REPRESENTATIONS OF COMPACT GROUPS, CUNTZ-KRIEGER ALGEBRAS, AND GROUPOID C^*-ALGEBRAS(*)

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Doplicher and Roberts have recently showed how to recover a compact Lie group G from a single faithful representation ρ of G in $SU_n(\mathbf{C})$, via a C^*-algebra $\mathcal{O}_\rho$, constructed from the intertwiners of the tensor powers of ρ, and an endomorphism of $\mathcal{O}_\rho$ [3, 4]. The key idea is that the tensor powers ρ^n contain every irreducible representation $\pi \in \hat{G}$, so their intertwiners should contain information about the decompositions of $\pi_1 \otimes \pi_2$ for all $\pi_i \in \hat{G}$, and hence characterise G. We found it intriguing that the theory is based on just one representation ρ, apparently randomly chosen, and attempted to understand how this works. As a first step, we investigated the structure of the algebra $\mathcal{O}_\rho$, and how it depends on ρ.

Our first plan was to identify $\mathcal{O}_\rho$ as the C^*-algebra of a locally compact groupoid $\mathcal{P}$, and exploit the theory of groupoid C^*-algebras [7]. Since $\mathcal{O}_\rho$ is constructed from finite-dimensional pieces, and in particular has a large AF core, we looked at the Bratteli diagram of this core. It has a good deal of vertical symmetry — indeed, one can identify the vertices at each level with the set $\hat{G}$. Thus the path space X of the diagram carries a natural shift, and the groupoid $\mathcal{P}$ is a subset of the groupoid semidirect product $X \times X \times \mathbf{Z}$ with an appropriate topology. Next, we noticed that by enlarging the path space X, we obtained a similar groupoid whose C^*-algebra was the Cuntz-Krieger

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algebra $\mathcal{O}_{A_\rho}$ of a $\{0,1\}$-matrix A_ρ naturally associated to ρ; this suggested that we analyse $\mathcal{O}_\rho$ by relating it to the relatively well-understood algebra $\mathcal{O}_{A_\rho}$.

It turned out that one can bypass the groupoid construction, and relate $\mathcal{O}_\rho$ directly to $\mathcal{O}_{A_\rho}$. For finite groups, this works beautifully: $\mathcal{O}_\rho$ is isomorphic to a corner in the simple C^*-algebra $\mathcal{O}_{A_\rho}$, and hence in particular has the same K-theory. Since Cuntz has computed $K_*(\mathcal{O}_A)$, this immediately gives us interesting invariants of the $\mathcal{O}_\rho$, and we can deduce that the structure of $\mathcal{O}_\rho$ does indeed vary considerably with ρ.

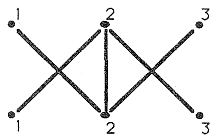
The details of this direct approach were worked out in [5]; in §1, we discuss the main ideas, and what might be involved in extending this analysis to cover the case of compact G. In §2, we outline our original construction of the groupoid $\mathcal{P}$. Even though this realisation may not at present provide as much information about $\mathcal{O}_\rho$, it does raise some interesting side questions, which may have more general significance for the groupoid approach to C^*-algebras.

Throughout, ρ will be a faithful representation of a compact group in a Hilbert space H_ρ with $1 < \dim H_\rho < \infty$; often we shall also require that $\rho(G) \subset SU(H_\rho)$ or G is finite. Let ρ^n denote the nth tensor power of ρ, acting in H_ρ^n, and let (ρ^m, ρ^n) denote the space of intertwining operators $T : H_\rho^n \to H_\rho^m$. If we also have $S \in (\rho^n, \rho^p)$, then $T \circ S$ lies in (ρ^m, ρ^p), and by identifying $T \in (\rho^m, \rho^n)$ with $T \otimes 1 \in (\rho^{m+1}, \rho^{n+1})$, composition extends to give a multiplication on ${}^0\mathcal{O}_\rho = \bigcup_{m,n}(\rho^m, \rho^n)$. With the natural involution $T \mapsto T^*$, which maps (ρ^m, ρ^n) to (ρ^n, ρ^m), ${}^0\mathcal{O}_\rho$ is a *-algebra, and the **Doplicher-Roberts algebra** $\mathcal{O}_\rho$ is its C^*-enveloping algebra.

