *l.s.c.* functional on X^* (see Zeidler [8], Section 51.3).

More generally one may regularize this problem by defining $\mathcal{V}_\mathcal{E} : X \to [0,\infty]$ by

$$\mathcal{V}_\varepsilon(\rho) = \int [\frac{\varepsilon}{\gamma}|\rho|^\gamma + \frac{1}{2}\rho V\rho]\, dx \text{ with } \quad \varepsilon > 0. \tag{6.4}$$

Then $\mathcal{V}_\varepsilon^*(u)$ will be a well-defined convex, continuous and weakly *l.s.c.* functional.

Consider the relaxed Lagrangean functional $\mathcal{L}_\varepsilon : D \times X^* \to [-\infty, \infty]$ defined by

$$\mathcal{L}_\varepsilon\ (\rho, u) = E_1(\rho) + T(\rho) + \int (\frac{\varepsilon}{\gamma}\rho^\gamma - \rho u) dx + \mathcal{V}_\varepsilon^*(u) \tag{6.5}$$

with $\varepsilon \geq 0$, and the associated variational principle $(\mathcal{Q})$ of minimizing $\mathcal{L}_\varepsilon$ on $D \times X^*$.

The basic facts about this functional may be summarized as follows.

THEOREM 3. *Suppose $\mathcal{L}_\varepsilon$ is defined by (6.5), $\gamma > \frac{6}{5}$ and (D4) holds. Then*

(i) $\mathcal{L}_\varepsilon(.,u)$ is strictly convex and weakly l.s.c. on D for each u in X^*,

(ii) $\mathcal{L}_\varepsilon\ (\rho, .)$ is convex and weakly l.s.c. on X^* for each ρ in D, and

(iii) $\mathcal{E}(\rho) = inf_{u \epsilon X^*}\ \mathcal{L}_\varepsilon\ (\rho, u)$ for each ρ in D.

Proof. (i) E_1, T and $\int\ \rho^\gamma\ dx$ are convex and weakly *l.s.c.* on D from results 3.2, 3.4, and 3.5. Also $\int \rho u$ is linear and weakly continuous for each u in X^*, so (i) holds.

(ii) $\mathcal{V}_\varepsilon^*$ is convex and weakly *l.s.c.* from the properties of dual functionals while $\int \rho u$ is linear and weakly continuous on X^* when ρ is fixed, so (ii) follows.

$$\begin{aligned}\inf_{u \epsilon X^*}\ \mathcal{L}_\varepsilon\ (\rho, u) &= E_1(\rho) + T(\rho) + \frac{\varepsilon}{\gamma} \int |\rho|^\gamma\ dx + \inf_{u \epsilon X^*}(\mathcal{V}_\varepsilon^*(u) - \int \rho u) \\ &= E_1(\rho) + T(\rho) + \frac{\varepsilon}{\gamma} \int |\rho|^\gamma\ dx - \mathcal{V}_\varepsilon(\rho)\end{aligned}$$

as $\mathcal{V}_\varepsilon^{**} = \mathcal{V}_\varepsilon$ when $\mathcal{V}_\varepsilon$ is a proper *l.s.c.* convex function (see [8], Theorem 51.6). Using (6.4) this right hand side is, in fact, $\mathcal{E}(\rho)$. □

From (5.1) and (iii) of this theorem

$$\alpha = \inf_{\rho \epsilon D}\ \inf_{u \epsilon X^*} \mathcal{L}_\varepsilon(\rho, u) \tag{6.6}$$

so the value α of the variational principle $(\mathcal{Q})$ equals that of the original problem $(\mathcal{P})$. The next result relates the minimizers of $(\mathcal{P})$ and $(\mathcal{Q})$.

