CLIFFORD MARTINGALES, THE $T(b)$ THEOREM AND CAUCHY INTEGRALS

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1. INTRODUCTION

Clifford analysis has proved in the last few years to be particularly useful in the study of singular integrals on curves and surfaces [**13**], [**14**], [**16**], [**17**], [**18**] . While Clifford algebras have enjoyed a deal of popularity and success in mathematical physics [**2, Notes to Chap. 1**], [**11**] and were a subject of much attention from the 1930's onwards [**20**], the impetus for using them in the study of analysis on curves and surfaces is recent, and due to R. Coifman.

The purpose of this article is to give some indications of recent work by the author, in collaboration with R. Long and T. Qian, on Clifford martingales and their application to the proof of a suitable version of the $T(b)$ theorem for Clifford-valued functions, and to the L^2-boundedness of the Cauchy principal value integral on Lipschitz surfaces. The full details are to appear elsewhere. The results are not new, but the methods of proof are. They are inspired by the paper of Coifman, Jones and Semmes [**3**]. One half of their paper gives a proof of the boundedness of the Cauchy integral on Lipschitz curves using dyadic partitions with respect to arc-length, and the corresponding Haar functions and martingales. There are significant differences in the present context, in that it is necessary to construct a novel dual system of left- and right-martingales in order to cope with the noncommutativity of the Clifford algebra. Full details are to appear elsewhere [**11**].

It is a pleasure to acknowledge the influence of Michael Cowling and Alan McIntosh, who encouraged us to carry through this programme of research.

1991 *Mathematics Subject Classification.* 31B05 42B20 47S10

2. THE CLIFFORD ALGEBRA $\mathbb{A}_d$

We begin with the Euclidean space $\mathbb{R}^{1+d}$, with basis $e_0, e_1, \ldots, e_d$. The Clifford algebra $\mathbb{A}_d$ is the 2^d-dimensional algebra generated by $\{e_0, e_1, \ldots, e_d\}$, subject to the relations

(i) $e_0 = 1$;

(ii) $e_j^2 = -1$ for $1 \le j \le d$;

(iii) $e_j e_k = -e_k e_j$ for $1 \le j < k \le d$.

A basis of $\mathbb{A}_d$. This is formed as follows. Take any subset $S \subseteq \{1, \ldots, d\}$. Define the element e_S of $\mathbb{A}_d$ as follows. If $S = \varnothing$, put $e_\varnothing = e_0 = 1$. Otherwise, write $S = \{j_1 < j_2 < \ldots < j_s\}$, and put $e_S = e_{j_1} e_{j_2} \cdots e_{j_s}$.

Inner product structure. We declare the set $\{e_S : S \subseteq \{1, \ldots, d\}\}$ to be orthonormal. The length of the element $x = \sum x_S e_S$ is $|x| = (\sum x_S^2)^{1/2}$.

Notable special cases of the Clifford algebra are: $d = 1$, $\mathbb{A}_1 \simeq \mathbb{C}$; $d = 2$, $\mathbb{A}_2 = \mathbb{H}$, the algebra of quaternions; $d = 3$, $\mathbb{A}_3 =$ the *Pauli algebra*.

Conjugation. This is the real-linear mapping whose action on the basis is given by

$$\overline{e}_S = \begin{cases} e_S & \text{if } |S| \equiv 0, 3 \pmod 4 \\ -e_S & \text{if } |S| \equiv 1, 2 \pmod 4 \end{cases}$$

In particular,

$$\Big(\sum_{j=0}^{d} x_j e_j\Big)^- = x_0 e_0 - x_1 e_1 - \cdots - x_d e_d.$$

One of the most important elementary reasons for the utility of the algebra $\mathbb{A}_d$ is that it is possible to invert nonzero vectors within $\mathbb{A}_d$:

$$x^{-1} = \frac{\overline{x}}{|x|^2}.$$

If G is the group generated by the nonzero elements of $\mathbb{R}^{1+d}$—the so-called *Clifford group*—then $|xy| = |x||y|$ for all $x, y \in G$ [**1**].

