ALMOST PERIODIC BEHAVIOUR OF UNBOUNDED SOLUTIONS OF DIFFERENTIAL EQUATIONS

BOLIS BASIT AND A. J. PRYDE

Abstract. A key result in describing the asymptotic behaviour of bounded solutions of differential equations is the classical result of Bohl-Bohr: If $\phi : \mathbb{R} \to \mathbb{C}$ is almost periodic and $P\phi(t) = \int_0^t \phi(s)\, ds$ is bounded then $P\phi$ is almost periodic too. In this paper we reveal a new property of almost periodic functions: If $\psi(t) = t^N \phi(t)$ where ϕ is almost periodic and $P\psi(t)/(1 + |t|)^N$ is bounded then $P\phi$ is bounded and hence almost periodic. As a consequence of this result and a theorem of Kadets, we obtain results on the almost periodicity of the primitive of Banach space valued almost periodic functions. This allows us to resolve the asymptotic behaviour of unbounded solutions of differential equations of the form $\sum_{j=0}^m b_j u^{(j)}(t) = t^N \phi(t)$. The results are new even for scalar valued functions. The techniques include the use of reduced Beurling spectra and ergodicity for functions of polynomial growth.

Keywords: almost periodic, almost automorphic, ergodic, reduced Beurling spectrum, primitive of weighted almost periodic functions, Esclangon-Landau.

1. Introduction, Notation and Preliminaries

A problem arising naturally from a theorem of Bohl-Bohr-Kadets [21], (see also [4], [9, Sections 5, 6] and references therein) is to investigate the almost periodicity of the primitive $P\psi$ when $\psi = t^N\phi$, where $\phi : \mathbb{R} \to X$ is almost periodic, X is a Banach space, and N is a non-negative integer. More generally we describe the asymptotic behaviour of solutions $u : \mathbb{R} \to X$ of differential equations of the form $\sum_{j=0}^m b_j u^{(j)}(t) = t^N \phi(t)$ where $b_j \in X$ and $m \in \mathbb{N}$.

We begin by introducing some notation. The function $w(t) = w_N(t) = (1 + |t|)^N$ is a weight on $\mathbb{R}$, satisfying in particular $w(s + t) \le w(s)w(t)$. By J we will mean $\mathbb{R}$, $\mathbb{R}_+$ or $\mathbb{R}_-$. A function $\phi : J \to X$ is called w-bounded if ϕ/w is bounded and $BC_w(J, X)$ is the space of all continuous w-bounded functions, a Banach space with norm $||\phi||_{w,\infty} = \sup_{t\in\mathbb{R}} \frac{||\phi(t)||}{w(t)}$. Following Reiter [28, p. 142], ϕ is *w-uniformly continuous* if $\|\Delta_h\phi\|_{w,\infty} \to 0$ as $h \to 0$ in J. Here $\Delta_h\phi$ denotes the *difference* of ϕ by h defined by $\Delta_h\phi(t) = \phi(h+t) - \phi(t)$. The closed subspace of $BC_w(J,X)$ consisting of all w-uniformly continuous functions is denoted $BUC_w(J,X)$. It is not hard to show that ϕ is w-uniformly continuous if and only if ϕ/w is uniformly continuous. Furthermore, $\|\phi_{t+h} - \phi_t\|_{w,\infty} \leq w(t)\,\|\phi_h - \phi\|_{w,\infty}$ and so

$$\text{if } \phi \in BUC_w(J,X) \text{ then the function}$$
$$t \to \phi_t : J \to BUC_w(J,X) \text{ is continuous.} \tag{1.1}$$

When $N = 0$ or equivalently $w = 1$ we will drop the subscript w from the names of various spaces.

As an example, note that for $\lambda, N \neq 0$, the function $\phi(t) = t^N e^{i\lambda t}$ is not bounded or uniformly continuous. However, ϕ is both w-bounded and w-uniformly continuous and so $\phi \in BUC_w(\mathbb{R},X)$.

We define $TP_w(\mathbb{R},X) = \text{span}\{t^j e^{i\lambda t}\colon\ 0 \leq j \leq N\ \lambda \in \mathbb{R}\}$ and $AP_w(\mathbb{R},X)$ to be the closure in $BUC_w(\mathbb{R},X)$ of $TP_w(\mathbb{R},X)$.

These are natural generalizations of the spaces $TP(\mathbb{R},X)$ of X-valued trigonometric polynomials and $AP(\mathbb{R},X)$ of almost periodic functions which correspond to the case $N = 0$.

Suppose now that $u' = \psi$ where $u \in BUC_w(\mathbb{R},X)$, $\psi \in AP_w(\mathbb{R},X)$ and $X \not\supseteq c_0$, that is X does not contain a subspace isomorphically isometric to c_0. Kadets proved that necessarily $u \in AP_w(\mathbb{R},X)$ when $N = 0$. However, the following example shows that this is not the case for general N. Indeed, we will show below that the general case is more delicate.

Example 1.1. *Take* $X=\mathbb{C}$, $N=1$, $w(t)=1+|t|$, $\psi(t)=\frac{t}{w(t)}\cos\log w(t)$ *and* $u(t) = \frac{1}{2}w(t)\cos\log w(t) + \frac{1}{2}w(t)\sin\log w(t) - \sin\log w(t) - \frac{1}{2}$. *Then* $u \in BUC_w(\mathbb{R},X)$, $\psi \in AP_w(\mathbb{R},\mathbb{C})$ *and* $u' = \psi$. *However,* $u \notin AP_w(\mathbb{R},\mathbb{C})$.

The proof of this assertion requires some further theory and will be given in Remark 4.4.

2. Some function spaces.

In [12] a function $\phi : J \to X$ is called *Maak-ergodic* with mean $M\phi = x \in X$ (see also [25], [19], [20], [11]) if for each $\varepsilon > 0$ there is a finite subset $F \subseteq J$ with $\|R_F\phi - x)\| < \varepsilon$ where $R_F\phi = \frac{1}{|F|}\sum_{t\in F}\phi_t$. Moreover $E(J,X)$ is the closed subspace of $BC(J,X)$ consisting of Maak-ergodic functions and $E_0(J,X) = \{\phi \in E(J,X) : M\phi = 0\}$. If $M : E(J,X) \to X$ is the function $\phi \to M\phi$, it follows that M is linear and continuous and $E(J,X) = E_0(J,X) \oplus X$.

In [12] we also defined a notion of ergodicity that applies to unbounded functions. This ergodicity differs from both that of Maak and that of Basit-Günzler [9]. Indeed, the space of *w-ergodic* functions is defined by $E_w(J,X) = \{ \phi \in BC_w(J,X) : \phi/w \in E(J,X)\}$. Both $E_w(J,X)$ and $E_{w,0}(J,X) = \{\phi \in E_w(J,X) : M(\phi/w) = 0\}$ are closed subspaces of $BC_w(J,X)$. It is convenient to introduce an even larger class. For this we need $P_w(J,X)$ the closed subspace of $BC_w(J,X)$ consisting of polynomials on J with coefficients in X. We shall say a function $\phi : J \to X$ is *w -polynomially ergodic* with *w-mean* $p \in P_w(J,X)$ if $(\phi - p)/w \in E_0(J,X)$. The space of all such ϕ is denoted $PE_w(J,X)$ and satisfies $PE_w(J,X) = E_{w,0}(J,X) + P_w(J,X)$. For $N \neq 0$ a w-mean is not unique and this last sum is not direct.

Of course $P_w(J,\mathbb{C})$ is finite dimensional and so $PE_w(J,\mathbb{C})$ is a closed subspace of $BC_w(J,\mathbb{C})$. Moreover, we can choose a subspace $P_w^M(J,\mathbb{C})$ of $P_w(J,\mathbb{C})$ such that $PE_w(J,\mathbb{C}) = E_{w,0}(J,\mathbb{C}) \oplus P_w^M(J,\mathbb{C})$. The (continuous) projection map $M_w : PE_w(J,\mathbb{C}) \to P_w^M(J,\mathbb{C})$ then provides a unique w-polynomial mean $M_w(\phi)$ for each $\phi \in PE_w(J,\mathbb{C})$. Now set $P_w^M(J,X) = P_w^M(J,\mathbb{C}) \otimes X$ and define $M_w : PE_w(J,X) \to P_w^M(J,X)$ by $M_w(\phi) = \sum_{j=1}^k M_w(p_j) \otimes x_j$ where $\phi \in PE_w(J,X)$ has w-polynomial mean $p = \sum_{j=1}^k p_j \otimes x_j \in P_w(J,\mathbb{C}) \otimes X$.