THEOREM 4. *Let $\mathcal{L}_\varepsilon$ be defined by (6.5) with $\varepsilon \geq 0$, and assume $\gamma > \frac{4}{3}$ and (D4) holds. Then there exists $\hat{\rho}$ in $D, \hat{u}$ in X^* such that $\mathcal{L}_\varepsilon\ (\hat{\rho}, \hat{u}) = \alpha$ as defined by (6.6). Moreover $(\hat{\rho}, \hat{u})$ obey*

$$u(x) = \varepsilon\rho^{\gamma-1}\ (x) + V\rho(x)\ , \text{and} \tag{6.7}$$

$$(\gamma K_o + \varepsilon)\ \rho\ (x)^{\gamma-1} + W_\rho(r) \geq \lambda + u(x) \tag{6.8}$$

a.e. on B_R with equality holding in (6.8) on any open set where $\rho(x)$ is positive.

Proof. Choose $\hat{\rho}$ to be the minimizer of $\mathcal{E}$ on D; from theorem 2 this exists when $\gamma > \frac{4}{3}$. Define $\mathfrak{G} : X^* \to (-\infty,\ \infty]$ by

$$\mathfrak{G}(u) = \mathcal{V}_\varepsilon^*(u) - \int \hat{\rho}u.$$

From (6.4) and (3.9), $\mathcal{V}_\varepsilon(\rho) \leq \dfrac{\varepsilon}{\gamma}\ ||\rho||_\gamma^\gamma + C_o||\rho||_\gamma^2$ where C_o is a constant depending on R. Let $\phi(s) = \frac{\varepsilon}{\gamma}s^\gamma + C_o s^2$, then its convex conjugate function is

$$\phi^*\ (t) = \sup_{s\geq 0}\ (st - \frac{\varepsilon}{\gamma}s^\gamma - C_o s^2).$$

ϕ^* is a non-negative, monotone increasing function and, when $\varepsilon > 0$,

$$\lim_{t\to\infty} \frac{\phi^*(t)}{t^p} = \nu > 0 \quad \text{where} \quad p = min(\frac{\gamma}{\gamma-1}, 2). \tag{6.9}$$

Moreover from [3], Lemma 2.2, it follows that

$$\mathfrak{G}(u) \geq \phi^*\ (||u||_*) - ||\hat{\rho}||_\gamma\ ||u||_*.$$

Thus $\mathfrak{G}$ is coercive so the problem of minimizing $\mathfrak{G}$ on X^* has a solution. Hence there exists $\hat{u}$ in X^* such that

$$\mathcal{L}_\varepsilon\ (\hat{\rho}, \hat{u}) = \inf_{u\epsilon X^*} \mathcal{L}_\varepsilon(\hat{\rho}, u).$$

But $\alpha = \mathcal{E}(\hat{\rho}) = \inf_{u \epsilon X^*} \mathcal{L}_\varepsilon(\hat{\rho}, u)$ from (iii) of theorem 3. Therefore $\alpha = \mathcal{L}_\varepsilon(\hat{\rho}, \hat{u})$ so there are minimizers of the problem $\mathcal{Q}$.

When $\varepsilon = 0$, this result still holds with $p = 2$ in (6.9).

At a minimizer $(\hat{\rho}, \hat{u})$ of $\mathcal{L}_\varepsilon$ on $D \times X^*$ one must have

$$0 \in \partial_\rho \mathcal{L}_\varepsilon\ (\hat{\rho}, \hat{u}) \quad \text{and} \quad 0 \in \partial_u\ \mathcal{L}_\varepsilon\ (\hat{\rho}, \hat{u}).$$

where ∂_ρ and ∂_u denote the partial subdifferentials of $\mathcal{L}_\varepsilon$ with respect to ρ, u respectively.

From (6.6), $w\ \in \partial_u\ \mathcal{L}(\rho, u)$ if and only if $w \in \{-\rho + v : v \in \partial \mathcal{V}_\varepsilon^*(u)\ \}$.

Since $\mathcal{V}_\varepsilon$ is convex, and weakly $l.s.c., v \in\ \partial \mathcal{V}_\varepsilon^*(u)$ if and only if $u \in \partial\ \mathcal{V}_\varepsilon(v)$, see [8], theorems 51.2 and 51.6. Thus $0 \in \partial_u\ \mathcal{L}(\rho, u)$ if $\rho \in\ \partial\ \mathcal{V}_\varepsilon^*(u)$ or $u \in \partial\ \mathcal{V}_\varepsilon\ (\rho)$.