1. CUNTZ-KRIEGER ALGEBRAS.

Let A be an $N \times N$ matrix with entries in $\{0,1\}$. We define an infinite graph $\mathcal{G}$ as follows. First, we define a building block by taking two sets of N vertices, at two different levels, with an edge joining i at the upper level to j at the lower exactly when $A(i,j) = 1$. Thus, for example,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{yields}$$



Now $\mathcal{G}$ is obtained by sticking infinitely many copies of this block below each other, continuing to label the vertices at each level by $\{1, 2, \cdots, N\}$. The matrix A is **irreducible** if, for each pair i, j, there is a connected path in $\mathcal{G}$ joining i at the top level to j at some lower level, and **aperiodic** if there is an infinite aperiodic path going down the graph.

Cuntz and Krieger proved that, if A is irreducible and aperiodic, then the C^*-algebra generated by N non-zero partial isometries S_i satisfying

$$S_i^* S_i = \textstyle\sum_j A(i,j) S_j S_j^* \tag{1}$$

is unique up to isomorphism, and is simple [2]. The resulting uniquely-defined C^*-algebra is now called the **Cuntz-Krieger algebra**, and is denoted $\mathcal{O}_A$. Later, Cuntz [1] calculated the K-theory of $\mathcal{O}_A$:

$$K_1(\mathcal{O}_A) = \ker\big((1 - A^t) : \mathbf{Z}^N \to \mathbf{Z}^N\big)$$

$$K_0(\mathcal{O}_A) = \operatorname{coker}\big((1 - A^t) : \mathbf{Z}^N \to \mathbf{Z}^N\big).$$

The proof of the Cuntz-Krieger theorem is roughly as follows. Equation (1) implies that the S_i have orthogonal ranges, so that $S_j^* S_i = 0$ for $j \neq i$; thus one can use (1) to move all adjoints S_j^* to the right of any S_i's, and each word in S_j^* and S_i equals one of the form $S_\mu S_\nu^* = \big(S_{\mu_1} S_{\mu_2} \cdots S_{\mu_m}\big)\big(S_{\nu_n}^* \cdots S_{\nu_1}^*\big)$. The product S_μ is non-zero precisely when $A(\mu_i, \mu_{i+1}) = 1$ for $i = 1, \cdots, m-1$ — that is, precisely when the vertices $\mu_1, \mu_2, \cdots, \mu_m$ lie along a connected path in the infinite graph $\mathcal{G}$ associated to A. Then

$C^*(S_i) = \overline{\text{sp}}\{S_\mu S_\nu^*\}$ is naturally graded by the subspaces $\mathcal{O}_A^k = \overline{\text{sp}}\{S_\mu S_\nu^* : n-m = k\}$ for $k \in \mathbf{Z}$, and Cuntz-Krieger mimic O'Donovan's proof of simplicity for crossed products $B \times \mathbf{Z}$ [6] to deduce that $C^*(S_i)$ is simple.