Proposition 2.1. *The map $M_w : PE_w(J,X) \to P_w^M(J,X)$ is well-defined and continuous. Moreover, for each $\phi \in PE_w(J,X)$, $M_w(\phi)$ is a w-polynomial mean for ϕ and for each of its translates. Finally, $PE_w(J,X)$ is a closed translation invariant subspace of $BC_w(J,X)$ and $PE_w(J,X) = E_{w,0}(J,X) \oplus P_w^M(J,X)$.*

Proof. Let $\phi \in PE_w(J,X)$ have means $p = \sum_{j=1}^k p_j \otimes x_j$ and $q = \sum_{j=1}^m q_j \otimes y_j$. Then $p - q \in E_{w,0}(J,X)$ and so $x^* \circ (p-q) \in E_{w,0}(J,\mathbb{C})$ for all $x^* \in X^*$. Hence $M_w(x^* \circ (p-q)) = 0 = x^* \circ (\sum_{j=1}^k M_w(p_j) \otimes x_j - \sum_{j=1}^m M_w(q_j) \otimes y_j)$ which gives $\sum_{j=1}^k M_w(p_j) \otimes x_j = \sum_{j=1}^m M_w(q_j) \otimes y_j$ showing M_w is well-defined. Also, $p_j - M_w(p_j) \in E_{w,0}(J,\mathbb{C})$ and so by Lemma 2.2(a) below $p - M_w(p) \in E_{w,0}(J,X)$. Hence $M_w(\phi)$ is a mean for ϕ. Moreover, $\|x^* \circ M_w(\phi)\|_{w,\infty} = \|M_w(x^* \circ \phi)\|_{w,\infty} \leq c\,\|x^* \circ \phi\|_{w,\infty} = c \sup_{t \in J} \|x^* \circ \phi(t)\| / w(t) \leq c\,\|x^*\|\,\|\phi\|_{w,\infty}$. Hence, $|||M_w(\phi)||| \leq c\,\|\phi\|_{w,\infty}$ where $|||\psi||| = \sup_{x^* \in X^*} \frac{\|x^* \circ \psi\|_{w,\infty}}{\|x^*\|}$ for $\psi \in P_w(J,X)$. By Lemma 2.2(b) below, M_w is continuous. If (ϕ_n) is a sequence in $PE_w(J,X)$ converging to ϕ in $BC_w(J,X)$, let $p_n = M_w(\phi_n)$. Then (p_n) converges to some $p \in P_w^M(J,X)$ and so $(\frac{\phi_n - p_n}{w})$ converges to $\frac{\phi - p}{w}$ in $BC(J,X)$. By the continuity of the Maak mean function, $M(\frac{\phi-p}{w}) = 0$ and so $PE_w(J,X)$ is closed. That $PE_w(J,X) =$

$E_{w,0}(J,X) + P_w^M(J,X)$ is clear and that the sum is direct follows from the Hahn-Banach theorem. Finally, for each $t \in J$ we have $\frac{\phi_t - p}{w} = \frac{\Delta_t \phi}{w} + \frac{\phi - p}{w}$ and so, by Lemma 2.2(c) below, p is a w-polynomial mean of ϕ_t and $PE_w(J,X)$ is translation invariant. □

Lemma 2.2.

(a) *If* $p \in P_w(J,X)$ *and* $x^* \circ p \in E_{w,0}(J,\mathbb{C})$ *for each* $x^* \in X^*$ *then* $p \in E_{w,0}(J,X)$.

(b) *On* $P_w(J,X)$ *the norms* $\|\phi\|_{w,\infty}$ *and* $|||\phi||| = \sup_{x^* \in X^*} \frac{\|x^* \circ \phi\|_{w,\infty}}{\|x^*\|}$ *are equivalent.*

(c) *If* $\phi \in BC_w(J,X)$ *then* $\Delta_t \phi \in E_{w,0}(J,X)$ *for all* $t \in J$.

(d) *If* $\phi \in PE_w(\mathbb{R},X)$ *has* w*-mean* p*, then* $\phi|_J \in PE_w(J,X)$ *and* $\phi|_J$ *has* w*-mean* $p|_J$.

(e) $P_w(J,X) \subseteq BUC_w(J,X)$.

(f) *Let* $\phi \in BUC_w(\mathbb{R},X)$*,* $f \in L^1_w(\mathbb{R})$ *and suppose* $\phi|_J$ *is* w*-polynomially ergodic with* w*-mean* $p|_J$ *where* $p \in P_w(\mathbb{R},X)$*. Then* $(\phi * f)|_J$ *is* w*-polynomially ergodic with* w*-mean* $(p * f)|_J$ *.*

Proof. (a) We can choose $q_1, ..., q_m \in P_w(J,\mathbb{C})$ and linearly independent unit vectors $x_1, ..., x_m \in X$ such that $p = \sum_{j=1}^m q_j \otimes x_j$. Also choose unit vectors $x_j^* \in X^*$ such that $\langle x_j^*, x_i \rangle = \delta_{i,j}$. Given $\varepsilon > 0$ there are finite subsets F_j of J such that $\left\| R_{Fj}(x_j^* \circ p/w) \right\| < \varepsilon/m$. Setting $F = F_1 + ... + F_m$ we find

$$\begin{aligned} \|R_F(p/w)\| &= \left\| \sum_{j=1}^m R_F(q_j/w) \otimes x_j \right\| \le \sum_{j=1}^m \|R_F(q_j/w)\| \\ &= \sum_{j=1}^m \left\| R_F(x_j^* \circ p/w) \right\| \le \sum_{j=1}^m \left\| R_{F_j}(x_j^* \circ p/w) \right\| < \varepsilon. \end{aligned}$$

This proves that $p \in E_{w,0}(J,X)$

(b) Let $\{p_1, ..., p_k\}$ be a basis of $P_w(J,\mathbb{C})$ consisting of unit vectors for the norm $\|p\|_{w,\infty}$. If $p = \sum_{j=1}^k c_j p_j$, where $c_j \in \mathbb{C}$ then $\|p\|_{w,\infty} \sim \sum_{j=1}^k \|c_j\|$, $\sim$ denoting equivalence of norms. Every $\phi \in P_w(J,X)$ has a unique representation $\phi = \sum_{j=1}^k p_j \otimes x_j$, where $x_j \in X$, and by the closed graph theorem, $\|\phi\|_{w,\infty} \sim \sum_{j=1}^k \|x_j\|$. Hence, $|||\phi||| = \sup \left\| \sum_{j=1}^k p_j \langle x^*, x_j \rangle \right\| / \|x^*\| \le \sum_{j=1}^k \|x_j\| \sim \|\phi\|_{w,\infty}$. Conversely, choose j_0 such that $\|x_j\| \le \|x_{j_0}\|$ for each j. Then choose $x^* \in X^*$ such that $\langle x^*, x_{j_0} \rangle = \|x_{j_0}\|$ and $\|x^*\| = 1$. Hence,

$$|||\phi||| \geq \left\| \sum_{j=1}^{k} p_j \langle x^*, x_j \rangle \right\|_{w,\infty}$$
$$\sim \sum_{j=1}^{k} |\langle x^*, x_j\rangle| \geq \|x_{j_0}\| \geq \frac{1}{k} \sum_{j=1}^{k} \|x_j\| \sim \|\phi\|_{w,\infty}.$$

(c) We have $\frac{\Delta_t \phi}{w} = \Delta_t(\frac{\phi}{w}) + (\frac{\phi}{w})_t \frac{\Delta_t w}{w}$ where $\Delta_t(\frac{\phi}{w}) \in E_0(J, X)$ by [11, Proposition 3.2] and $(\frac{\phi}{w})_t \frac{\Delta_t w}{w} \in C_0(J, X)$.

(d) We prove the case $J = \mathbb{R}_+$. Given $\varepsilon > 0$ there is a finite subset $F = \{t_1, ..., t_m\} \subseteq \mathbb{R}$ such that $\left\| \frac{1}{m} \sum_{j=1}^{m} (\frac{\phi - p}{w})(t_j + t) \right\| < \varepsilon$ for all $t \in \mathbb{R}$. choose $u_j, v_j \in \mathbb{R}_+$ such that $t_j = u_j - v_j$. Let $v = v_1 + ... + v_m$ and set $s_j = t_j + v$. So $s_j \in \mathbb{R}_+$ and $\left\| \frac{1}{m} \sum_{j=1}^{m} (\frac{\phi - p}{w})(s_j + t) \right\| < \varepsilon$ for all $t \in \mathbb{R}_+$.

