UNITARY APPROXIMATION AND SUBMAJORIZATION

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0. Introduction

We begin by considering the following inequality for complex numbers:

$$\text{(i) } |x-1| \;\le\; |x-u| \;\le\; |x+1|, \quad \forall\, 0 \le x \in \mathbb{R} \text{ and } u \in \mathbb{C} \text{ with } |u| = 1,$$

and the equivalent inequality:

$$\text{(ii) } \big|\, |z|-1 \,\big| \;\le\; |vz-1| \;\le\; \big|\, |z|+1 \,\big|, \quad \forall\, z \in \mathbb{C} \text{ and } v \in \mathbb{C} \text{ with } |v| = 1.$$

It has been shown by Ky Fan and A. J. Hoffman [FH] that the inequality (i) remains valid if x is replaced by a given $n \times n$ Hermitian positive semi-definite matrix, u by any $n \times n$ unitary matrix and the modulus of a complex number is replaced by a unitarily invariant norm. Subsequently (ii) was shown to hold by D. J. van Riemsdijk [vR] for a certain class of symmetric norms, with z a bounded linear operator on a separable Hilbert space H, v any partial isometry with initial space containing the range of z. It is a well-known fact, due to Ky Fan [Fa], that metric inequalities in symmetrically normed ideals of compact operators are consequences of corresponding submajorization inequalities for singular values, and this indeed is the approach of [FH]. The approach of [vR] is based on an extension to arbitrary bounded linear operators of the notion of singular value sequence of a compact operator given in the monograph of Gohberg and Krein [GK], and again the metric inequalities given in [vR] are derived from corresponding submajorization inequalities. We mention further that special cases of the results of [vR] have also been given, in the setting of the Schatten p−classes and using methods of independent interest, by Aiken, Erdos and Goldstein [AEG], to which the reader is referred for an illuminating discussion of the relation of such inequalities to quantum chemistry.

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* Research supported by A.R.C.

1980 *Mathematics Subject Classification.* Primary 46E30; Secondary 46L50, 47B55.

It is our intention in this paper to indicate that the submajorization inequalities of [FH] and [vR] may be extended to the general setting of measurable operators affiliated with a semi-finite von Neumann algebra equipped with a distinguished semi-finite normal trace. Such inequalities immediately imply metric inequalities for a class of symmetric operator norms which are known as fully symmetric norms and we recover as special cases the operator extensions of the inequalities (i) and (ii) given by [FH] and [vR]. While our method follows that of [vR], the principal new ingredient required is a very general version of a submajorization inequality established earlier by the present authors and Ben de Pagter [DDP1], which is due to A.S. Markus [Ma] in the case of compact operators, and which goes back to Lidskii [Li] and Wielandt [Wi] for finite matrices.

In section 1 below, we gather the relevant terminology and essential properties of generalized singular value functions of measurable operators that will be required in the sequel. In section 2, we present the main results in the form of submajorization inequalities, while in section 3 we show how certain metric inequalities are related to submajorization. The metric inequalities of section 3 imply that the identity operator is a best unitary approximant to a given measurable self-adjoint operator, for any fully symmetric norm. In section 4, we consider the question of uniqueness of such best approximants, and show that the identity is the unique best unitary approximant for the fully symmetric norms arising from the familiar L^p- spaces, for $1 < p < \infty$.

1. Preliminaries

We denote by $\mathcal{M}$ a semi-finite von Neumann algebra in the Hilbert space H with given normal faithful semi-finite trace τ. If x is a (densely defined) self-adjoint operator in H and if $x = \int_{(-\infty,\infty)} s de_s^x$ is its spectral decomposition then, for any Borel subset $B \subseteq \mathbb{R}$, we denote by $\chi_B(x)$ the corresponding spectral projection $\int_{(-\infty,\infty)} \chi_B(s) de_s^x$. A closed densely defined linear operator z in H affiliated with

$\mathcal{M}$ is said to be *τ-measurable* if and only if there exists a number $s \geq 0$ such that

$$\tau(\chi_{(s,\infty)}(|z|)) \;<\; \infty.$$

The set of all τ-measurable operators will be denoted by $\widetilde{\mathcal{M}}$. The set $\widetilde{\mathcal{M}}$ is a *-algebra with sum and product being the respective closures of the algebraic sum and product. For $z \in \widetilde{\mathcal{M}}$, the *generalized singular value function* (or *decreasing rearrangement*) $\mu.(z)$ of z is defined by

$$\mu_t(z) \;=\; \inf\{s \geq 0 \;:\; \tau(\chi_{(s,\infty)}(|z|)) \leq t\}, \qquad t \geq 0.$$

It follows that $\mu(z)$ is a decreasing, right continuous function on the half-line $\mathbb{R}^+ = [0, \infty)$.