Now suppose $\rho : G \to U_n(\mathbf{C})$ is a representation of a finite group G. Associated to ρ is another bipartite building block. Here the vertices are two copies of $\hat{G}$, and the number of edges from π_1 at the top level to π_2 at the lower is the multiplicity of π_2 in $\pi_1 \otimes \rho$; if e is such an edge, we write $s(e) = \pi_1$, $r(e) = \pi_2$. Again we form an infinite graph $\mathcal{G}_\rho$ by sticking copies of this block on top of each other. This time each finite path of length n starting at the trivial representation ι represents an irreducible summand of ρ^n: the first edge gives a summand π of ρ, the second a summand of $\pi \otimes \rho \subset \rho^2$, and so on. We can make this precise by letting the edges e from π_1 to π_2 represent a fixed family of isometric intertwiners $T_e : H_{\pi_2} \to H_{\pi_1} \otimes H_\rho$, such that $\bigoplus\{\text{range}\, T_e : s(e) = \pi_1\} = H_{\pi_1} \otimes H_\rho$. Then the path $x = (x_1, x_2, \cdots, x_m)$ starting at ι determines an isometric intertwiner

$$T_x = \left(T_{x_1} \otimes 1_{m-1}\right) \circ \left(T_{x_2} \otimes 1_{m-2}\right) \circ \cdots \circ T_{x_m} : H_{r(x_m)} \to H_\rho^m,$$

and the set $\{T_x T_y^* : |x| = m, |y| = n\}$ is a basis for (ρ^m, ρ^n).

Thus $\mathcal{O}_\rho$ is also spanned by a family $\{T_x T_y^*\}$ of partial isometries parametrised by pairs of finite paths in an infinite graph. There are two differences, though: here the paths all start at a fixed vertex ι, and the paths are determined by sequences of edges rather than vertices. The second of these is easily dealt with by passing to the dual graph: we let E be the set of edges in the bipartite block, and define an $E \times E$ matrix $A = A_\rho$ by

$$A_\rho(e, f) = \begin{cases} 1 & \text{if } r(e) = s(f) \\ 0 & \text{otherwise.} \end{cases}$$

The graph $\mathcal{G}$ associated to A_ρ has the same paths as $\mathcal{G}_\rho$, and setting $\phi(T_x T_y^*) = S_x S_y^*$

gives a homomorphism ϕ of ${}^0\mathcal{O}_\rho = \bigcup(\rho^m, \rho^n)$ onto the subspace

$$\bigcup \mathrm{sp}\{S_x S_y^* : s(x_1) = s(y_1) = \iota, |x| = m, |y| = n\}$$

of $\mathcal{O}_{A_\rho} = C^*(S_e)$. The requirement that the paths start at ι means that ϕ maps ${}^0\mathcal{O}_\rho$ into the corner $P\mathcal{O}_{A_\rho}P$, where $P = \sum_{\{e:s(e)=\iota\}} S_e S_e^*$.

THEOREM 1. [5] *If G is finite, the map ϕ induces an isomorphism of $\mathcal{O}_\rho$ onto the corner $P\mathcal{O}_{A_\rho}P$.*

Since the $\{T_x T_y^*\}$ form a basis for each (ρ^m, ρ^n), it is not hard to see that ϕ is an isomorphism on each graded piece ${}^0\mathcal{O}_\rho^k = \bigcup(\rho^n, \rho^{n+k})$. It is not so obvious that ϕ is an isomorphism on ${}^0\mathcal{O}_\rho = \bigoplus_k {}^0\mathcal{O}_\rho^k$; for this one has to check that the images $\phi({}^0\mathcal{O}_\rho^k)$ are independent in $\mathcal{O}_A$, and this seems to depend on some of the non-trivial properties of $\mathcal{O}_A$ established by Cuntz and Krieger. Even when one knows that ϕ is an isomorphism on ${}^0\mathcal{O}_\rho$, one still has to prove that the C^*-enveloping norm agrees with the one inherited from $\mathcal{O}_A$; this is established in [5, §3] by showing how to realise every representation of $\phi({}^0\mathcal{O}_\rho)$ as a compression of a representation of $\mathcal{O}_A$ to a subspace of finite codimension. The result, however, is that one can deduce properties of $\mathcal{O}_\rho$ from those of $\mathcal{O}_A$.