Some vector analysis. Let Ω be an open subset of $\mathbb{R}^{1+d}$, and $f = \sum_S f_S e_S$ a $C^{(1)}$-function with values in $\mathbb{A}_d$. The *left Dirac operator*

$$D_l = \sum_{j=0}^{d} e_j \frac{\partial}{\partial x_j}$$

acts on f as follows:

$$D_l f = \sum_{j=0}^{d} \sum_S \frac{\partial f_S}{\partial x_j} e_j e_S.$$

The function f is said to be *left monogenic* if $D_l f \equiv 0$ in Ω. The concepts of *right Dirac operator* and *right monogenic function* are defined similarly. In particular,

$$D_r f = \sum_{j=0}^{d} \sum_S \frac{\partial f_S}{\partial x_j} e_S e_j.$$

We write Df in place of $D_l f$ and fD in place of $D_r f$.

Each Dirac operator has a *conjugate operator*. For instance, $\bar{D}_l = \sum_{j=0}^{d} (\partial/\partial x_j)\bar{e}_j$. A basic fact about a Dirac operator and its conjugate (for both operators acting on the left, or both on the right) is that

$$D\bar{D} = \bar{D}D = \Delta_{d+1},$$

where Δ_{d+1} is the Laplacian on $\mathbb{R}^{1+d}$.

A function is said to be *monogenic in the open set* $\Omega \subseteq \mathbb{R}^{1+d}$ if it is both left- and right-monogenic in Ω.

THEOREM (GREEN'S THEOREM). *Let $\Omega \subseteq \mathbb{R}^{1+d}$ be an open set, U an open set whose closure lies in Ω, and is such that ∂U is an orientable d-manifold, with consistently defined exterior normal $n(y)$ ($y \in \partial U$). Then if $f, g \in C^{(1)}(\Omega; \mathbb{A}_d)$,*

$$\int_{\partial U} f(y)n(y)g(y)\, d\sigma(y) = \int_U (fD)g \;+\; f(Dg)\, dx.$$

The *Caucy kernel* is the kernel

$$C(x) = c_d^{-1} \frac{\bar{x}}{|x|^{1+d}},$$

where c_d is the volume of the unit d-sphere. The Cauchy kernel is monogenic where defined, and is a fundamental solution for D: $DC = CD = \delta_0$ in the distributional sense.

THEOREM. *The notation being as in the preceding theorem, we have*

$$\int_{\partial U} C(y-x)n(y)f(y)\,d\sigma(y) - \int_U C(y-x)(Df)(y)\,dy = \begin{cases} f(x), & x \in U \\ 0, & x \in \Omega\backslash\bar{U} \end{cases}$$

COROLLARY (CAUCHY'S THEOREM). *If f is left-monogenic, then*

$$\int_{\partial U} C(y-x)n(y)f(y)\,d\sigma(y) = \begin{cases} f(x), & x \in U \\ 0, & x \in \Omega\backslash\bar{U} \end{cases}$$

Proofs of these and other results about basic Clifford analysis can be found in [**2**].

3. THE CAUCHY SINGULAR INTEGRAL OPERATOR

Let Σ be an oriented d-dimensional surface in $\mathbb{R}^{1+d}$, $n(y)$ $(y \in \Sigma)$ the unit normal consistent with the orientation. The Cauchy (principal value) singular integral is given, for suitable $\mathbb{A}_d$-valued functions on Σ, by the formula

$$T_\Sigma f(x) = \text{p.v.} \int_\Sigma \frac{\overline{y-x}}{|y-x|^{1+d}} n(y)f(y)\,d\sigma(y).$$

For simplicity, we drop "p.v." from the various integrals in the sequel.

The Cauchy singular integral lies at the heart of the boundary behaviour of harmonic functions and their conjugates [**16**].

The theorem, which is proved by Clifford martingale methods, is the following.

THEOREM. *The Cauchy singular integral operator is bounded on $L^2(\Sigma;\mathbb{A}_d)$ if Σ is a Lipschitz graph.*

The manifold Σ is a *Lipschitz graph* if $\Sigma = \{A(v)e_0 + v : v \in \mathbb{R}^d\}$ and A is a scalar-valued function $\mathbb{R}^d$ such that $|\nabla A(v)| \leq M$. Note that if $\phi(v) = A(v)e_0 + v$, then n can be chosen to be

$$n(\phi(v)) = \frac{e_0 - \nabla A(v)}{\sqrt{1+|\nabla A|^2}}$$

and

$$T_\Sigma f(\phi(u)) = \int_{\mathbb{R}^d} \frac{\overline{\phi(v)-\phi(u)}}{|\phi(v)-\phi(u)|^{1+d}} \psi(v) f(\phi(v))\, dv,$$

where $\psi(v) = e_0 - \nabla A(v)$. So we study the mapping

$$h \longmapsto \int_{\mathbb{R}^d} K(u,v)\psi(v)h(v)\, dv = Th(u),$$

noting that $|\psi|$ is both bounded and bounded away from 0.