If $\mathcal{M}$ is the space $\mathcal{L}(H)$ of all bounded linear operators on H and τ is the standard trace, then $\widetilde{\mathcal{M}} = \mathcal{L}(H)$ and $z \in \mathcal{L}(H)$ is compact if and only if $\mu_t(z) \to 0$ as $t \to \infty$, in which case for each $n = 0, 1, 2, \ldots$,

$$\mu_n(z) \;=\; \mu_t(z), \qquad t \in [n, n+1),$$

and $\{\mu_n(z)\}_{n=0}^{\infty}$ is the usual singular value sequence of z in decreasing order, counted according to multiplicity [GK].

We identify the space $L^\infty(\mathbb{R}^+)$ of all bounded complex-valued Lebesgue measurable functions on the half-line $\mathbb{R}^+$ as a commutative von Neumann algebra acting by multiplication on the Hilbert space $L^2(\mathbb{R}^+)$ with trace given by integration with respect to Lebesgue measure. In this case the τ-measurable operators coincide with those complex measurable functions f on $\mathbb{R}^+$ which are bounded except on a set of finite measure. In this example, the generalized singular value function, which we continue to denote by $\mu(f)$, coincides with the familiar right-continuous decreasing rearrangement of the function $|f|$. See, for example, [KPS].

If z, $w \in \widetilde{\mathcal{M}}$, we say that z is *submajorized* by w, written $z \prec\!\!\prec w$, if and only if

$$\int_0^\alpha \mu_t(z)dt \;\leq\; \int_0^\alpha \mu_t(w)dt, \qquad \text{for all} \quad \alpha \geq 0.$$

The following result is proved in [DDP1].

$$\textit{If } z,\ w \in \widetilde{\mathcal{M}},\ \textit{then } \mu(z) - \mu(w) \ \prec\prec\ \mu(z-w).$$

We remark that the submajorization is, of course, with respect to the von Neumann algebra $L^\infty(\mathbb{R}^+)$. This submajorization inequality generalizes a similar inequality for compact operators proved by A.S. Markus [Ma] and accordingly we shall refer to it as the *generalized Markus inequality*. We note, as a simple consequence, that if $z, w \in \widetilde{\mathcal{M}}$ then

$$\mu(z+w) \ \prec\prec\ \mu(z) + \mu(w).$$

We define

$$\widetilde{\mathcal{M}}_0 \ = \ \{z \in \widetilde{\mathcal{M}} \ : \ \mu_t(z) \to 0 \quad \text{as} \quad t \to \infty\}.$$

It is clear that $z \in \widetilde{\mathcal{M}}_0$ if and only if $\tau(\chi_{(s,\infty)}(|z|)) < \infty$ for all $s > 0$. The operator norm on $\mathcal{M}$ will be denoted by $\|\cdot\|_\infty$. We denote by $\left(L^1(\mathcal{M}), \|\cdot\|_1\right)$ the Banach space of all operators $z \in \widetilde{\mathcal{M}}$ for which $\|z\|_1 = \int_0^\infty \mu_t(z)dt < \infty$.

2. Singular Value Inequalities

We first gather some basic properties of generalized singular value functions, the proofs of which may be found in [FK].

Proposition 2.1. (a) $\mu(z) \ = \mu(z^*), \quad \forall\, z \in \widetilde{\mathcal{M}}$,
(b) $\mu(uz) \ = \mu(z), \quad \forall\, z \in \widetilde{\mathcal{M}}$, *unitary* $u \in \mathcal{M}$,
(c) $\mu(xyz) \ \le \ \|x\|_\infty\ \mu(y)\ \|z\|_\infty, \quad \forall\, x, z \in \mathcal{M}, \quad y \in \widetilde{\mathcal{M}}$,
(d) $\mu(f(|z|)) \ = \ f(\mu(z)),\ \forall\, z \in \widetilde{\mathcal{M}}$ *and for any continuous increasing function* f *on* $[0,\infty)$ *with* $f(0) = 0$.