That is $\quad u = \varepsilon|\rho|^{\gamma-2}\ \rho + V\rho$ from the definition (6.4) of $\mathcal{V}_\varepsilon(\rho)$. Since $\rho(x) \geq 0$ a.e. for ρ in D, (6.7) follows. Using the same argument as in theorem 1, (6.8) follows as the extremality condition for minimizing $\mathcal{L}\ (\rho, \hat{u})$ over ρ in D. □

7. A Descent Algorithm.

The minimization of $\mathcal{E}$ on D was shown in the last section to be equivalent to minimizing $\mathcal{L}_\varepsilon$ on $D \times X^*$. The two problems have related minimizers and the same values.

Computationally however it is advantageous to work with $\mathcal{L}_\varepsilon$ instead of $\mathcal{E}$. The original problem $(\mathcal{P})$ a requires the minimization of the non-convex function $\mathcal{E}$ over the closed, convex set D. To solve $(\mathcal{Q})$, however, one minimizes the function $\mathcal{L}_\varepsilon$ which is convex in each of ρ and u, over the convex domain $D \times X^*$. This can be regarded as a sequence of convex programming problems and leads to the following algorithm.

Given $\rho^{(0)}$ in D, for $k \geq 0$

(1) compute $\quad u^{(k)} = \varepsilon(\rho^{(k)})^{\gamma-1}\ + V(\rho^{(k)}).$ (7.1)

(2) find $\rho^{(k+1)}$ in D obeying

$$\mathcal{L}_\varepsilon\ (\rho^{(k+1)},\ u^{(k)}) = \inf_{\rho \varepsilon D} \mathcal{L}_\varepsilon(\rho, u^{(k)}) \tag{7.2}$$

(3) if $\rho^{(k+1)} \ = \ \rho^{(k)}$ stop, else go to 1.

Step 1 chooses $u^{(k)}$ to be a minimizer of the convex function $\mathcal{L}_\varepsilon\ (\rho^{(k)}, .)$ on X^*. From the expression (6.5) for $\mathcal{L}_\varepsilon$, this minimizer obeys

$$\rho^{(k)} \ \in \ \partial\, \mathcal{V}_\varepsilon^*\ (u^{(k)}).$$

Since $\mathcal{V}_\varepsilon$ is proper, weakly *l.s.c.* and X is reflexive, this holds if and only if

$$u^{(k)} \ \in \ \partial\, \mathcal{V}_\varepsilon\ (\rho^{(k)})$$

which is (7.1) as $\mathcal{V}_\varepsilon$ is differentiable. Thus (7.1) is the explicit formula for the solution of this minimization problem, and is the iterative version of (6.7).

Similarly step 2 specifies $\rho^{(k+1)}$ to be the minimizer of a strictly convex, coercive function on the closed, convex set D. This minimizer exists, is unique and there is a constant λ_{k+1} such that $\rho^{(k+1)}$ obeys

$$(\gamma K_o + \varepsilon)\ \rho(x)^{\gamma-1} \ + \ W_\rho(r) \ \geq \ \lambda_{k+1} \ + u^{(k)}\ (x) \tag{7.3}$$

a.e. on B_R with equality on any open set where $\rho^{(k+1)}$ is positive. This is proven just as in theorems 1 and 4.

If $\rho^{(k+1)} \ = \ \rho^{(k)}$, then $(\rho^{(k)},\ u^{(k)})$ is a solution of the system (6.7) – (6.8) so $\rho^{(k)}$ will be a solution of (4.2). The analysis of section 4 shows that any solution of (4.2) defines a classical solution of our problem so the stopping criteria in step 3 is justified.

When $\rho^{(k+1)} \neq \rho^{(k)}$, then since $\mathcal{L}_\varepsilon\ (.,u^{(k)})$ is strictly convex on D, it follows that

$$\mathcal{L}_\varepsilon\ (\rho^{(k+1)}\ ,u^{(k)}) < \mathcal{L}_\varepsilon\ (\rho^{(k)},\ u^{(k)}) \leq \mathcal{L}_\varepsilon\ (\rho^{(k)},\ u^{(k-1)}) \tag{7.4}$$

so $\{(\rho^{(k)},\ u^{(k)}) : k \geq 0\}$ is a strict descent sequence for $\mathcal{L}_\varepsilon$.