(e) Given $p \in P^n(J, X)$ we may choose $p_j \in P^n(J, X)$ and $q_j \in P^n(J, \mathbb{C})$ with $q_j(0) = 0$ such that $\Delta_h p(t) = \sum_{j=1}^{k} p_j(t) q_j(h)$ for all $h, t \in J$. Hence $\|\Delta_h p(t)\| \leq c w(t) \sum_{j=1}^{k} |q_j(h)|$, where $c = \sup_j \|p_j\|_{w,\infty}$, and so $p \in BUC_w(\mathbb{R}, X)$.

(f) If χ is the characteristic function of a compact set $K \subseteq \mathbb{R}$ then $(\phi - p) * \chi(s) = \int_{-K} (\phi - p)_t(s) dt$ for each $s \in \mathbb{R}$. But for each $t \in \mathbb{R}$, $(\phi - p)_t = \Delta_t(\phi - p) + (\phi - p)$ and so by (c), $(\phi - p)_t|_J \in E_{w,0}(J, X)$. Also, by (e), $\phi - p \in BUC_w(\mathbb{R}, X)$. By (1.1), the function $t \to (\phi - p)_t|_J : \mathbb{R} \to E_{w,0}(J, X)$ is continuous and hence weakly measurable and separably-valued on $-K$. The integral $\int_{-K} (\phi - p)_t|_J dt$ is therefore a convergent Haar-Bochner integral and so belongs to $E_{w,0}(J, X)$. As evaluation at $s \in J$ is continuous on $E_{w,0}(J, X)$ we conclude that $((\phi - p) * \chi)|_J \in E_{w,0}(J, X)$. Hence also $((\phi - p) * \sigma)|_J \in E_{w,0}(J, X)$ for any step function $\sigma : \mathbb{R} \to \mathbb{C}$ By [28, p. 83] the step functions are dense in $L^1_w(\mathbb{R})$ and so $((\phi - p) * f)|_J \in E_{w,0}(J, X)$ for any $f \in L^1_w(\mathbb{R})$. □

The difference theorem below, included here in order to characterize $E_w(J, X)$ will also be used later. We use the notation $C_{w,0}(J, X) = \{w\xi : \xi \in C_0(J, X)\}$, clearly a closed subspace of $BUC_w(J, X)$.

Theorem 2.3. *Let $\mathcal{F}$ be any translation invariant closed subspace of $BC_w(J, X)$. If $\phi \in PE_w(J, X)$ has w-mean p and $\Delta_t \phi \in \mathcal{F}$ for each $t \in J$, then $\phi - p \in \mathcal{F} + C_{w,0}(J, X)$. If also $w = 1$, then $\phi - p \in \mathcal{F}$.*

Proof. For any finite subset $F \subseteq J$, $\frac{\phi - p}{w} - R_F(\frac{\phi - p}{w}) = -\frac{1}{|F|} \sum_{t \in F} \Delta_t(\frac{\phi - p}{w})$ and so $\phi - p = w R_F(\frac{\phi - p}{w}) - \frac{1}{|F|} \sum_{t \in F} \Delta_t \phi + \frac{1}{|F|} \sum_{t \in F} (\frac{\phi - p}{w})_t \Delta_t w + \frac{1}{|F|} \sum_{t \in F} \Delta_t p$. The first term on the right may be made arbitrarily small in norm by suitable choice of F. The second term is in $\mathcal{F}$ by assumption, the third and fourth terms are in $C_{w,0}(J,X)$ since $\Delta_t p$, $\Delta_t w \in C_{w,0}(J,X)$ for $t \in J$. If $w = 1$ then $\Delta_t w = \Delta_t p = 0$ which shows $\phi - p \in \mathcal{F}$. □

We are now able to characterize w-polynomially ergodic functions. Denote by $D_{w,0}(J,X)$ the closed span of $\{\Delta_t\phi : t \in J, \phi \in BC_w(J,X)\} \cup C_{w,0}(J,X)$ and by $D_w(J,X)$ the closed span of $\{\Delta_t\phi : t \in J, \phi \in BC_w(J,X)\}$.

Corollary 2.4. *$E_{w,0}(J,X) = D_{w,0}(J,X)$. If $w = 1$, then $D_{w,0}(J,X) = D_w(J,X) = E_0(J,X)$.*

Proof. Since $C_{w,0}(J,X) \subset E_{w,0}(J,X)$, by Lemma 2.2 (c) and the closedness of $E_{w,0}(J,X)$, we have $D_{w,0}(J,X) \subset E_{w,0}(J,X)$. Conversely, let $\phi \in E_{w,0}(J,X)$. Then $\Delta_t\phi \in D_w(J,X)$ for all $t \in J$ and by Theorem 2.3, $\phi \in D_{w,0}(J,X)$. If $w = 1$, then for any $\psi \in C_{w,0}(J,X)$ and any finite subset F of J, we have $\psi = -\frac{1}{|F|}\sum_{t\in F}\Delta_t\psi + R_F\psi$. As $\|R_F\psi\|_{w,\infty}$ may be made arbitrarily small, we conclude that $\psi \in D_w(J,X)$ and hence $D_{w,0}(J,X) = D_w(J,X)$. □

We conclude this section with a characterization of $AP_w(\mathbb{R},X)$.

Theorem 2.5. *If $N \geq 1$, then $AP_w(\mathbb{R},X) = t^N AP(\mathbb{R},X) \oplus C_{w,0}(\mathbb{R},X)$.*

Proof. Note firstly that the sum on the right is direct. For suppose $t^N\psi_1 + \xi_1 = t^N\psi_2 + \xi_2$ for $\psi_j \in AP(\mathbb{R},X)$ and $\xi_j \in C_{w,0}(\mathbb{R},X)$. Set $q(t) = (1+t)^N - t^N$ and $J = \mathbb{R}_+$. Then $(\psi_1 - \psi_2)|_J = \frac{1}{w}(\xi_2 - \xi_1 - q\psi_2 + q\psi_1)|_J \in C_0(J,X)$. This is impossible unless $\psi_1 = \psi_2$ (see [31] or [5, Proposition 2.1.6). The sum is also topological. For suppose $\phi_n = t^N\psi_n + \xi_n$ where $\psi_n \in AP(\mathbb{R},X)$ and $\xi_n \in C_{w,0}(\mathbb{R},X)$ and (ϕ_n) converges to ϕ in $BC_w(\mathbb{R},X)$. Then $\frac{\phi_n}{w}|_J = \psi_n|_J + \frac{\xi_n - q\psi_n}{w}|_J \in AP(\mathbb{R},X)|_J \oplus C_0(J,X)$. But this last sum is a topological direct sum (see [5, 31]) and so $(\psi_n|_J)$ converges to $\psi|_J$ for some $\psi \in AP(\mathbb{R},X)$. Hence (ψ_n) converges to ψ in $AP(\mathbb{R},X)$ and $\left(t^N\psi_n\right)$ converges to $t^N\psi$ in $AP_w(\mathbb{R},X)$. It follows that (ξ_n) converges to some ξ in $C_{w,0}(\mathbb{R},X)$ and that $\phi = t^N\psi + \xi$.

Next, given $\phi \in AP_w(\mathbb{R},X)$ we may choose a sequence $(\pi_n) \subset TP_w(\mathbb{R},X)$ converging to ϕ in $AP_w(\mathbb{R},X)$. But $\pi_n = t^N\psi_n + \xi_n$ where $\psi_n \in TP(\mathbb{R},X)$ and $\xi_n \in C_{w,0}(\mathbb{R},X)$. It follows from the previous paragraph that $\phi = t^N\psi + \xi$ for some $\psi \in AP(\mathbb{R},X)$ and $\xi \in C_{w,0}(\mathbb{R},X)$.