To illustrate the utility of the notion of generalized singular value function, we note the following immediate consequences.

Proposition 2.2. (i) *If* $z \in \widetilde{\mathcal{M}}$, *then*

$$\mu(\mathrm{Re}z) \prec\!\!\prec \mu(z), \qquad \mu(\mathrm{Im}z) \prec\!\!\prec \mu(z).$$

(ii) *If* $z, x \in \widetilde{\mathcal{M}}$ *and if* x *is self-adjoint, then*

$$\mu(z - \mathrm{Re}z) \prec\!\!\prec \mu(z - x), \qquad \mu(z - i\mathrm{Im}z) \prec\!\!\prec \mu(z - ix).$$

(iii) *If* $x, y \in \widetilde{\mathcal{M}}$ *are self-adjoint and* $y \geq 0$, *then*

$$\mu(x + x^-) \leq \mu(x + y), \qquad \mu(x - x^+) \leq \mu(x - y).$$

(iv) *If* $x, y \in \widetilde{\mathcal{M}}$ *are self-adjoint, then*

$$\frac{1}{2}\mu(\ (x-i)(x+i)^{-1} - (y-i)(y+i)^{-1}\ ) \leq \mu(x - y).$$

One of the chief sources of technical difficulty in proving operator extensions of simple numerical inequalities is the failure of the triangle inequality for the absolute value, even for two-by-two matrices. Let us illustrate this point further as follows. If x, y are real numbers and if $-x \leq y \leq x$, then of course it follows that $|y| \leq x$. The corresponding assertion fails for self-adjoint operators, if $|y|$ is interpreted as $\sqrt{y^*y}$, as may be observed by setting

$$x = \begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This observation may be found in [Ha]. Nonetheless, it is possible to prove an appropriate metric version, using generalized singular values. The proposition which follows was proved for finite von Neumann algebras in [CMS], Lemma 2.1. The present proof for arbitrary semi-finite von Neumann algebra was obtained jointly with Ben de Pagter, and is based on a refinement of an interpolation formula given in [DDP3].

Proposition 2.3. (i) *If* $x, y \in \widetilde{\mathcal{M}}$ *are self-adjoint and* $-x \leq y \leq x$, *then*

$$\mu(y) \prec\!\!\prec \mu(x).$$

(ii) *If* $0 \le x, y \in \widetilde{\mathcal{M}}$, *then*

$$\mu(x-y) \prec\!\!\prec \mu(x+y).$$

Proof. (i) Let the projection $e \in \mathcal{M}$ be such that $ey = ye$ and

$$y^+ = ey, \qquad y^- = -(1-e)y.$$

Since $-x \le y \le x$, it follows that

$$y^+ \le exe, \qquad y^- \le (1-e)x(1-e),$$

and so

$$|y| = y^+ + y^- \le exe + (1-e)x(1-e),$$

which implies that

$$\mu(y) \le \mu(\ exe + (1-e)x(1-e)\ ).$$

Now by [DDP3], Corollary 4.11, for every $w \in \widetilde{\mathcal{M}}$ and every number $\alpha > 0$,

$$\int_0^\alpha \mu_t(w)\, dt = \inf\{\|u\|_1 + \alpha\|v\|_\infty \ :\ |w| = u + v,\ 0 \le u \in L^1(\mathcal{M}),\ 0 \le v \in \mathcal{M}\}.$$