Using (iii) of theorem 3, and (7.1), one has

$$\mathcal{E}\ (\rho^k) = \mathcal{L}_\varepsilon\ (\rho^{(k)},\ u^{(k)})$$

as $u^{(k)}$ minimizes $\mathcal{L}_\varepsilon\ (\rho^{(k)},.)$ on X^*. Thus the sequence $\{\rho^{(k)}\ : k \geq 0\}$ generated by this algorithm is a strict descent sequence for $\mathcal{E}$.

These results may be summarized as follows.

<u>THEOREM 5</u>. *Suppose (D4) holds and* $\{\rho^{(k)}\ : k \geq 0\}$ *is defined by this algorithm. If* $\rho^{(k+1)} \neq \rho^{(k)}$, *then* $\mathcal{E}\ (\rho^{(k+1)}) < \mathcal{E}(\rho^{(k)})$ *while if* $\rho^{(k+1)} = \rho^{(k)}$, *then* $\rho^{(k)}$ *is a solution of (4.2). If* $\gamma > \frac{4}{3}$ *this sequence is bounded in* X *and has at least one weak limit point.*

Proof. The descent results were proven above. When $\gamma > \frac{4}{3}$, since $\mathcal{E}(\rho^{(k)}) < \mathcal{E}(\rho^{(o)})$ for all $k > 0$, (5.3) implies that $||\rho^{(k)}||_\gamma$ is uniformly bounded.

Since X is reflexive, this implies that $\{\rho^{(k)}\ : k \geq 0\}$ is a subset of a weakly compact set, so it has a weak limit point. □

One would like to prove that this weak limit point is at least a local minimizer of $\mathcal{E}$ on D. This remains an open question but the following holds.

<u>THEOREM 6</u>. *Suppose* $\gamma > \frac{4}{3}$, *(D4) holds and* $\{\rho^{(k)}\ : k \geq 0\}$ *is defined by (7.1) – (7.2). If this sequence converges strongly to a* $\hat{\rho}$ *in* D, *then* $\hat{\rho}$ *is a solution of (4.2).*

Proof. From (7.1) and (7.3), upon multiplying both sides by $\rho^{(k+1)}$, one has

$$(\gamma K_o + \varepsilon)\ (\rho^{(k+1)}\ (x))^{\gamma} + \rho^{(k+1)}\ [W_{k+1}\ (r) - \lambda_{k+1}\ - \varepsilon(\rho^{(k)}(x))^{\gamma-1} - V\rho^{(k)}_{(x)}] = 0$$

$$\text{where} \qquad W_{k+1}\ (r) = W_{\rho^{(k+1)}}\ (r). \tag{7.5}$$

Since $\rho^{(k)}$ converges strongly to $\hat{\rho}$ in X, then $m^{(k)}$ converges uniformly to $\hat{m}$ on $[0, R]$ and so W_{k+1} converges uniformly to $\hat{W}$ on $[\delta, R]$ for each $\delta > 0$.

Choose $A = \{x \in B_R : r(x) \geq r_1\}$ where r_1 is chosen so that $\int\limits_A \hat{\rho}\ dx = \frac{M}{2}$. Then $r_1 > 0$. Integrate (7.5) over A and let k go to ∞, then

$$\lim_{k\to\infty} \int_A \{\gamma K_0\ (\rho^{(k+1)}\ (x))^{\gamma-1}\ + W_{k+1}\ (r) - V\rho^{(k)}(x)\}\rho^{(k+1)}\ (x)dx$$
$$= \int_A\ [\gamma K_0(\hat{\rho}(x)^{\gamma-1}\ + \hat{W}(r) - V\hat{\rho}(x)]\hat{\rho}(x)dx = \lim_{k\to\infty}\ \lambda_{k+1} \int\limits_A \rho^{(k+1)}\ (x)dx.$$

Thus there is a $\hat{\lambda}$ in $\mathbb{R}$ such that λ_k converges to $\hat{\lambda}$ as $k \to \infty$ since this last integral converges to $\frac{1}{2}M$.