The boundedness of the Cauchy singular integral operator has already been proved by a number of other authors, beginning with the breakthrough in [4], using different approaches. See e.g. [6], [18] and the references given there.

4. CLIFFORD MARTINGALES

Let X be a set, $\mathcal{B}$ a σ-field of subsets of X, ν a nonnegative measure on $\mathcal{B}$, and $\{\mathcal{F}_n\}_{-\infty}^{\infty}$ a nondecreasing family of σ-fields satisfying the following conditions:

(i) $\bigcup_{-\infty}^{\infty} \mathcal{F}_n$ generates $\mathcal{B}$;

(ii) $\bigcap_{-\infty}^{\infty} \mathcal{F}_n = \{\varnothing, X\}$;

(iii) the measure ν is σ-finite on $\mathcal{B}$, and on each $\mathcal{F}_n$.

The standard conditional expectation. If $\mathcal{F}$ is a sub-σ-field of $\mathcal{B}$ such that ν is σ-finite on $\mathcal{F}$, then we can write $X = \bigcup_j U_j$ where $U_j \in \mathcal{F}$ and $\nu(U_j) < +\infty$. If f is an $\mathbb{A}_d$-valued $\mathcal{B}$-measurable function which is integrable on each set of finite ν-measure, we say that it is *locally integrable* and write $f \in L^1_{\text{loc}}(\mathcal{B}, \nu, \mathbb{A}_d)$ or, when $\mathcal{B}$ is understood, $f \in L^1_{\text{loc}}(X, \mathbb{A}_d)$. We can define the standard conditional expectation of such a function "given $\mathcal{F}$" $\tilde{E}(f|\mathcal{F})$ by defining it on each U_j. We get this way a function that is $\mathcal{F}$-measurable, and such that

$$\int_A \tilde{E}(f|\mathcal{F})\, d\nu = \int_A f\, d\nu \tag{1}$$

for each set $A \in \mathcal{F}$ of finite ν-measure. If f is integrable, (1) holds for all $A \in \mathcal{F}$.

Clifford right- and left-conditional expectations. Keeping in mind the discussion in §3, let $\psi \in L^\infty(X; \mathbb{A}_d)$, and suppose that $\tilde{E}(\psi|\mathcal{F}) \neq 0$ a.e.. If $f \in L^1_{\text{loc}}(\mathcal{B}, \nu, \mathbb{A}_d)$, the *left- and right-conditional expectations* E^l and E^r of f with respect to $\mathcal{F}$ are given by:

$$E^l(f) = E^l(f|\mathcal{F}) = \tilde{E}(\psi|\mathcal{F})^{-1}\tilde{E}(\psi f|\mathcal{F})$$
$$E^r(f) = E^r(f|\mathcal{F}) = \tilde{E}(f\psi|\mathcal{F})\tilde{E}(\psi|\mathcal{F})^{-1}.$$

If h and k are appropriately restricted $\mathbb{A}_d$-valued functions, we define

$$\langle h, k\rangle_\psi = \int_X h\psi k \, d\nu.$$

The pseudo-accretivity condition. Naturally enough, the operators E^l and E^r do not behave well without some further restriction on ψ.

PROPOSITION 1. *If* $1 \leq p \leq \infty$, *the operator* E^l *(resp.* E^r*) is bounded on* L^p *if and only if there exists a constant* $C_0 > 0$ *such that*

(2) $$C_0^{-1} \leq |\tilde{E}(\psi|\mathcal{F})(x)| \leq C_0 \qquad \textit{for a.e. } x.$$

A function that satisfies the condition (2) is called *pseudo-accretive.* We assume that the function ψ is pseudo-accretive with respect to all of our sub-σ-fields of $\mathcal{B}$, with a uniform constant C_0. Under this assumption, standard martingale formulas have their analogues in the new setting, with appropriate changes. E.g.

(i) $\int_A \psi E^l(f)\, d\nu = \int_A \psi f \, d\nu \qquad (f \in L^1(X;\mathbb{A}_d), A \in \mathcal{F};$ or $f \in L^1_{\text{loc}}(X;\mathbb{A}_d), A \in \mathcal{F}, \nu(A) < +\infty)$.