If $x = u + v$ where $0 \le u \in L^1(\mathcal{M})$ and $0 \le v \in \mathcal{M}$, then

$$exe + (1-e)x(1-e) = eue + (1-e)u(1-e) + eve + (1-e)v(1-e),$$

$$eue + (1-e)u(1-e) \in L^1(\mathcal{M}), \qquad eve + (1-e)v(1-e) \in \mathcal{M},$$

moreover,

$$\|eue + (1-e)u(1-e)\|_1 = \tau(eue) + \tau((1-e)u(1-e)) = \tau(eu) + \tau((1-e)u) = \|u\|_1,$$

and

$$\|eve + (1-e)v(1-e)\|_\infty \le \|\ \|v\|_\infty e + \|v\|_\infty (1-e)\ \|_\infty = \|v\|_\infty,$$

hence for every number $\alpha > 0$,

$$\int_0^\alpha \mu_t(\ exe + (1-e)x(1-e)\ )\, dt \le \int_0^\alpha \mu_t(x)\, dt,$$

that is

$$\mu(\ exe + (1-e)x(1-e)\ ) \prec\!\!\prec \mu(x)$$

and the proof of (i) is complete.

(ii) is obtained by replacing x by $x+y$ and y by $x-y$ in (i). $\square$

We now turn to the main result of this section. We need the following lemma.

Lemma 2.4. (i) *If* $0 \le x \in \widetilde{\mathcal{M}}$, *then*

$$\mu(x+1) \ = \mu(x) + \mu(1).$$

(ii) *If* $0 \le x \in \widetilde{\mathcal{M}}_0$ *then*

$$\mu(x-1) \ = \ \mu(\ \mu(x) - \mu(1)\ ).$$

(iii) *If* $e \in \mathcal{M}$ *is a projection such that* $\tau(e) < \infty$ *and if* $0 \le x \in \widetilde{\mathcal{M}}$, *then*

$$\mu(exe - e) \ = \ \mu(\ \mu(exe) - \mu(e)\ ).$$

The proof of (i) is straightforward, and we refer to [DD] Proposition 2.3 and Lemma 2.2 for the proof of (ii) and (iii).

It is worth noting that the equality asserted by (ii) of the preceding Lemma may fail in general. In fact, let $\mathcal{M}$ be $L^\infty(\mathbb{R}^+)$ and let x be given by setting

$$x(t) \ = \ 1 - e^{-t}, \qquad t \in \mathbb{R}^+.$$

It is clear that $\mu(\ \mu(x) - \mu(1)\ ) = 0$; on the other hand

$$\mu_t(x-1) \ = \ e^{-t}, \qquad t \in \mathbb{R}^+.$$

The main result of this section now follows. It is an extension of [DD] Theorem 3.1, which we state as Corollary 2.6(ii) below, and the proof follows similar lines. We outline the proof and indicate the modifications that need to be made to the argument of [DD]. For $z \in \widetilde{\mathcal{M}}$, we write $r(z)$ for the closure of the range of z.

Theorem 2.5. *If* $z \in \widetilde{\mathcal{M}}$, *then*

$$\mu(|z|-1) \prec\!\!\prec \mu(vz-1) \prec\!\!\prec \mu(|z|+1),$$

for all partial isometries $v \in \mathcal{M}$ *such that* $v^*vH \supseteq r(z)$.

Proof. Let $z \in \widetilde{\mathcal{M}}$ and $v \in \mathcal{M}$ be a partial isometry such that $v^*vH \supseteq r(z)$. By part (i) of the Lemma and the generalized Markus inequality

$$\mu(vz-1) \prec\!\!\prec \mu(vz)+\mu(1) \leq \mu(z)+\mu(1) = \mu(|z|+1).$$

To obtain the left hand inequality we first observe that

$$\mu(vze) = \mu^{\frac{1}{2}}(ez^*v^*vze) = \mu^{\frac{1}{2}}(e|z|^2e) = \mu(|z|e), \qquad \text{for all projections } e \in \mathcal{M},$$

and specially,

$$\mu(vz) = \mu(|z|).$$

If $\tau(1) < \infty$, then part (ii) of the Lemma and the generalized Markus inequality imply that

$$\mu(|z|-1) = \mu(\,\mu(|z|)-\mu(1)\,) = \mu(\,\mu(vz)-\mu(1)\,) \prec\!\!\prec \mu(vz-1).$$

Assume then that $\tau(1) = \infty$. Let $\alpha > 0$, $\epsilon > 0$ be numbers. It follows from the proof of Theorem 3.1 of [DD] that there exists $\tilde{z} = \tilde{z}(\alpha, \epsilon) \in \widetilde{\mathcal{M}}$ and a projection $e = e(\alpha, \epsilon) \in \mathcal{M}$ such that

(i) $\tilde{z} \geq 0, \quad e\tilde{z} = \tilde{z}e,$

(ii) $\alpha \leq \tau(e) < \infty,$

(iii) $\mu(e\tilde{z}e - e) = \mu(|z|-1)\, \chi_{[0,\tau(e))}$,

(iv) $\|\tilde{z}e - |z|e\|_\infty \leq \frac{\epsilon}{\alpha}$.