Conversely, let $\phi = t^N\psi + \xi$ for some $\psi \in AP(\mathbb{R},X)$ and $\xi \in C_{w,0}(\mathbb{R},X)$. We may choose $\alpha_n \in TP(\mathbb{R},X)$ such that $\|\psi(t) - \alpha_n(t)\| \leq \frac{1}{n}$ (see[1, (1.2), p. 15] or [23]). Moreover, $\frac{\xi}{w} \in C_0(\mathbb{R},X)$ and $\|\xi(s)\| \leq$

$\|\xi\|_{\infty,w} w(s)$ for all s. Hence we may choose $\widetilde{t}_n > 0$ such that $\|\xi(t)\| \le \frac{1}{n} w(t)$ for all $|t| \ge \widetilde{t}_n$. Then choose $t_n > \widetilde{t}_n$ such that $\|\xi\|_{\infty,w} w(\widetilde{t}_n) \le \frac{1}{n} w(t_n)$. It follows that $\|\xi(t)\| \le \frac{1}{n} w(t)$ and $\|\xi(s)\| \le \frac{1}{n} w(t_n)$ for all $|t| \ge t_n$ and all $|s| \le t_n$. Since ξ is continuous we may choose $\beta_n \in TP(\mathbb{R}, X)$ such that $\|\xi(s) - \beta_n(s)\| \le \frac{1}{n}$ and $\|\beta_n(t)\| \le \frac{1}{n} w(t_n) + \frac{1}{n}$ for all $|s| \le t_n$ and all t. Thus $\|\xi(s) - \beta_n(s)\| \le \frac{3}{n} w(s)$ for all s. Set $\pi_n = t^N \alpha_n + \beta_n \in TP_w(\mathbb{R}, X)$. Then (π_n) converges to ϕ in $BC_w(\mathbb{R}, X)$ and so $\phi \in AP_w(\mathbb{R}, X)$. □ □

Corollary 2.6. *$AP_w(\mathbb{R}, X) \subset PE_w(\mathbb{R}, X)$.*

Proof. Let $\phi = t^N \psi + \xi$ where $\psi \in AP(\mathbb{R}, X)$ and $\xi \in C_{w,0}(\mathbb{R}, X)$. Set $p = t^N M\psi$ and $\psi_0 = \psi - M\psi$. Then $\frac{\phi - p}{w} = \psi_0 - \frac{\psi_0}{w} + \frac{\xi}{w}$ on $\mathbb{R}_+$ and $\frac{\phi - p}{w} = -\psi_0 + \frac{\psi_0}{w} + \frac{\xi}{w}$ on $\mathbb{R}_-$. Hence $\frac{\phi - p}{w}|_J \in E_0(J, X)$ for $J = \mathbb{R}_+$ or $\mathbb{R}_-$ and thus $\frac{\phi - p}{w} \in E_0(\mathbb{R}, X)$. □

3. Spectral analysis

Throughout this section we will assume that $\mathcal{F}$ is a BUC_w-invariant closed subspace of $BC_w(J, X)$. A subspace $\mathcal{F}$ of $BC_w(J, X)$ is called BUC_w-invariant (see [12]) if $\phi_t|_J \in \mathcal{F}$ whenever $\phi \in BUC_w(\mathbb{R}, X)$, $\phi|_J \in \mathcal{F}$ and $t \in \mathbb{R}$. Numerous examples are provided in [12].

The dual group of $\mathbb{R}$ is denoted $\widehat{\mathbb{R}} = \{\gamma_s : \gamma_s(t) = e^{ist} \text{ for } s, t \in \mathbb{R}\}$ and the Fourier transform of $f \in L^1(\mathbb{R})$ by $\hat{f}(\gamma_s) = \int_{-\infty}^{\infty} f(t)\gamma_s(-t)dt$.

Let $\phi \in BC_w(\mathbb{R}, X)$. The set $I_w(\phi) = \{f \in L^1_w(\mathbb{R}) : \phi * f = 0\}$ is a closed ideal of $L^1_w(\mathbb{R})$ and the *Beurling spectrum* of ϕ is defined to be $sp_w(\phi) = \operatorname{cosp}(I_w(\phi)) = \{\gamma \in \widehat{\mathbb{R}} : \hat{f} = 0 \text{ for all } \gamma \in I_w(\phi)\}$. More generally, following [5, Section 4], the set $I_{\mathcal{F}}(\phi) = \{f \in L^1_w(\mathbb{R}) : (\phi * f)|_J \in \mathcal{F}\}$ is a closed translation invariant subspace of $L^1_w(\mathbb{R})$ and therefore an ideal. We define the *spectrum of ϕ relative to $\mathcal{F}$*, or the *reduced Beurling spectrum*, to be $sp_{\mathcal{F}}(\phi) = \operatorname{cosp}\,(I_{\mathcal{F}}(\phi))$.

The following proposition contains some basic properties of these spectra. The proofs are the same as for the Beurling spectrum. See for example [17, p. 988] or [29] also [6], [15], [27].

Proposition 3.1. *Let $\phi, \psi \in BC_w(\mathbb{R}, X)$.*

(a) *$sp_{\mathcal{F}}(\phi_t) = sp_{\mathcal{F}}(\phi)$ for all $t \in \mathbb{R}$.*

(b) *$sp_{\mathcal{F}}(\phi * f) \subseteq sp_{\mathcal{F}}(\phi) \cap supp(\hat{f})$ for all $f \in L^1_w(\mathbb{R})$.*

(c) *$sp_{\mathcal{F}}(\phi + \psi) \subseteq sp_{\mathcal{F}}(\phi) \cup sp_{\mathcal{F}}(\psi)$.*

(d) *$sp_{\mathcal{F}}(\gamma\phi) = \gamma sp_{\mathcal{F}}(\phi)$, provided $\mathcal{F}$ is invariant under multiplication by $\gamma \in \widehat{\mathbb{R}}$.*

(e) *If* $f \in L^1_w(\mathbb{R})$ *and* $\hat{f} = 1$ *on a neighbourhood of* $sp_{\mathcal{F}}(\phi)$*, then* $sp_{\mathcal{F}}(\phi * f - \phi) = \emptyset$.

The following theorem is proved in [12](see also [10], [11]). It gives our motivation for introducing $sp_{\mathcal{F}}(\phi)$.

Theorem 3.2. *Let* $\phi \in BUC_w(\mathbb{R}, X)$.

(a) *If* $f \in L^1_w(G)$ *and* $\phi|_J \in \mathcal{F}$*, then* $(\phi * f)|_J \in \mathcal{F}$.

(b) $sp_{\mathcal{F}}(\phi) = \emptyset$ *if and only if* $\phi|_J \in \mathcal{F}$.

(c) *If* $\Delta^k_t \phi|_J \in \mathcal{F}$ *for all* $t \in \mathbb{R}$ *and some* $k \in \mathbb{N}$*, then* $sp_{\mathcal{F}}(\phi) \subseteq \{1\}$.

(d) $sp_{\mathcal{F}}(\phi) \subseteq \{\gamma_1, ..., \gamma_n\}$ *if and only if* $\phi = \psi + \sum_{j=1}^n \eta_j \gamma_j$ *for some* $\psi, \eta_j \in BUC_w(\mathbb{R}, X)$ *with* $\psi|_J \in \mathcal{F}$ *and* $\Delta_t \eta_j|_J \in \mathcal{F}$ *for each* $t \in \mathbb{R}^{N+1}$.

4. Primitives and Derivatives

Throughout this section we assume that $\mathcal{F}$ is a translation invariant closed subspace of $BUC_w(J, X)$. Examples of such classes are $P_w(J, X)$, $C_{w,0}(J, X)$, $AP_w(\mathbb{R}, X)$, $E_{w,0}(J, X) \cap BUC_w(J, X)$ and $PE_w(J, X) \cap BUC_w(J, X)$.

We define the *primitive* $P\phi$ of a function $\phi \in BC_w(\mathbb{R}, X)$ by $P\phi(t) = \int_0^t \phi(s) ds$.

Theorem 4.1.

(a) *If* $\mathcal{F}_w$ *denotes any of* $BC_w(J, X)$, $C_{w,0}(J, X)$, $E_{w,0}(J, X)$, $P_w(J, X)$, $PE_w(J, X)$ *or* $AP_w(\mathbb{R}, X)$ *then* P *maps* $\mathcal{F}_w$ *continuously into* $\mathcal{F}_{ww_1}$.

(b) *If* $\phi \in E_{w,0}(J, X)$ *then* $P\phi \in C_{ww_1,0}(J, X)$.

(c) *If* $\phi \in AP_w(J, X)$ *has* w*-mean* p*. Then* $P(\phi - p) \in C_{ww_1,0}(\mathbb{R}, X)$.