Standard facts from the representation theory of finite groups show that $A = A_\rho$ is irreducible and aperiodic, so $\mathcal{O}_A$ is simple. Thus the corner $P\mathcal{O}_A P \cong \mathcal{O}_\rho$ is stably isomorphic to $\mathcal{O}_A$, and hence has the same K-theory. At least for finite G, therefore, we have

$$K_1(\mathcal{O}_\rho) = \ker\big((1 - A_\rho^t) : \mathbf{Z}^N \to \mathbf{Z}^N\big)$$

$$K_0(\mathcal{O}_\rho) = \mathrm{coker}\big((1 - A_\rho^t) : \mathbf{Z}^N \to \mathbf{Z}^N\big),$$

where N is the cardinality of the set E. It turns out that one can quite easily compute A_ρ from a character table for G, and then, provided N is small enough, one can solve the equation $(1 - A_\rho^t)u = v$ by hand. This is done for several examples in [5, §4], where there is also a discussion of some useful shortcuts. One result is that for $G = A_5$, there are distinct irreducible, faithful representations ρ_1, ρ_2 with

$$K_1(\mathcal{O}_{\rho_1}) = \mathbf{Z}, \qquad K_0(\mathcal{O}_{\rho_1}) = \mathbf{Z} \times \mathbf{Z}_2 \times \mathbf{Z}_2,$$

$$K_1(\mathcal{O}_{\rho_2}) = 0, \qquad K_0(\mathcal{O}_{\rho_2}) = \mathbf{Z}_4.$$

In particular, the algebras $\mathcal{O}_{\rho_1}, \mathcal{O}_{\rho_2}$ are not even stably isomorphic or Morita equivalent.

For compact G, we can still deduce that ϕ induces a homomorphism of $\mathcal{O}_\rho$ onto $PC^*(S_e)P$, which has to be an isomorphism because we know from [3] that $\mathcal{O}_\rho$ is simple. However, this is a little unsatisfactory: our goal was to glean new insight into the structure of $\mathcal{O}_\rho$, and we could at least hope to have its basic properties emerge as corollaries. So far, though, our process for extending representations from $\phi({}^0\mathcal{O}_\rho)$ to $\mathcal{O}_A$ only works for finite groups. And there are more pressing problems: we don't know enough about the Cuntz-Krieger algebras $\mathcal{O}_A$ for infinite A. Cuntz and Krieger did assert that their results extend to infinite A, but the method we have found of doing this does not enable us to extend Cuntz's calculation of $K_*(\mathcal{O}_A)$. So our current goal is to compute $K_*(\mathcal{O}_A)$ by methods which will work when A is infinite.

2. GROUPOID C^*-ALGEBRAS.

We consider the space X of infinite paths in the graph $\mathcal{G}_\rho$ which start at the trivial representation ι, viewing them as sequences of edges, so

$$X = \{x \in \prod_{i=1}^{\infty} E : s(x_1) = \iota, r(x_n) = s(x_{n+1}) \text{ for } n \geq 1\}.$$

The path groupoid $\mathcal{P}$ is the set

$$\mathcal{P} = \{(x, y, k) \in X \times X \times \mathbf{Z} : x_n = y_{n-k} \text{ for large } n\},$$

with range, source maps defined by $r(x, y, k) = x$, $s(x, y, k) = y$, and

$$(x, y, k)(y, z, l) = (x, z, k + l)$$

$$(x, y, k)^{-1} = (y, x, -k).$$

The path space X is compact in the product topology, because at each level only finitely many edges are accessible from ι. For finite paths α, β of length $|\alpha|, |\beta|$ starting at ι, and with $r(\alpha_{|\alpha|}) = r(\beta_{|\beta|})$, we let

$$Z(\alpha, \beta) = \{(x, y, k) \in \mathcal{P} : k = |\beta| - |\alpha|, x_i = \alpha_i \text{ for } i \leq |\alpha|, \ \ y_j = \beta_j \text{ for } j \leq |\beta|\}.$$