Now (iv) implies that

$$\int_0^\alpha \mu_t(\tilde{z}e - |z|e)dt \leq \alpha\|\tilde{z}e - |z|e\|_\infty \leq \epsilon,$$

and so, by the Lemma and the generalized Markus inequality, it follows that

$$\begin{aligned}
\int_0^\alpha \mu_t(|z| - 1)dt &= \int_0^\alpha \mu_t(e\tilde{z}e - e)dt \\
&= \int_0^\alpha \mu_t(\ \mu(e\tilde{z}e) - \mu(e)\ )dt \\
&\leq \int_0^\alpha \mu_t(\ \mu(\tilde{z}e) - \mu(|z|e)\ )dt + \int_0^\alpha \mu_t(\ \mu(|z|e) - \mu(e)\ )dt \\
&\leq \int_0^\alpha \mu_t(\tilde{z}e - |z|e)dt + \int_0^\alpha \mu_t(\ \mu(vze) - \mu(e)\ )dt \\
&\leq \epsilon + \int_0^\alpha \mu_t(vze - e)dt \\
&\leq \epsilon + \int_0^\alpha \mu_t(vz - 1)dt.
\end{aligned}$$

It follows again that

$$\mu(|z| - 1) \prec\!\!\prec \mu(vz - 1). \qquad \square$$

Corollary 2.6. (i) *If* $z \in \widetilde{\mathcal{M}}$ *is such that* $z = w|z|$ *with* $w \in \mathcal{M}$ *being unitary, then*

$$\mu(z - w) \prec\!\!\prec \mu(z - u) \prec\!\!\prec \mu(z + w), \qquad \textit{for all unitary } u \in \mathcal{M}.$$

(ii) *If* $0 \leq x \in \widetilde{\mathcal{M}}$, *then*

$$\mu(x - 1) \prec\!\!\prec \mu(x - u) \prec\!\!\prec \mu(x + 1), \qquad \textit{for all unitary } u \in \mathcal{M}.$$

The assertion (i) follows immediately from the Theorem, for if $z = w|z|$ with $w \in \mathcal{M}$ being unitary, then

$$\mu(z - w) = \mu(|z| - 1) \quad \text{and} \quad \mu(z - u) = \mu(u^*z - 1).$$

The assertion (ii) follows easily from (i).

The following special case of Theorem 2.5 seems worth separate mention.

Corollary 2.7. *If* $z \in \widetilde{\mathcal{M}}$, *then*

$$\mu(|z|-1) \prec\!\!\prec \mu(z-1) \prec\!\!\prec \mu(|z|+1).$$

3. Inequalities For Fully Symmetric Norms

An extended norm ρ on $\widetilde{\mathcal{M}}$ is called *fully symmetric* if and only if

$$z, w \in \widetilde{\mathcal{M}},\ \mu(z) \prec\!\!\prec \mu(w) \implies \rho(z) \le \rho(w).$$

We now indicate how the theory of function norms may be used to construct a large class of fully-symmetric extended norms on $\widetilde{\mathcal{M}}$. If $\mathcal{M} = L^\infty(\mathbb{R}^+)$, then we write $\mathcal{S}$ for $\widetilde{\mathcal{M}}$. The space $\mathcal{S}$ consists of those (complex) measurable functions on $\mathbb{R}^+$ which are bounded except on a set of finite measure. An extended functional $\rho : \mathcal{S} \to [0, \infty]$ is called a *function norm* on $\mathcal{S}$ if and only if ρ is positively homogeneous, subadditive, absolute in the sense that $\rho(f) = \rho(|f|)$ for all $f \in \mathcal{S}$, and monotone on the positive cone of $\mathcal{S}$ that is, $0 \le f \le g \in \mathcal{S}$ implies $\rho(f) \le \rho(g)$. The function norm ρ is called *rearrangement invariant* if $f, g \in \mathcal{S}$, $\mu(f) = \mu(g)$ implies $\rho(f) = \rho(g)$.