Proof. Take $J = \mathbb{R}_+$, the other cases being proved similarly. If $\phi \in BC_w(J, X)$ and $t \in J$ then $||P\phi(t)|| \le t.||\phi||_{w,\infty} w(t)$. Hence P maps $BC_w(J, X)$ continuously into $BC_{ww_1}(J, X)$. If also $\phi \in C_{w,0}(J, X)$ then given $\varepsilon > 0$ there exists $t_0 > 0$ such that $||\phi(t)|| < \varepsilon w(t)$ whenever $t > t_0$. For these t we have $||P\phi(t)|| \le \int_0^{t_0} ||\phi(s)||\, ds + \varepsilon w(t)(t - t_0)$ and so P maps $C_{w,0}(J, X)$ into $C_{ww_1,0}(J, X)$. Next, $P(\Delta_t \phi) = \Delta_t(P\phi) - P\phi(t)$ and since P is continuous it follows from Corollary 2.4 that P maps $E_{w,0}(J, X)$ into $E_{ww_1,0}(J, X)$. The result for $P_w(J, X)$ is clear and so therefore is the result for $PE_w(J, X)$. For (b) note that $||\Delta_t\, P\phi(s)|| \le t.w(t)w(s)\, ||\phi||_{w,\infty}$ for all $s \in J$. Hence $\Delta_t(P\phi) \in C_{ww_1,0}(J, X)$. If $\phi \in E_{w,0}(J, X)$, we can apply Theorem 2.3 to $P\phi$ to obtain $P\phi \in C_{ww_1,0}(J, X)$. Finally, (c) follows from (b) using Corollary 2.6, and then (a) with $\mathcal{F}_w = AP_w(\mathbb{R}, X)$ follows from (c) using Theorem 2.5. □

Proposition 4.2.

(a) *If* $\phi \in BC_w(\mathbb{R}, X)$ *and* $sp_w(\phi)$ *is compact, then* $\phi^{(j)} \in BUC_w(\mathbb{R}, X)$ *for all* $j \geq 0$.

(b) *If* $\phi \in \mathcal{F}$ *and* ϕ' *is* w*-uniformly continuous, then* $\phi' \in \mathcal{F}$.

(c) *If* $\phi \in BC_w(J, X)$ *and* ϕ' *is* w *-uniformly continuous, then* $\phi' \in E_{w,0}(J, X) \cap BUC_w(J, X)$.

(d) *If* $\phi, \phi' \in BC_w(\mathbb{R}, X)$ *then* $sp_w(\phi') \subseteq sp_w(\phi) \subseteq sp_w(\phi') \cup \{1\}$.

(e) *If* $\phi, \phi' \in BC_w(J, X)$ *then* $\phi \in BUC_w(J, X)$.

Proof. (a) Choose $f \in S(\mathbb{R})$, the Schwartz space of rapidly decreasing functions, such that f has compact support and is 1 on a neighbourhood of $sp_w(\phi)$. Then $f^{(j)} \in L^1_w(\mathbb{R})$ for all $j \geq 0$. Moreover, $\phi = \phi * f$ and so $\phi^{(j)} = \phi * f^{(j)}$ for all $j \geq 0$. Hence $\phi^{(j)} \in BUC_w(\mathbb{R}, X)$.

(b) If $\psi_n = n\Delta_{1/n}\phi$ then $\psi_n \in \mathcal{F}$. Moreover, by the w-uniform continuity of ϕ', given $\varepsilon > 0$ there exists n_ε such that

$$\|\psi_n(t) - \phi'(t)\| = \left\| n \int_0^{1/n} (\phi'(t+s) - \phi'(t))ds \right\| < \varepsilon w(t)$$

for all $t \in J$ and $n > n_\varepsilon$. Hence $\phi' \in \mathcal{F}$.

(c) With the notation used in the proof of (b), $\psi_n \in E_{w,0}(J, X) \cap BUC_w(J, X)$ by Lemma 2.2(c). Hence, so does ϕ'.

(d) For any $f \in L^1_w(\mathbb{R})$ we have $(\phi * f)' = \phi' * f$ and so $I_w(\phi') \supseteq I_w(\phi)$. Hence, $sp_w(\phi') \subseteq sp_w(\phi)$. For the second inclusion, let $g(t) = \exp(-t^2)$ so that $g, g' \in L^1_w(\mathbb{R})$ and $\hat{g}$ is never zero. Now take $\gamma \in \widehat{\mathbb{R}}\ (sp_w(\phi') \cup \{1\})$. So $\gamma(t) = e^{ist}$ for some $s \neq 0$ and there exists $f \in L^1_w(\mathbb{R})$ such that $\phi' * f = 0$ but $\hat{f}(\gamma) \neq 0$. Let $h = f * g' \in L^1_w(\mathbb{R})$. Then $\phi * h = \phi * f * g' = \phi' * f * g = 0$ whereas $\hat{h}(\gamma) = is\hat{f}(\gamma)\hat{g}(\gamma) \neq 0$. So $\gamma \notin sp_w(\phi')$ showing $sp_w(\phi) \subseteq sp_w(\phi') \cup \{1\}$.

(e) For any $h, t \in J$ we have $\|\Delta_h\phi(t)\| = \left\| \int_t^{t+h} \phi'(s)ds \right\| \leq |h| \cdot \|\phi'\|_{w,\infty} w(h)w(t)$ from which it follows that ϕ is w-uniformly continuous. □

Proposition 4.3. *Let* $\phi \in \mathcal{F}$ *and assume that* $\mathcal{F}$ *is* BUC_w*-invariant.*

(a) *If* $P\phi$ *is* w*-polynomially ergodic with* w*-mean* p, *then* $P\phi - p \in \mathcal{F} + C_{w,0}(J, X)$.

(b) *If* $\mathcal{F} = AP_w(\mathbb{R}, X)$ *and* $P\phi \in PE_w(\mathbb{R}, X)$, *then* $P\phi \in AP_w(\mathbb{R}, X)$.

(c) *If* $\mathcal{F} = AP_w(\mathbb{R}, X)$ *and* $P\phi \in BUC_w(\mathbb{R}, X)$, *then* $sp_{\mathcal{F}}(P\phi) \subseteq \{1\}$.

(d) *If* $\mathcal{F} = C_0(\mathbb{R}_+, X)$ *and* $P\phi$ *is ergodic with mean* c, *then* $P\phi - c \in C_0(\mathbb{R}_+, X)$.

Proof. (a) Take $J = \mathbb{R}_+$, the other cases being proved similarly. Extend ϕ to an even function $\widetilde{\phi} \in BUC_w(\mathbb{R}, X)$. For $t \geq 0$ set $\chi_t = \chi_{[-t,0]}$ so that $\Delta_t P\phi = \left(\widetilde{\phi} * \chi_t\right)|_J = \int_{\mathbb{R}} \left(\widetilde{\phi}_{-s}\right)|_J \chi_t(s)ds$. Since $\widetilde{\phi} \in BUC_w(\mathbb{R}, X)$ the integral converges as a Lebesgue-Bochner integral. Since $\mathcal{F}$ is BUC_w-invariant $\left(\widetilde{\phi}_{-s}\right)|_J \in \mathcal{F}$ and therefore $\Delta_t P\phi \in \mathcal{F}$. The result follows from Theorem 2.3.

(b) In view of Theorem 2.5, this follows from (a).

(c) Let $s, t \in \mathbb{R}$ With χ_s as in the previous proof, $(\Delta_s P\phi)_t = \phi_t * \chi_s$ and by Proposition 3.2 (a), $(\Delta_s P\phi)_t \in \mathcal{F}$. By Proposition 3.2(c), $sp_{\mathcal{F}}(P\phi) \subseteq \{1\}$.

(d) This is a special case of part (a). □

Remark 4.4. *Recall* $u(t) = \frac{1}{2}w(t)\cos\log w(t) + \frac{1}{2}w(t)\sin\log w(t) - \sin\log w(t) - \frac{1}{2}$ *from Example 1.1. So* $u'(t) = \frac{t}{w(t)}\cos\log w(t)$ *and therefore* $u' \in C_{w,0}(\mathbb{R}, \mathbb{C}) \subset AP_w(\mathbb{R}, \mathbb{C}) \subset PE_w(\mathbb{R}, \mathbb{C})$. *However,* $u \notin PE_w(\mathbb{R}, \mathbb{C})$. *Indeed, if* $u \in PE_w(\mathbb{R}, \mathbb{C})$ *then for* $t \in \mathbb{R}_+$ *set* $\xi(t) = w(t)\cos\log w(t) + w(t)\sin\log w(t)$. *So* $\xi \in PE_w(\mathbb{R}_+, \mathbb{C})$ *and for some polynomial* $p(t) = at + b$ *we have* $(\xi - p)/w \in E_0(\mathbb{R}_+, \mathbb{C})$. *Thus* $\eta = \xi/w \in E(\mathbb{R}_+, \mathbb{C})$. *But* $\eta'(t) = [-\sin\log w(t) + \cos\log w(t)]/w(t)$ *and so* $\eta' \in C_0(\mathbb{R}, \mathbb{C})$. *By Proposition 4.3(d) we conclude* $\eta \in C_0(\mathbb{R}_+, \mathbb{C}) + \mathbb{C}$ *which is false.*

Lemma 4.5. *For natural numbers* m, N *and non-negative integers* j, k *set* $a(m,j) = (-1)^j \binom{N}{j}\binom{m-1+j}{j} j!$.