LEMMA 2. *The sets $Z(\alpha, \beta)$ are a basis of compact open sets for a locally compact topology on $\mathcal{P}$, and $\mathcal{P}$ is then a locally compact amenable groupoid for which the counting measures form a Haar system.*

This lemma is not quite as innocuous as it looks. The idea is that if $\Omega \subset \prod_{-\infty}^{\infty} E$ is the space of two-sided paths, then

$$\mathcal{S} = \{(x, y) \in \Omega \times \Omega : a_n = b_n \text{ for large } n\}$$

is a groupoid, and $\mathcal{P}$ is a reduction of the semidirect product of $\mathcal{S}$ by the shift homeomorphism. If E is finite, $\mathcal{S}$ can be made into a locally compact amenable groupoid, and this property is preserved by taking semidirect products and reducing [7, p.96, p.92]. If E is infinite — as is the case for compact G — the space Ω is not even locally compact in the product topology, and one must first compactify the space E of edges, using a modification of the construction in [7, p.139].

There is a natural map ϕ of the Doplicher-Roberts algebra ${}^0\mathcal{O}_\rho$ into $C_c(\mathcal{P})$ which sends the intertwiner $T_{\alpha,\beta} \in (\rho^m, \rho^n)$ to the characteristic function $1_{Z(\alpha,\beta)} \in C_c(\mathcal{P})$; this is well-defined on each (ρ^m, ρ^n) since the $T_{\alpha,\beta}$ form a basis, and respects the embeddings of (ρ^m, ρ^n) in (ρ^{m+1}, ρ^{n+1}) because

$$T_{\alpha,\beta} \otimes 1 = \sum_{\{e : s(e) = r(\alpha_{|\alpha|})\}} (T_{\alpha,\beta} \otimes 1) \circ T_e T_e^* = \sum T_{\alpha e, \beta e}$$

maps into

$$\sum 1_{Z(\alpha e, \beta e)} = 1_{\bigcup Z(\alpha e, \beta e)} = 1_{Z(\alpha, \beta)}.$$

LEMMA 3. *The map ϕ is a *-isomorphism of ${}^0\mathcal{O}_\rho$ onto the *-subalgebra of $C_c(\mathcal{P})$ spanned by the functions $1_{Z(\alpha,\beta)}$.*

As in §1, the slight subtlety here concerns the grading of ${}^0\mathcal{O}_\rho$: it is not obvious that the images of ${}^0\mathcal{O}_\rho^k$ are independent in $C_c(\mathcal{P})$, i.e. that $\sum_{\alpha,\beta}\phi(T_{\alpha,\beta}) = 0$ implies $\sum_{|\beta|-|\alpha|=k}\phi(T_{\alpha,\beta}) = 0$ for all k. However, if we define $\beta_z(f)(x,y,k) = z^k f(x,y,k)$, then β_z is a *-automorphism of $C_c(\mathcal{P})$ which is isometric for the norm $\|\cdot\|_I$ (see [7, p.50]), and hence extends to a *-automorphism of $C^*(\mathcal{P})$, which is by definition the enveloping algebra of $C_c(\mathcal{P})$ with respect to $\|\cdot\|_I$-bounded representations. The map $z \to \beta_z(f)$ is continuous for the inductive limit topology on $C_c(\mathcal{P})$, hence for the C^*-norm topology, and β is a continuous action of $\mathbf{T}$ on $C^*(\mathcal{P})$. We have

$$\beta_z\big(1_{Z(\alpha,\beta)}\big) = z^{|\beta|-|\alpha|}1_{Z(\alpha,\beta)},$$

and hence the inequality $\|\int z^{-k}\beta_z(b)\,dz\| \le \|b\|$ translates into

$$\Big\|\sum_{|\beta|-|\alpha|=k}\phi(T_{\alpha,\beta})\Big\| \le \Big\|\sum_{\alpha,\beta}\phi(T_{\alpha,\beta})\Big\|$$

for all finite sums. Since the canonical map of $C_c(\mathcal{P})$ into $C^*(\mathcal{P})$ is injective [7, Proposition II.1.11], this shows that ϕ is injective.