Suppose now that ρ is a rearrangement invariant function norm on $\mathcal{S}$. The *associate norm* ρ' is defined by setting

$$\rho'(f) = \sup\{\int_0^\infty |f(t)g(t)|dt \ : g \in \mathcal{S},\ \rho(g) \le 1\}.$$

It may be shown [KPS], [Lux] that ρ' is a rearrangement invariant function norm on $\mathcal{S}$ and that

$$\rho'(f) = \sup\{\int_0^\infty \mu_t(f)\mu_t(g)dt \ : g \in \mathcal{S},\ \rho(g) \le 1\}.$$

Similarly, we can define the second associate norm $\rho'' = (\rho')'$ and it is clear that $\rho'' \le \rho$. The function norm on $\mathcal{S}$ is said to have the *σ-Fatou property* if and only if $0 \le f_n \uparrow_n f \in \mathcal{S}$ implies that $\rho(f_n) \uparrow_n \rho(f)$. The importance of this property lies in the well-known theorem of G.G. Lorentz and W.A.J. Luxemburg [Za] that the equality $\rho'' = \rho$ holds if and only if the function norm ρ has the σ-Fatou property. This implies that any rearrangement invariant function norm ρ on $\mathcal{S}$ which has the σ-Fatou property admits the representation

$$\rho(f) = \sup\{\int_0^\infty \mu_t(f)\mu_t(g)dt \ : g \in \mathcal{S},\ \rho'(g) \le 1\}.$$

for all $f \in \mathcal{S}$. Such a representation implies that the function norm ρ is fully symmetric. This is a direct consequence the following lemma, due to Hardy [KPS] II 2.18, which we recall for the convenience of the reader.

Hardy's Lemma *If* $0 \le f, g$ *are locally integrable functions on* $\mathbb{R}^+$ *such that*

$$\int_0^\alpha f(t)dt \le \int_0^\alpha g(t)dt$$

for all $\alpha > 0$ *and if* h *is a decreasing non-negative function on* $\mathbb{R}^+$, *then*

$$\int_0^\infty f(t)h(t)dt \le \int_0^\infty g(t)h(t)dt.$$

We now remark that if ρ is an extended function norm on $\mathcal{S}$ with the σ-Fatou property, then ρ induces a fully symmetric norm on $\widetilde{\mathcal{M}}$ in a natural way by setting $\|z\|_\rho = \rho(\mu(z))$ for all $z \in \widetilde{\mathcal{M}}$. It is, of course, a simple matter to check whether a given function norm has the σ-Fatou property. For example, if $1 \le p < \infty$ and if $\rho_p(f) = \left(\int_0^\infty |f(t)|^p dt\right)^{1/p}$, $f \in \mathcal{S}$, then it is clear that ρ_p has the σ-Fatou property and consequently the corresponding extended norm induced on $\widetilde{\mathcal{M}}$ is fully symmetric. Similar remarks apply to the case that $p = \infty$ and to the extended function norms derived from the familiar Lorentz and Marcinkiewicz spaces on $\mathbb{R}^+$, given, for example in [KPS] Chapter II. With these remarks in mind, we now turn to the following metric inequalities which are an immediate consequence of Proposition 2.3, Theorem 2.5 and Corollaries 2.6, 2.7, and which extends the metric inequalities given in [FH], [GK], [vR] and [AEG].

Theorem 3.1. *Let $\|\cdot\|$ be a fully symmetric norm defined on $\widetilde{\mathcal{M}}$. Then the following inequalities hold.*

(i) *If $z \in \widetilde{\mathcal{M}}$, then*

$$\| \, |z| - 1 \, \| \leq \|vz - 1\| \leq \| \, |z| + 1 \, \|,$$

for all partial isometries $v \in \mathcal{M}$ such that $v^*vH \supseteq r(z)$.