(ii) If $g \in \mathrm{L}^\infty(\mathcal{F}, d\nu; \mathbb{A}_d)$, then
$$E^l(fg) = E^l(f)g.$$

(iii) We have $\langle \Delta_n^r f, \Delta_m^l g\rangle_\psi = 0, \quad$ for all $n \neq m$, and $f, g \in \mathrm{L}^2(d\nu;\mathbb{A}_d)$.

Denote by $E_n^l(f)$ the left-conditional expectation $E^l(f|\mathcal{F}_n)$.

Definition (Littlewood-Paley square function). If $f \in L^1_{\text{loc}}(X;\mathbb{A}_d)$, the *left-martingale with respect to* $\{\mathcal{F}_n\}_{-\infty}^{\infty}$ *generated by* f is the sequence $\{f_n^l\}_{-\infty}^{\infty} = \{E_n^l(f)\}_{-\infty}^{\infty}$. The left-Littlewood-Paley square function is

$$S^l(f) = \left(|f_{-\infty}^l|^2 + \sum_{-\infty}^{\infty} |\Delta_n^l f|^2\right)^{\frac{1}{2}}$$

if the limit $f_{-\infty}^l = \lim_{n\to-\infty} E_n^l(f)$ exists pointwise a.e.. (If $\nu(X) < +\infty$, $f_{-\infty}^l$ is constant; if $\nu(X) = +\infty$, and f is integrable, then $f_{-\infty}^l = 0$.)

Everything that we have described for left martingales applies, with appropriate changes, to right martingales.

Littlewood-Paley estimates for L^2. In the case of the standard martingales $\{\tilde{E}_n f\}_{-\infty}^{\infty}$, L^2-estimates for the Littlewood-Paley square function are quite straightforward to prove. In the Clifford case, there are some technical difficulties to overcome. This can be done by using variants of techniques that appear in [9], and in Garsia's book [**10**]. See also [**5**] and [**15**].

THEOREM (LITTLEWOOD-PALEY). *There exists a constant $C > 0$, depending only on C_0 and d, such that*

$$C^{-1}\|S(f)\|_{\mathrm{L}_2} \le \|f\|_{\mathrm{L}_2} \le C\|S(f)\|_{\mathrm{L}_2}, \tag{3}$$

for all $f \in \mathrm{L}^2(X;\mathbb{A}_d)$, where S denotes either S^l or S^r.

There are variants of (3) for L^p $(1 < p < +\infty)$, but we shall not need them.

5. A PAIR-BASIS CLIFFORD-HAAR SYSTEM IN $\mathbb{R}^d$

Let $X = \mathbb{R}^d$, $\mathcal{B}$ be the Borel σ-field, and ν be Lebesgue measure. Start with $\mathcal{F}_0$ the σ-field generated by the family $\mathcal{I}_0$ of cubes of side length 1, having corners at the points of the integer lattice.

Bisect each cube $I \in \mathcal{I}_0$: $I = I_1 \cup I_2$, as shown in Fig. 1 (illustration in $\mathbb{R}^2$):

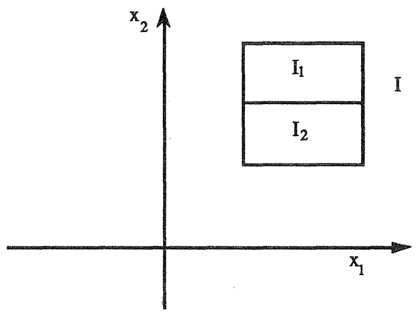

Fig. 1.

Let $\mathcal{I}_1$ be the collection of such sets I_j, $(j = 1, 2)$ formed from atoms $I \in \mathcal{I}_0$, and let $\mathcal{F}_1 = \sigma(\mathcal{I}_1)$.

Next divide each atom $I \in \mathcal{F}_1$ by dissection by a hyperplane orthogonal to the x_2-axis, as illustrated in Fig. 2.

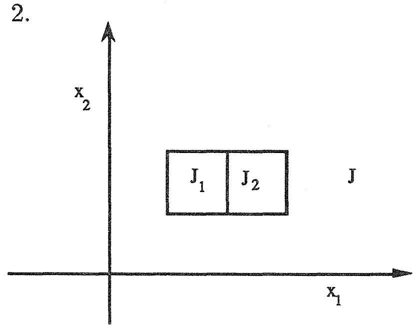

Fig. 2.