THEOREM 4. *If G is finite, or if G is compact and $\rho : G \to SU_n$, then the Doplicher-Roberts algebra is isomorphic to $C^*(\mathcal{P})$.*

Since ${}^0\mathcal{O}_\rho$ has a unique C^*-seminorm (by [3, Theorem 2.12] in the compact case, our Theorem 1 in the finite case), the isomorphism ϕ must be isometric and extend to an isomorphism of the completion $\mathcal{O}_\rho$ into $C^*(\mathcal{P})$. However, since the sets $Z(\alpha,\beta)$ are compact and open, standard arguments allow one to approximate a function f in $C_c(\mathcal{P})$ uniformly on its support by a combination of $1_{Z(\alpha,\beta)}$'s, so the image of ${}^0\mathcal{O}_\rho$ is dense in $C_c(\mathcal{P})$, and the image of $\mathcal{O}_\rho$ must be all of $C^*(\mathcal{P})$.

As in the previous section this proof is rather unsatisfying: one would prefer the basic facts about $\mathcal{O}_\rho$ to be consequences of the general theory of groupoid C^*-algebras. Under either set of hypotheses on G and ρ, one can use standard representation theory to see that the groupoid $\mathcal{P}$ is essentially principal, as in [7, p.100], and hence it follows from [7, Proposition II.4.6] that $C^*(\mathcal{P})$ is simple. It would still take some work to deduce from this and Lemma 3 that $\mathcal{O}_\rho$ is simple: $\phi({}^0\mathcal{O}_\rho) = \mathrm{sp}\{1_{Z(\alpha,\beta)}\}$ is not necessarily all of $C_c(\mathcal{P})$, and one would have to show that $\phi({}^0\mathcal{O}_\rho)$ and $C_c(\mathcal{P})$ have the same enveloping algebra.

For finite G, the Cuntz-Krieger algebra $\mathcal{O}_A$ is also a groupoid C^*-algebra $C^*(\mathcal{G}_A)$ — one just replaces the space X by the space of all paths in $\prod_{i=1}^{\infty} E$ — and one can prove Theorem 1 by identifying $C^*(\mathcal{P})$ with a corner in $C^*(\mathcal{G}_A)$. For more general compact G, the matrix A is infinite, the path space is not locally compact, and, although we have tried quite hard, we have been unable to find a *locally compact* groupoid whose C^*-algebra is $\mathcal{O}_A$. Thus it seems that, at least for the purpose of calculating $K_*(\mathcal{O}_\rho)$ via computations of $K_*(\mathcal{O}_A)$, the approach in §1 is more promising.

REFERENCES.

1. J. Cuntz, A class of C^*-algebras and topological Markov chains II: reducible chains and the Ext-functor for C^*-algebras, *Invent. Math.* **63** (1981), 25–40.
2. J. Cuntz and W. Krieger, A class of C^*-algebras and topological Markov chains, *Invent. Math.* **56** (1980), 251–268.
3. S. Doplicher and J.E. Roberts, Duals of compact Lie groups realised in the Cuntz algebras, and their actions on C^*-algebras, *J. Funct. Anal.* **74** (1987), 96–120.
4. S. Doplicher and J.E. Roberts, Endomorphisms of C^*-algebras, cross products and duality for compact groups, *Ann. of Math.* **130** (1989), 75–119.
5. M.H. Mann, I. Raeburn and C.E. Sutherland, Representations of finite groups and Cuntz-Krieger algebras, *Bull. Austral. Math. Soc.*, to appear.
6. D.P. O'Donovan, Weighted shifts and covariance algebras, *Trans. Amer. Math. Soc.* **208** (1975), 1-25.
7. J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Math., vol. 793, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

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