(ii) *If $z \in \widetilde{\mathcal{M}}$ has polar decomposition $z = w|z|$ with $w \in \mathcal{M}$ being unitary, then*

$$\|z - w\| \leq \|z - u\| \leq \|z + w\|, \qquad \text{for all unitary } u \in \mathcal{M}.$$

(iii) *If $0 \leq x \in \widetilde{\mathcal{M}}$, then*

$$\|x - 1\| \leq \|x - u\| \leq \|x + 1\|, \qquad \text{for all unitary } u \in \mathcal{M}.$$

(iv) *If $z \in \widetilde{\mathcal{M}}$, then*

$$\| \, |z| - 1 \, \| \leq \|z - 1\| \leq \| \, |z| + 1 \, \|.$$

(v) *If $x, y \in \widetilde{\mathcal{M}}$ are self-adjoint and $-x \leq y \leq x$, then*

$$\|y\| \leq \|x\|.$$

(vi) *If $0 \leq x, y \in \widetilde{\mathcal{M}}$, then*

$$\|x - y\| \leq \|x + y\|.$$

We remark further that each of the submajorizations in Proposition 2.2 gives rise to a corresponding metric inequality for extended fully symmetric norms on $\widetilde{\mathcal{M}}$.

We now characterize submajorization in terms of metric inequalities. We suppose that ρ is a rearrangement invariant function norm on $\mathcal{S}$. We set $E_\rho = \{f \in \mathcal{S} : \rho(f) < \infty\}$. A real convex function Φ on E_ρ is called rearrangement invariant if $f, g \in E_\rho$, $\mu(f) = \mu(g)$ implies $\Phi(f) = \Phi(g)$.

Proposition 3.2. *Let ρ be a rearrangement invariant function norm on $\mathcal{S}$ with the σ-Fatou property. If $f, g \in E_\rho$ then the following statements are equivalent.*

$$\text{(i)} \qquad f \prec\prec g$$

$$\text{(ii)} \qquad \Phi(f) \leq \Phi(g)$$

for all rearrangement invariant real convex functions Φ *on* E_ρ *which are* $\sigma(E_\rho, E_{\rho'})$ *lower semicontinuous.*

The preceding proposition is a variant of [Lux] Theorem 13.3 in the setting of finite measure spaces and goes back to Hardy, Littlewood and Polya [HLP]. The proof follows the general outline given by [Lux] and is based, in part, on the well known fact that each lower semicontinuous function on a locally convex space is the supremum of its subgradients. The other ingredient in the proof is the fact ([BS] 2.2.7, [KPS] II 2.14) that if $f, g \in \mathcal{S}$ then

$$\int_0^\infty \mu_t(f)\mu_t(g)dt = \sup \int_0^\infty |f(t)h(t)|dt,$$

where the supremum is taken over all functions $h \in \mathcal{S}$ for which $\mu(h) = \mu(g)$. These facts then imply that any rearrangement invariant real convex function Φ on E_ρ which is $\sigma(E_\rho, E_{\rho'})$ lower semicontinuous admits a representation of the form

$$\Phi(f) = \sup\{\int_0^\infty \mu_t(f)\mu_t(g_i)dt \ + \beta_i\}$$

for some family $\{g_i\} \subseteq E_{\rho'}$ and corresponding family $\{\beta_i\}$ of real scalars. The implication (i) implies (ii) then follows from Hardy's Lemma. The implication (ii) implies (i) follows from the observation that, if $\alpha > 0$, and if

$$\Phi(f) = \int_0^\alpha \mu_t(f)dt, \quad f \in E_\rho,$$

then Φ satisfies the conditions of (ii). More generally, we remark that if ϕ is any increasing convex function on $\mathbb{R}^+$ with $\phi(0) = 0$, if $\alpha > 0$ and if

$$\Phi(f) = \int_0^\alpha \phi\left(\mu_t(f)\right) dt, \quad f \in E_\rho,$$

then it is also not difficult to see that Φ is convex, rearrangement invariant and $\sigma(E_\rho, E_{\rho'})$ lower semicontinuous.

4. Uniqueness of Best Approximation

In the following discussion, it will be convenient to reformulate the result of Theorem 3.1 (iii) in terms of the construction of symmetric operator spaces given elsewhere [DDP1], rather than in the setting of extended function norms. For the convenience of the reader, we recall the necessary special case of this construction.