(a) $P^m(t^N\phi) = \sum_{j=0}^{N} a(m,j)\, t^{N-j} P^{m+j}\phi$ *for any* $\phi \in L^1_{loc}(J, X)$.

(b) $$\sum_{j=0}^{N} \frac{a(m,j)}{(j+k)!} = \begin{cases} \binom{N+k-m}{N}\frac{N!}{(N+k)!} & \text{if } m \leq k \\ 0 & \text{if } k+1 \leq m \leq k+N \\ (-1)^N \binom{m-k-1}{N}\frac{N!}{(N+k)!} & \text{if } m > k+N \end{cases}.$$

(c) $\sum_{j=0}^{N} a(m,j)\, t^{N-j} P^{j+1} r = P\sum_{j=0}^{N} a(m-1,j)\, t^{N-j} P^j\, r$ *for any* $r \in P_{m-2}(J, X)$.

Proof. (a) For $N = 1$ the claim is readily proved by induction on m. The general case is then proved by induction on N.

(b) For $m \geq k+1$ we have

$$\sum_{j=0}^{N} \frac{a(m,j)}{(j+k)!} = \sum_{j=0}^{N} (-1)^j \binom{N}{j}\binom{m-1+j}{j} \frac{j!}{(j+k)!}$$

$$= \frac{1}{(m-1)!}\sum_{j=0}^{N}(-1)^j\binom{N}{j}\ \frac{(m-1+j)!}{(k+j)!}$$
$$= \frac{1}{(m-1)!}\sum_{j=0}^{N}(-1)^j\binom{N}{j}\ D^{m-k-1}t^{m+j-1}|_{t=1}$$
$$= \frac{1}{(m-1)!}D^{m-k-1}t^{m-1}\sum_{j=0}^{N}(-1)^j\binom{N}{j}\ t^j|_{t=1}$$
$$= \frac{1}{(m-1)!}D^{m-k-1}\ t^{m-1}(1-t)^N|_{t=1}.$$

For $m-k-1<N$ this last expression is 0 and for $m-k-1\geq N$ it is

$$\frac{1}{(m-1)!}\binom{m-k-1}{N}(D^{m-k-1-N}\ t^{m-1})D^N(1-t)^N|_{t=1}$$
$$= \frac{1}{(m-1)!}\binom{m-k-1}{N}\frac{(m-1)!}{(N+k)!}(-1)^N N!$$

as claimed. For $m\leq k$ the claim follows readily by substituting $\phi(t)=t^{k-m}$ in (a).

(c) It follows readily from (b) that $\sum_{j=0}^{N}\frac{a(m-1,j)}{(j+k)!}=(N+k+1)\times\sum_{j=0}^{N}\frac{a(m,j)}{(j+k+1)!}$ if $0\leq k\leq m-2$. So setting $r(t)=\sum_{k=0}^{m-2}c_k t^k$ we find

$$\sum_{j=0}^{N}a(m,j)\ t^{N-j}\ P^{j+1}r(t)$$
$$= \sum_{k=0}^{m-2}c_k k!t^{N+k+1}\sum_{j=0}^{N}\ \frac{a(m,j)}{(j+k+1)!}$$
$$= \sum_{k=0}^{m-2}c_k k!t^{N+k+1}\frac{1}{N+k+1}\sum_{j=0}^{N}\ \frac{a(m-1,j)}{(j+k)!}$$
$$= P\sum_{k=0}^{m-2}c_k k!t^{N+k}\sum_{j=0}^{N}\ \frac{a(m-1,j)}{(j+k)!}$$
$$= P\sum_{j=0}^{N}a(m-1,j)\ t^{N-j}P^j r(t).$$
□

Our main result is the following:

Theorem 4.6. *Assume $\phi\in AP(\mathbb{R},X)$ and that $\sum_{j=0}^{N}b_j\,t^{N-j}P^{j+1}\phi\in BUC_{w_N}(\mathbb{R},X)$ for some $b_j\in\mathbb{C}$, $b_0\neq 0$.*

(a) *$P\phi \in BUC(\mathbb{R}, X)$ and if $\sum_{j=0}^{N} \frac{b_j}{(j+1)!} \neq 0$ then $M\phi = 0$.*

(b) *If $X \not\supseteq c_0$ then $P(\phi - M\phi) \in AP(\mathbb{R}, X)$.*

Proof. Let $a = M\phi$ and $\psi = \sum_{j=0}^{N} b_j\, t^{N-j} P^{j+1}\phi$. Then we have $\psi = \sum_{j=0}^{N} b_j t^{N-j} P^{j+1}(\phi - a) + t^{N+1} a \sum_{j=0}^{N} \frac{b_j}{(j+1)!}$. By Theorem 4.1(c), $\psi - \sum_{j=0}^{N} b_j\, t^{N-j} P^{j+1}(\phi - a) \in C_{w_{N+1},0}(\mathbb{R}, X)$ and so either $a = 0$ or $\sum_{j=0}^{N} \frac{b_j}{(j+1)!} = 0$. To prove the rest of the theorem, we may assume $a = 0$. By Theorem 4.1(c), $P^j\phi(t)/w_j(t) \to 0$ as $t \to \infty$. Since ϕ is almost periodic we may choose $(t_n) \subset \mathbb{R}$ such that $t_n \to \infty$ and $\phi_{t_n} \to \phi$ uniformly on $\mathbb{R}$. Moreover, as $M\phi = 0$, by Theorem 4.1(c), $P^j\phi(s+t_n)/w_j(s+t_n) \to 0$ uniformly on $\mathbb{R}$ for $j > 0$. Given $x^* \in X^*$, it follows that $x^* \circ P\phi_{t_n} \to x^* \circ P\phi$ locally uniformly. Moreover, by passing to a subsequence if necessary, we may assume $x^* \circ \psi(t_n)/w_N(t_n) \to b$ for some $b \in \mathbb{C}$. By Theorem 4.1(c) again, we obtain

$$\begin{aligned}\psi(t + t_n) &= \sum_{j=0}^{N} b_j\, (t + t_n)^{N-j} [\int_0^t P^j\phi(s + t_n) ds + P^{j+1}\phi(t_n)] \\ &= \psi(t_n) + b_0 t_n^N P\phi(t + t_n) + o(t_n^N).\end{aligned}$$

Therefore $x^* \circ \psi(t + t_n)/w_N(t + t_n) \to b + b_0 x^* \circ P\phi(t)$ for each $t \in \mathbb{R}$. Hence, since ψ/w_N is bounded, so too is $x^* \circ P\phi$. Since x^* is arbitrary, $P\phi$ is weakly bounded and therefore bounded. From Proposition 4.2(e) it follows that $P\phi \in BUC(\mathbb{R}, X)$. If also $X \not\supseteq c_0$ then by Kadet's theorem [21] (see also [4]), $P\phi$ is almost periodic. □

Corollary 4.7. *Assume $\phi \in AP(\mathbb{R}, X)$ and $P(t^N\phi) \in BUC_{w_N}(\mathbb{R}, X)$.*

(a) *$P\phi \in BUC(\mathbb{R}, X)$ and $M\phi = 0$.*

(b) *If $X \not\supseteq c_0$ then $P\phi \in AP(\mathbb{R}, X)$.*

Proof. Since $P^m(t^N\phi) = \sum_{j=0}^{N} a(m, j)\, t^{N-j} P^{m+j}\phi$ the result follows from Theorem 4.6 and Lemma 4.5. □