Multiply both sides of (7.3) by a non-negative function h in X obeying (H3), and integrate over B_R, then

$$\int [(\gamma K_0 + \varepsilon)(\rho^{(k+1)})^{\gamma-1}\ + W_{k+1}\ - V\rho^{(k)}\ - \varepsilon(\rho^{(k)})^{\gamma-1}\ ]\ h(x)dx \geq \lambda_{k+1} \int\ h\ dx$$

Let k go to infinity here, then

$$\int\ [\gamma K_0\hat{\rho}^{\gamma-1}\ + \hat{W} - V\hat{\rho}]\ h(x)dx \geq \hat{\lambda} \int\ h\ dx$$

There are sufficiently many such h that

$$\gamma K_0(\hat{\rho}(x))^{\gamma-1}\ + \hat{W}(r) - V\hat{\rho}(x) \geq \lambda \quad a.e. \text{ on } B_R$$

upon letting δ decrease to 0.

Let A be an open set in B_R such that $\hat{\rho}(x) \geq \mu_1 > 0$ on A and choose h in X to obey (H3) and have support in A. Choose a subsequence $\{\rho^{(k_j)} : j \geq 0\}$ if necessary, such that $\rho^{(k_j)}(x)$ converges to $\hat{\rho}(x)$ *a.e.* on B_R. Then multiplying (7.3) by h and integrating over B_R, one finds

$$\int h[(\gamma K_0 + \varepsilon)\,(\rho^{(k_j)})^{\gamma-1} + W - \lambda_{k_j} - V\rho^{(k_j-1)}) - \varepsilon(\rho^{(k_j-1)})^{\gamma}\,]dx = \delta_j$$

where δ_j is the integral of this left hand side over

$$E_j = \{x \in \text{ supp } h \;:\; \rho^{(k_j)}(x) = 0\}$$

The measure of E_j may be made arbitrarily small using Egoroff's theorem and this integrand is integrable so $\delta_j \to 0$ as $j \to \infty$. Taking this limit

$$\int h\;[\gamma K_0 \hat{\rho}^{\gamma-1} - \hat{W} - V\hat{\rho}]\;dx = \hat{\lambda}\int h\;dx$$

There are enough such h to conclude that equality holds in (4.2) *a.e.* on any open set where $\hat{\rho}(x)$ is positive and $r(x) \geq \delta$. Thus $\hat{\rho}$ is a solution of (4.2) as claimed. □

The theorems in section 6 of [3] show that an iteration of this type will converge if one can guarantee sufficient descent at each step of the algorithm. That is, one has a good descent estimate for these iterates. It would be interesting to know whether such estimates hold for this algorithm, or if there are other methods of proving convergence of this algorithm directly.

References

1. Auchmuty, J.F.G., and R. Beals, *Variational Solution of some Nonlinear Free Boundary Problems,* Arch. Rat. Mech. and Anal. **43,** (1971), 255-271

2. Auchmuty, G., *Duality for Non-convex Variational Principles*, J. Diff. Eqns. **50**, (1983), 80-145.

3. Auchmuty, G., *Duality Algorithms for Non-convex Variational Principles*, J. Numer. Funct. Anal. and Optimiz. **10**, (1989), 211-264.

4. Auchmuty, G., *The Global Branching of Rotating Stars,* Arch. Rat. Mech. and Anal. (to appear).

5. Chandrasekhar, S., *Introduction to the Study of Stellar Structure,* University of Chicago Press (1939), Chicago.

6. Sobolev, S.L., *Applications of Functional Analysis in Mathematical Physics,* American Math. Soc. (1963), Providence, 1963.

7. Tassoul, J.L., *Theory of Rotating Stars,* Princeton University Press. Princeton, 1979.

8. Zeidler, E., *Nonlinear Functional Analysis and its Applications, III. Variational Methods and Optimization,* Springer Verlag, Berlin, 1985.

Department of Mathematics
University of Houston
Houston, TX 77204-3476