Let $\mathcal{I}_2$ be the collection of such sets I_j, $(j = 1, 2)$, and let $\mathcal{F}_2 = \sigma(\mathcal{I}_2)$. Continue in this manner, passing at each stage to dissection by a hyperplane orthogonal to the next axis, proceeding cyclically around the set of axes.

The atoms in the σ-field $\mathcal{F}_{-1}$ are constructed by doubling atoms of $\mathcal{F}_0$ in the x_d-direction, and then generating the corresponding σ-field $\mathcal{F}_{-1}$; then doubling the resulting atoms in the x_{d-1}-direction—and so on. The collection of all of the atoms (of the various sizes) is denoted $\mathcal{I}$. Its elements are called *dyadic quasi-cubes.*

Haar functions: classical case. If I is a dyadic interval of $\mathbb{R}$, of length 2^{-n}, which is bisected as shown in Fig. 3,

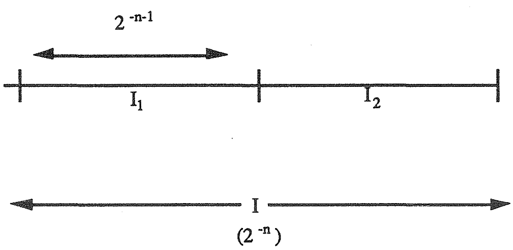

Fig. 3.

then the standard martingale difference on I is given by

$$\begin{aligned}\tilde{E}_{n+1}f - \tilde{E}_n f &= \langle 2^{n/2}\chi_{I_1} - 2^{n/2}\chi_{I_2}, f\rangle(2^{n/2}\chi_{I_1} - 2^{n/2}\chi_{I_2}) \\ &= \alpha_I\langle\alpha_I, f\rangle\end{aligned}$$

say.

Clifford Haar system (left-martingales). We consider a typical atom $I \in \mathcal{I}_{n-1}$, which has been bisected as illustrated in Fig. 1: $I = I_1 \cup I_2$, $(I_1, I_2 \in \mathcal{I}_n)$.

LEMMA. *For each $I \in \mathcal{I}_{n-1}$, $I = I_1 \cup I_2, (I_j \in \mathcal{I}_n)$, there is a* pair *$\alpha_I, \beta_I$ of $\mathbb{A}_d$-valued functions and a positive constant C such that*

(i) $$\alpha_I = a_1\chi_{I_1} + a_2\chi_{I_2}, \quad (a_j \in \mathbb{A}_d);$$

$$\beta_I = b_1\chi_{I_1} + b_2\chi_{I_2}, \quad (b_j \in \mathbb{A}_d);$$

(ii) *for all* $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{A}_d)$,

$$\Delta_n f(x) = \alpha_I(x)\langle \beta_I, f\rangle_\psi \qquad (x \in I);$$

(iii) $C^{-1}|I|^{-\frac{1}{2}} \leq |\alpha_I(x)| \leq C|I|^{-\frac{1}{2}}$ *and* $C^{-1}|I|^{-\frac{1}{2}} \leq |\beta_I(x)| \leq C|I|^{-\frac{1}{2}}$ *for all* $x \in I$ *and all* I;

(iv) $\int \psi\alpha_I\, dx = \int \beta_I\psi\, dx = 0.$

Proof. This amounts to choosing the coefficients a_1, a_2 and b_1, b_2 so that (ii) holds, and then normalising so that (iii) holds. Care needs to be taken with noncommutativity.

6. PROOF OF L^2-BOUNDEDNESS OF THE CAUCHY OPERATOR

Recall that we are dealing with the operator

$$f \longmapsto \int_{\mathbb{R}^d} K(u,v)\psi(v)f(v)\, dv = T(\psi f),$$

say. If $f \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{A}_d)$, then by the Littlewood-Paley theorem,

$$f = \sum_{-\infty}^{\infty} \Delta_n f = \sum_I \alpha_I\langle \beta_I, f\rangle_\psi$$

and, at least formally,

$$\begin{aligned} T(\psi f) &= \sum_{J\in\mathcal{I}} T(\psi\alpha_J)\langle\beta_J, f\rangle_\psi = \sum_{J,I} \alpha_I\langle\beta_I, T(\psi\alpha_J)\rangle_\psi\langle\beta_J, f\rangle_\psi \\ &= \sum_I \alpha_I \sum_J \langle\beta_I, T(\psi\alpha_J)\rangle_\psi\langle\beta_J, f\rangle_\psi. \end{aligned}$$