Theorem 4.8. *Assume $\phi \in AP(\mathbb{R}, X)$, $X \not\supseteq c_0$, $P^m(t^N\phi) + p \in BUC_{w_N}(\mathbb{R}, X)$ for natural numbers m, N and some $p \in P_{m-1}(\mathbb{R}, X)$. Then $P^j(t^N\phi) + p^{(m-j)} \in AP_{w_N}(\mathbb{R}, X)$ for $1 \le j \le m$. Moreover, there is a polynomial $q \in P_{m-1}(\mathbb{R}, X)$ such that $P^j\phi + q^{(m-j)} \in AP(\mathbb{R}, X)$ for $1 \le j \le m$ and, if $p(t) = \sum_{k=0}^{m-1} b_k t^k$ then $\sum_{j=0}^{N} a(m, j)\, t^{N-j} P^j q = \sum_{k=N+1}^{m-1} b_k t^k$.*

Proof. The proof is by induction on m. If $m = 1$ then, by Lemma 4.5 $\sum_{j=0}^{N} \frac{a(1,j)}{(j+1)!} = \frac{1}{N+1}$ and $P(t^N\phi) = \sum_{j=0}^{N} a(1, j)\, t^{N-j} P^{j+1}\phi \in$

$BUC_{w_N}(\mathbb{R},X)$. Therefore, by Theorem 4.6, $M\phi = 0$ and $P\phi \in AP(\mathbb{R},X)$. Moreover, by Lemma 4.5(b),

$$\sum_{j=0}^{N} a(1,j)\, t^{N-j} P^j(MP\phi) \;=\; (MP\phi)\, t^N \sum_{j=0}^{N} \frac{a_j}{j!} = 0.$$

Hence $P(t^N\phi) = \sum_{j=0}^{N} a(1,j)\, t^{N-j} P^j(P\phi - MP\phi)$ and by Theorem 4.1(c), $P(t^N\phi) \in AP_{w_N}(\mathbb{R},X)$. For $m>1$, Theorem 5.2 below shows $P^j(t^N\phi) + p^{(m-j)} \in BUC_{w_N}(\mathbb{R},X)$ for $1 \le j \le m$. Hence, as induction hypothesis we may assume there is a polynomial $r \in P_{m-2}(\mathbb{R},X)$ such that for $1 \le j \le m-1$ we have $P^j\phi + r^{(m-1-j)} \in AP(\mathbb{R},X)$, $P^j(t^N\phi) + p^{(m-j)} \in AP_{w_N}(\mathbb{R},X)$ and $\sum_{j=0}^{N} a(m-1,j)\, t^{N-j} P^j r = \sum_{k=N+2}^{m-1} k b_k t^{k-1}$. In particular, $\eta = P^{m-1}\phi + r + c \in AP(\mathbb{R},X)$ where the constant c is to be chosen. Moreover, by Lemma 4.5(d), $\sum_{j=0}^{N} a(m,j)\, t^{N-j} P^{j+1} r = \sum_{k=N+2}^{m-1} b_k t^k$.

Now set $q = P(r + c - M\phi)$ so that $P^m\phi + q = P(\eta - M\phi)$. By Theorem 4.6, to show $P^m\phi + q \in AP(\mathbb{R},X)$, it suffices to show that $\sum_{j=0}^{N} a(m,j)\, t^{N-j} P^{j+1}\eta \in BUC_{w_N}(\mathbb{R},X)$. By Lemma 4.5(a),

$$\begin{aligned} P^m(t^N\phi) &= \sum_{j=0}^{N} a(m,j)\; t^{N-j} P^{m+j}\phi \\ &= \sum_{j=0}^{N} a(m,j)\; t^{N-j} P^{j+1}(\eta - r - c). \end{aligned}$$

Since $P^m(t^N\phi) + p \in BUC_{w_N}(\mathbb{R},X)$, it suffices to show $\sum_{j=0}^{N} \{a(m,j) \times t^{N-j} P^{j+1}(r+c)\} = \sum_{k=N+1}^{m-1} b_k t^k$.

If $N > m-2$ we choose $c = 0$ as then both sides are 0. Otherwise $N \le m-2$ and by Lemma 4.5(c) we may choose c such that $\sum_{j=0}^{N} a(m,j)\; t^{N-j} P^{j+1} c = b_{N+1} t^{N+1}$, that is $c \sum_{j=0}^{N} \frac{a(m,j)}{(j+1)!} = b_{N+1}$. In this case also we have by Theorem 4.6, $M\eta = 0$. In either case $\sum_{j=0}^{N} a(m,j)\, t^{N-j} P^j q = \sum_{j=0}^{N} a(m,j)\; t^{N-j} P^{j+1}(r+c) = \sum_{k=N+1}^{m-1} b_k t^k$ and $\sum_{j=0}^{N} a(m,j)\, t^{N-j} P^{j+1} M\eta = 0$. Finally,

$$\begin{aligned} P^m(t^N\phi) + p &= \sum_{j=0}^{N} a(m,j)\; t^{N-j} P^{j+1}(\eta - r - c - M\phi) + p \\ &= \sum_{j=0}^{N} a(m,j)\; t^{N-j} P^{j}(P^m\phi + q) + \sum_{k=0}^{N} b_k t^k \end{aligned}$$

and by Theorem 4.1, $P^m(t^N\phi) + p \in AP_{w_N}(\mathbb{R},X)$. □

Remark 4.9. (a) *In Theorem 4.6(a) the space $AP(\mathbb{R},X)$ may be replaced by the class of Poisson stable functions. These are functions $\xi \in C(\mathbb{R},X)$ for which there exist sequences $(t_n) \subset \mathbb{R}$ such that $t_n \to \infty$ and $\xi_{t_n} \to \xi$ locally uniformly on $\mathbb{R}$. In part (b), $AP(\mathbb{R},X)$ may be replaced by any class for which Kadet's theorem remains valid. These include Poisson stable functions, almost automorphic functions and recurrent functions (see [4]).*

(b) *If $p = 0$ in Theorem 4.8 then $\sum_{j=0}^{N} a(m,j)\, t^{N-j} P^j q = 0$, which reduces to $q^{(k)}(0) = 0$ for $0 \le k \le m - N - 1$.*

(c) *Assume $\phi \in AP(\mathbb{R},X)$ where $X \not\supseteq c_0$. By Theorem 4.8, if $P^m(t^N\phi) + p \in BUC_{w_N}(\mathbb{R},X)$ for some p then $P^m\phi + q \in AP(\mathbb{R},X)$ for some q.*

The converse is also true. Indeed, $P^m(t^N\phi) = \sum_{j=1}^{N} \{a(m,j) \times t^{N-j} P^{m+j}\phi\} + t^N P^m\phi$ and the result follows from Theorem 4.1(c).

(d) *These results are dependent on the Poisson stability property of ϕ. Indeed, consider the function $\phi \in C_0(\mathbb{R},\mathbb{C})$ given by $\phi(t) = \frac{1}{1+|t|}$. Then $P(t\phi) = |t| - \ln(1+|t|)$ and $P\phi = sgn(t)\ln(1+|t|)$. Hence $P(t\phi) \in BUC_{w_1}(\mathbb{R},\mathbb{C})$ whereas $P\phi \notin BC(\mathbb{R},\mathbb{C})$.*

(e) *A well-known example to show that the condition $X \not\supseteq c_0$ may not be omitted from Theorem 4.6 is as follows. Let $X = c_0$ and $\phi(t) = (\frac{1}{n}\sin\frac{t}{n})_{n=1}^{\infty}$ so that $P\phi(t) = (2\sin^2\frac{t}{2n})_{n=1}^{\infty}$. Then $\phi \in AP(\mathbb{R},c_0)$ and $P\phi \in BUC(\mathbb{R},c_0)$. However, $P\phi$ does not have relatively compact range so it is not almost periodic.*

5. Esclangon-Landau theorem

In this section we use the abbreviations

$$Bu = \sum_{j=0}^{m} b_j u^{(j)} \tag{5.2}$$

and assume $b_m = 1$, $b_j \in \mathbb{C}$, $u : J \to X$.

We prove a theorem of Esclangon-Landau type ([18], [22], [14], [7], [16] and references therein).