Let $L^0(\mathbb{R}^+)$ be the linear space of all (equivalence classes of) complex-valued Lebesgue measurable functions on the half-line $\mathbb{R}^+$. A Banach space $E(\mathbb{R}^+)$ with norm $\|.\|_E$, which is a linear subspace of $L^0(\mathbb{R}^+)$, is called a *fully symmetric function space* on $\mathbb{R}^+$ if and only if

$$f \in L^0(\mathbb{R}^+),\ g \in E(\mathbb{R}^+) \quad \text{and} \quad \mu(f) \prec\!\!\prec \mu(g) \implies f \in E(\mathbb{R}^+) \quad \text{and} \quad \|f\|_E \leq \|g\|_E.$$

It is not difficult to see that each fully symmetric Banach function space on $\mathbb{R}^+$ is necessarily contained in $\mathcal{S}$. If $E = E(\mathbb{R}^+)$ is a fully symmetric Banach function space on $\mathbb{R}^+$, we define

$$E(\mathcal{M}) = \{z \in \widetilde{\mathcal{M}} : \mu(z) \in E\}$$

and set

$$\|z\|_{E(\mathcal{M})} = \|\mu(z)\|_E, \qquad z \in E(\mathcal{M}).$$

It can be shown (see [DDP1,2]) that $(E(\mathcal{M}), \|.\|_{E(\mathcal{M})})$ is a Banach space. Of course, the norm on $E(\mathcal{M})$ induces, in a natural way, an extended fully symmetric norm ρ on $\widetilde{\mathcal{M}}$ simply by setting $\rho(z) = \|z\|_{E(\mathcal{M})}$ if $z \in E(\mathcal{M})$ and by defining $\rho(z)$ to be ∞ otherwise.

We now express Theorem 3.1 (iii) as a result on the existence of a best unitary aproximant.

Theorem 4.1 *Let $E(\mathbb{R}^+)$ be a fully symmetric Banach function space. If $0 \leq x \in \widetilde{\mathcal{M}}$, if $u \in \widetilde{\mathcal{M}}$ is unitary and if $x - u \in E(\mathcal{M})$ then $x - 1 \in E(\mathcal{M})$ and*

$$\|x - 1\|_{E(\mathcal{M})} \leq \|x - u\|_{E(\mathcal{M})}.$$

We note that equality in Theorem 4.1 may hold for some unitary $u \neq 1$. In fact, if $\mathcal{M}$ is the von Neumann algebra $L^\infty[0,1]$, acting by multiplication on $L^2[0,1]$, set $x = \chi_{[0,\frac{1}{2}]}$ and $u = \chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}$. It is then easily verified that

$$\mu(x-1) \;=\; \mu(x-u).$$

Theorem 4.1 shows that if $0 \leq x \in \widetilde{\mathcal{M}}$ then 1 is a best unitary approximant to x, and as noted in the preceding example, unless further restrictions are imposed, then x need not have a unique best unitary approximant for any fully symmetric norm. Following the terminology of [AEG], if $0 \leq x \in \widetilde{\mathcal{M}}$, then x will be called *strictly positive* if $\ker x = \{0\}$. We will state a result which shows that under certain conditions the best unitary approximant is unique. This extends a similar result obtained in [AEG] for the special case of the Schatten p-classes. A proof is given in [DD] which is a suitable adaptation of the approach of [AEG].

Theorem 4.2. *Let $1 < p < \infty$. If $x \in \widetilde{\mathcal{M}}$ is strictly positive and if there exists a unitary operator u_0 such that $x - u_0 \in L^p(\mathcal{M})$, then it follows that $x - 1 \in L^p(\mathcal{M})$* and

$$\|x-1\|_p \;<\; \|x-u\|_p\,, \qquad \textit{for all unitary} \quad u \neq 1.$$

We remark that a special case of the preceding Theorems 4.1, 4.2 is given in [GK] VI Lemma 3.1 of section 3. For an interesting application of this special case to the study of bases in a separable Hilbert space which are quadratically close to an orthonormal basis, the reader is referred to [GK] Theorem VI 3.3.

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