So, if $f \sim \{\langle\beta_J, f\rangle_\psi\}_J$, then $T(\psi f)$ is determined by multiplication of the coefficient sequence $\{\langle\beta_J, f\rangle_\psi\}_J$ by the matrix $\left(\langle\beta_I, T(\psi\alpha_J)\rangle_\psi\right)_{I,J}$. Proving L^2-boundedness is therefore equivalent to proving boundedness, on $\ell^2(\mathcal{I}; \mathbb{A}_d)$, of the operator determined by the matrix $(u_{IJ}) = \left(\langle\beta_I, T(\psi\alpha_J)\rangle_\psi\right)_{I,J}$. In proving this, we use the following variant of Schur's lemma.

LEMMA (SCHUR'S LEMMA). *Suppose there exists a family of positive numbers* (ω_{IJ}) *and a constant* C *such that*

$$(i) \qquad \sum_J |\omega_J u_{IJ}| \le C\omega_I \qquad (I \in \mathcal{I})$$

and

$$(ii) \qquad \sum_I |\omega_I u_{IJ}| \le C\omega_J \qquad (J \in \mathcal{I}).$$

Then the matrix (u_{IJ}) *defines a bounded operator on* $\ell^2(\mathcal{I}; \mathbb{A}_d)$.

In fact, it is shown that the conditions (i) and (ii) hold, with $\omega_J = |J|^t$, where t is any number between $\frac{1}{2} - \frac{1}{d}$ and $\frac{1}{2}$.

Crucial properties of K (standard kernel conditions). The kernel K obtained by transporting the Cauchy integral to an integral on $\mathbb{R}^d$ (see Section 3) satisfies certain standard estimates, usually known as "standard kernel conditions". They are as follows:

(a) $\quad |K(x,y)| \le C|x-y|^{-d} \qquad (x \ne y);$

(b) $\quad |K(x,y) - K(x',y)| \le C\dfrac{|x-x'|}{|x-y|^{1+d}}, \qquad$ provided $0 < |x-x'| \le \frac{1}{2}|x-y|;$

(c) $\quad |K(y,x) - K(y,x')| \le C\dfrac{|x-x'|}{|x-y|^{1+d}} \qquad$ under the same conditions as in (b).

The verification of the Schur lemma conditions for $\left(\langle \beta_I, T(\psi\alpha_J)\rangle_\psi\right)$ rests on (a)–(c) and the monogenicity of the Cauchy kernel. One has to take careful account of the relative sizes of the atoms I, J, and their relative disposition (e.g. whether one meets the other). These estimates are lengthy. For complete details, see **[11]**.

7. THE CLIFFORD $T(B)$ THEOREM

The systematic use which we have made of the "coefficients"

$$\langle \alpha_I, T(\psi\beta_J)\rangle_\psi = \iint \alpha_I(x)\psi(x)K(x,y)\psi(y)\beta_J(y)\, dxdy$$

gives a pointer to the $T(b)$ theorem. (The expression above is of course only formal, as it stands.) See **[18]** for an account of various forms of the $T(b)$ theorem.

Let b_1 and b_2 be two pseudoaccretive functions. (In the Cauchy theorem setting, we had $b_1 = b_2 = \psi$.) Denote by $\mathcal{S}$ the span, over $\mathbb{A}_d$, of the set of characteristic functions of dyadic quasi-cubes. Suppose that T is a right-Clifford-linear mapping from $b_1\mathcal{S}$ into the space $(\mathcal{S}b_2)^*$ of left-Clifford-linear functionals on $\mathcal{S}b_2$.

Definition. Let $\Delta = \{(x,y) : x \neq y\}$. We say that T *is associated with a standard Calderón-Zygmund kernel* K if there is a C^∞ function K on $\mathbb{R}^d \times \mathbb{R}^d \backslash \Delta$, with values in $\mathbb{A}_d$, and a number δ: $0 < \delta \leq 1$, such that

(a) $$|K(x,y)| \leq C\frac{1}{|x-y|^d} \qquad (x \neq y);$$

(b) $$|K(x,y) - K(x,y_0)| + |K(y,x) - K(y_0,x)| \leq C\frac{|y-y_0|^\delta}{|x-y|^{d+\delta}}$$

if $0 < |y - y_0| < \frac{1}{2}|y-x|$; and

(c) $$T(b_1 f)(gb_2) = \iint g(x)b_2(x)K(x,y)b_1(y)f(y)\, dxdy$$

for all $f, g \in \mathcal{S}$ having disjoint supports. Note that, under these assumptions, there is no problem with the convergence of the right-hand side of (c).