Lemma 5.1. *If $Bu = \psi$ where $u, \psi \in BC_{w_N}(J,X)$ then $u^{(j)}(t) = O(|t|^{N+m-1})$ for $1 \le j \le m$.*

Proof. Since $u^{(m)} = \psi - \sum_{j=0}^{m-1} b_j u^{(j)}$, taking P^{m-k} we obtain

$$u^{(k)} = \sum_{j=1}^{m-k} P^{j-1}(u^{(j+k-1)}(0)) + P^{m-k}\psi - \sum_{j=0}^{m-1} b_j P^{m-k} u^{(j)}.$$

Setting $k = 1$ we conclude that $u'(t) = O(|t|^{N+m-1})$. In general $u^{(k)}(t) = O(|t|^{N+m-1}) - \sum_{j=m-k+1}^{m-1} b_j P^{m-k} u^{(j)}(t) = O(|t|^{N+m-1}) + \sum_{j=1}^{k-1} O(|u^{(j)}(t)|)$ from which the result follows by induction. □

Theorem 5.2. *If $Bu = \psi$ where $u, \psi \in BC_{w_N}(J, X)$ then $u^{(j)} \in BC_{w_N}(J, X)$ for $1 \le j \le m$.*

Proof. Take $J = \mathbb{R}_+$, the other cases being proved similarly. The proof is by induction on m. First, if $m = 1$ the equation becomes $u' + b_0 u = \psi$ showing $u' \in BC_{w_N}(J, X)$. For the general case we use the functions f and $\tilde{u}$ defined by $f(t) = \exp(-t)$ for $t \ge 0$, $f(t) = 0$ for $t < 0$, $\tilde{u}(t) = u(-t)$ for $-t \in J$ and $\tilde{u}(t) = 0$ for $-t \notin J$. It follows that $e^t \int_t^\infty e^{-s} u(s) ds = \int_0^\infty e^{-s} u(s+t) ds = \tilde{u} * f(-t)$ and $\tilde{u} * f \in BC_{w_N}(\mathbb{R}, X)$. Moreover, using repeated integration by parts and Lemma 5.1, we find $e^t \int_t^\infty e^{-s} u^{(k)}(s) ds = -\sum_{j=1}^{k-1} u^{(j)}(t) + \tilde{u} * f\ (-t)$. Hence the equation $B\phi = \psi$ may be transformed to the equation $\sum_{k=1}^m b_k \sum_{j=1}^{k-1} u^{(j)}(t) = (\sum_{k=0}^m b_k) \tilde{u} * f\ (-t) - \tilde{\psi} * f\ (-t)$. This is an equation of order $m-1$ and so by the induction hypothesis $u^{(j)} \in BC_{w_N}(J, X)$ for $1 \le j \le m-1$. Hence $u^{(m)} = \psi - \sum_{j=0}^{m-1} b_j u^{(j)} \in BC_{w_N}(J, X)$ which finishes the proof. □

6. Application

Again we use the abbreviation $Bu = \sum_{j=0}^m b_j u^{(j)}$ and assume $b_m = 1$. By p_B we denote the characteristic polynomial of the differential operator B. Thus $p_B(s) = \sum_{j=0}^m b_j (is)^j$ and for smooth f we have $\widehat{Bf}(\gamma_s) = p_B(s) \hat{f}(\gamma_s)$. The set of complex zeros of p_B is denoted $Z(B)$.

Lemma 6.1. *Assume $u \in BC_{w_N}(\mathbb{R}, X)$ and $\mathcal{F}$ is a BUC_{w_N}-invariant closed subspace of $BC_{w_N}(J, X)$. If $Bu = \psi$ where $\psi \in BUC_{w_N}(\mathbb{R}, X)$ and $\psi|_J \in \mathcal{F}$ then $sp_{\mathcal{F}}(u) \subset \{\gamma_s : s \in Z(B) \cap \mathbb{R}\}$.*

Proof. Take $s \in \mathbb{R}$ with $p_B(s) \ne 0$. Choose $f \in S(\mathbb{R})$ with $\hat{f}(\gamma_s) \ne 0$ and set $g = Bf$. Then $u * g = \psi * f$ and by Theorem 3.2(a), $(\psi * f)|_J \in \mathcal{F}$. Hence $g \in I_{\mathcal{F}}(u)$ whereas $\hat{g}(\gamma_s) = p_B(s) \hat{f}(\gamma_s) \ne 0$. So $\gamma_s \notin sp_{\mathcal{F}}(u)$ and the proof is completed. □

Theorem 6.2. *Suppose $Bu = \psi$ where $u \in BC_{w_N}(\mathbb{R}, X)$ and $\psi \in AP_{w_N}(\mathbb{R}, X)$.*

(a) *If $Z(B) \cap \mathbb{R} = \varnothing$ then $u^{(j)} \in AP_{w_N}(\mathbb{R}, X)$ for $0 \le j \le m$.*

(b) *If $Z(B) \cap \mathbb{R} \ne \varnothing$, but $X \not\supseteq c_0$ and $\psi = t^N \phi$ where $\phi \in AP(\mathbb{R}, X)$ then $u^{(j)} \in AP_{w_N}(\mathbb{R}, X)$ for $0 \le j \le m$.*

Proof. (a) Let $\mathcal{F} = AP_{w_N}(\mathbb{R}, X)$. By Lemma 6.1, $sp_{\mathcal{F}}(u) \subset Z(B) \cap \mathbb{R} = \varnothing$. Hence, by Theorem 3.2(b), $u \in \mathcal{F}$. The Esclangon-Landau Theorem 5.2 shows $u, u', ..., u^{(m)} \in BC_{w_N}(\mathbb{R}, X)$ and then Proposition 4.2(e) shows $u, u', ..., u^{(m-1)} \in BUC_{w_N}(\mathbb{R}, X)$. From Proposition 4.2(b) we conclude $u', ..., u^{(m-1)} \in \mathcal{F}$. Rearranging the differential equation, we obtain $u^{(m)} \in \mathcal{F}$.

(b) The proof is by induction on m. Note first that by Theorem 5.2, $u^{(j)} \in BC_{w_N}(\mathbb{R}, X)$ for $0 \leq j \leq m$. Let $\lambda \in Z(B) \cap \mathbb{R}$ and make the substitution $\eta(t) = \exp(-i\lambda t)u(t)$ so that $\eta^{(j)} \in BC_{w_N}(\mathbb{R}, X)$ for $0 \leq j \leq m$. If $m = 1$ the equation $u' - i\lambda u = t^N \phi$ reduces to $\eta' = \exp(-i\lambda t)t^N \phi$. From Theorem 4.8 we conclude $\eta \in \mathcal{F}$. Hence $u, u' \in \mathcal{F}$ as claimed. For general m, the equation $Bu = t^N \phi$ reduces to an equation of the form $\sum_{j=1}^{m} c_j \eta^{(j)} = \exp(-i\lambda t)t^N \phi$ where $c_m = 1$. This is a differential equation in η' of order $m - 1$. By the induction hypothesis, or by part (a) if the characteristic polynomial has no real zeros, $\eta^{(j)} \in \mathcal{F}$ for $1 \leq j \leq m - 1$. It remains to show $\eta \in \mathcal{F}$. For this, let $k = \min\{j : c_j \neq 0\}$. From $\sum_{j=k}^{m} c_j \eta^{(j)} = \exp(-i\lambda t)t^N \phi$ we obtain $\sum_{j=k}^{m} c_j \eta^{(j-k)} = P^k(\exp(-i\lambda t)t^N \phi) + p$ for some polynomial p of degree at most $k - 1$. But $\eta^{(j)} \in BC_{w_N}(\mathbb{R}, X)$ for $0 \leq j \leq m$ and so by Theorem 4.8 again we conclude $P^k(\exp(-i\lambda t)t^N \phi) \in \mathcal{F}$. Since $c_k \neq 0$ we can rearrange the differential equation and obtain $\eta \in \mathcal{F}$. □

Remark 6.3. *The asymptotic behaviour of bounded solutions of equations more general than (5.1) are investigated by numerous authors (see [2], [3], [6], [8], [13], [26], [27], [30]). In particular, it follows from [12, Theorem 4.7] that if $\phi \in BUC_w(J, X)$, $sp_{AP_w}(\phi)$ is countable and $\gamma^{-1}\phi \in E_w(J, X)$ for all $\gamma \in sp_{AP_w}(\phi)$, then $\phi \in AP_w(\mathbb{R}, X)$. In this paper for solutions of (5.1) we have replaced the ergodicity condition by $X \not\supset c_0$. This is satisfied, in particular, if X is finite dimensional or reflexive or weakly sequentially complete. So, the results of Theorems 4.6, 6.2 are new even for $X = \mathbb{R}$ or $\mathbb{C}$.*

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School of Mathematical Sciences, Monash University, PO Box 28M, VIC 3800, Australia

E-mail address: bbasit@vaxc.cc.monash.edu.au

School of Mathematical Sciences, Monash University, PO Box 28M, VIC 3800, Australia

E-mail address: a.pryde@sci.monash.edu.au