Dually, let T^t be a left-Clifford-linear mapping from $\mathcal{S}b_2$ into the space of right-linear functionals on $b_1\mathcal{S}$. Suppose that

$$T(b_1 f)(gb_2) = T^t(gb_2)(b_1 f)$$

for all $f, g \in \mathcal{S}$. Then T^t is associated, in the obvious way, with the kernel $K(y,x)$.

Definition of $T(b_1)$. The function b_1 is in L^∞, so $T(b_1)$ has no *a priori* meaning. Following David-Journé [7], it is defined as follows. Fix $g \in \mathcal{S}$; so g vanishes off a compact set. We want to make sense of $Tb_1(gb_2)$. This cannot be done, in general, but it can if $\int gb_2 \, dx = 0$. Choose $f_1 \in \mathcal{S}$ so that $f_1 \equiv 1$ on a neighbourhood of supp (g), say a set twice the size. See Fig. 5.

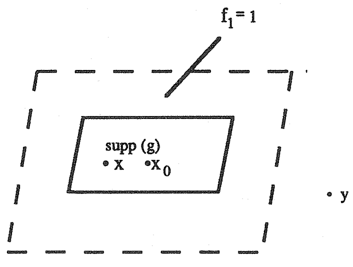

Fig. 4.

Then $T(b_1 f_1)$ makes sense, by hypothesis. So we have to define $T(b_1 f_2)(gb_2)$, where $f_2 = 1 - f_1$. This is where the condition $\int gb_2 \, dx = 0$ comes in. Fix $x_0 \in$ supp (g), and write formally

$$(\#) \qquad \int g(x)b_2(x)K(x,y)\, dx = \int_{\text{supp }(g)} g(x)b_2(x)[K(x,y) - K(x_0,y)]\, dx$$

Now the point is that the right side of $(\#)$ can be integrated against $b_1(y)f_2(y)$, because $|x - x_0| < \frac{1}{2}|x-y|$ and $|K(x,y) - K(x_0,y)|$ is $O(|x-y|^{d+\delta})$, while x ranges over the set enclosed in the solid line above. In summary, Tb_1 makes sense as a left-linear functional on the subspace $(\mathcal{S}b_2)_0$ consisting of functions having integral 0.

Definition. If $\phi \in \mathrm{L}^1_{\mathrm{loc}}(d\nu; \mathbb{A}_d)$, the BMO-norm of ϕ is defined to be

$$\|\phi\|_{\mathrm{BMO}} = \sup_n \|\tilde{E}_n(|\phi - \tilde{E}_{n-1}\phi|^2)\|_\infty^{\frac{1}{2}}.$$

Definition. We say that $Tb_1 \in$ BMO if there is a function ϕ, say, that is locally integrable, belongs to BMO, and is such that $\langle g, T(b_1)\rangle_{b_2} = \langle g, \phi\rangle_{b_2}$ for all $g \in (\mathcal{S}b_2)_0$. A similar interpretation applies to $T^t(b_2)$.

Definition. The operator T, associated with a standard Calderón-Zygmund kernel, is *weakly bounded* if there is a constant C such that

$$|Tb_1\chi_Q(\chi_Q b_2)| \le C|Q|$$

for all dyadic quasi-cubes Q.

THEOREM (CLIFFORD $T(B)$ THEOREM). *Let T and T^t be as above. Then the operator T is extendible to a bounded linear operator from $b_1\mathrm{L}^2(\mathbb{R}^d;\mathbb{A}_d))$ to $\mathrm{L}^2(\mathbb{R}^d;\mathbb{A}_d)b_2$ if and only if*

(a) $T(b_1), T^t(b_2) \in \mathrm{BMO}$;

(b) T is weakly bounded.

The proof uses rather similar estimates to those used to prove the boundedness of the Cauchy singular integral . See [**11**] for complete details. The one-dimensional case of the theorem appears in [**6**]. Some of our estimates follow the lines of arguments given in [**6**].

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