

SPECTRUM OF THE RUELLE OPERATOR AND ZETA FUNCTIONS FOR BROKEN GEODESIC FLOWS

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1. INTRODUCTION

Let K be an obstacle in \mathbb{R}^n , where $n \geq 3$ is odd, i.e. K is a compact subset of \mathbb{R}^n with C^∞ boundary ∂K such that

$$\Omega = \overline{\mathbb{R}^n \setminus K}$$

is connected. One of the main objects of study in scattering theory (by an obstacle) is the so called *scattering matrix* $S(z)$ related to the wave equation in $\mathbb{R} \times \Omega$ with Dirichlet boundary condition on $\mathbb{R} \times \Omega$. This is (cf. [LP], [M] or [Z]) a meromorphic operator-valued function

$$S(z) : L^2(\mathbf{S}^{n-1}) \longrightarrow L^2(\mathbf{S}^{n-1})$$

with poles $\{\lambda_j\}_{j=1}^\infty$ in the half-plane $\text{Im}(z) > 0$.

A variety of problems in scattering theory deal with finding geometric information about K from the distribution of the poles $\{\lambda_j\}$. In what follows we describe one particular problem of this type.

The obstacle K is called **trapping** if there exists an infinitely long bounded broken geodesic (in the sense of Melrose and Sjöstrand [MS]) in the *exterior domain* Ω . For example, if Ω contains a periodic broken geodesic (this is always the case when K has more than one connected component), then K is trapping.

It follows from results of Lax-Phillips (1971) and Melrose (1982) that if K is non-trapping, then $\{z : 0 < \text{Im}(z) < \alpha\}$ contains finitely many poles λ_j for any $\alpha > 0$ (cf. the Epilogue in [LP] for more precise information).

In the first edition of their monograph *Scattering Theory* published in 1967, Lax and Phillips conjectured that for trapping obstacles there should exist a sequence $\{\lambda_j\}$ of scattering poles such that $\text{Im}\lambda_j \rightarrow 0$ as $j \rightarrow \infty$. However M. Ikawa [I1] showed that this is not the case when K is a disjoint union of two strictly convex compact domains with smooth boundaries. It turned out that in this particular case the scattering matrix has poles approximately at the points $\frac{k\pi}{d} + i\delta$, $k = 0, \pm 1, \pm 2, \dots$, where d is the distance between the two connected

components K_1 and K_2 of K and $\delta > 0$ is a constant depending only on the curvatures of ∂K at the ends of the shortest segment connecting K_1 and K_2 . Substantial new information concerning the distribution of poles in this case was later given by C. Gerard [G].

Ikawa modified the initial conjecture of Lax and Phillips in the following way.

Lax-Phillips Conjecture (LPC) (in the form given by M. Ikawa): If K is trapping, then there exists $\alpha > 0$ such that the strip $\{z : 0 < \text{Im}(z) < \alpha\}$ contains infinitely many pôles λ_j .

So far results concerning this conjecture are only known in the case when K has the form

$$(1) \quad K = K_1 \cup K_2 \cup \dots \cup K_s ,$$

where K_i are strictly convex disjoint compact domains in \mathbb{R}^n (with C^∞ boundaries) satisfying the following condition introduced by M. Ikawa:

$$(H) \quad K_m \cap \text{convex hull}(K_i \cup K_j) = \emptyset \text{ for all } m \neq i \neq j \neq m.$$

As we mentioned above, the LPC holds in the case $s = 2$. There are partial results in the case $s \geq 3$ also due to Ikawa (cf. [I3], [I4]). Below we briefly describe Ikawa's approach in dealing with this case.

The starting point is the distribution

$$u(t) = \sum_j e^{i|t|\lambda_j}, \quad t \neq 0 ,$$

where as above $\{\lambda_j\}$ is the set of all poles of the scattering matrix $S(z)$. Guillemin and Melrose [GM] showed that the sum of the principal singularities of $u(t)$ on \mathbb{R}_+ is

$$(2) \quad \sum_\gamma (-1)^{m_\gamma} T_\gamma |\det(I - P_\gamma)|^{-1/2} \delta(t - d_\gamma) ,$$

where γ runs over the set of periodic broken geodesics in Ω_K , d_γ is the period (return time) of γ , T_γ the primitive period (length) of γ , and P_γ the linear Poincaré map associated to γ .

Applying the Laplace transform to (2), Ikawa [I3] introduced the following zeta function (which could be called the **scattering zeta function**)

$$Z(s) = \sum_{\gamma \in \Xi} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-sd_\gamma}, \quad s \in \mathbf{C} .$$

He then showed that the existence of analytic singularities of $Z(s)$ implies the existence of a band $0 < \text{Im}(z) < \alpha$ containing an infinite number of scattering pôles λ_j (i.e. LPC holds).

Clearly $Z(s)$ is a Dirichlet series. Let s_0 be its abscissa of absolute convergence. Assuming $n = 3$, Ikawa showed that there exists $\epsilon > 0$ such that in the region $s_0 - \epsilon < \text{Re}(s) \leq s_0$ the analytic singularities of $Z(s)$ coincide with these of the zeta function

$$Z_0(s) = \sum_{m=0}^{\infty} \sum_{\gamma} (-1)^{mr_{\gamma}} T_{\gamma} e^{m(-sT_{\gamma} + \delta_{\gamma})} ,$$

where γ runs over the set of primitive periodic broken geodesics in Ω , $r_{\gamma} = 0$ if γ has an even number of reflection points and $r_{\gamma} = 1$ otherwise, and $\delta_{\gamma} \in \mathbb{R}$ is determined by the spectrum of the linear Poincaré map related to γ . The function $Z_0(s)$ is rather similar to a dynamically defined zeta function (cf. the survey of Baladi [Ba] for general information on this topic). One of the main tools to study this sort of zeta function is the so called Ruelle operator (well known e.g. from the study of Gibbs measures in statistical mechanics and ergodic theory, cf. [R1] and [Si]). Ikawa [I4] succeeded to implement results of Pollicott (1986) and Haydn (1990) concerning the spectrum of the Ruelle operator and derived sufficient conditions for $Z_0(s)$ (and therefore $Z(s)$) to have a pole in a small neighbourhood of s_0 in \mathbb{C} . From this he derived the following.

Theorem 1. ([I4]) *If K is a finite union of disjoint balls in \mathbb{R}^3 with the same radius $\epsilon > 0$ and $\epsilon > 0$ is sufficiently small, then LPC holds.*

The study of the scattering zeta function itself seems to be rather difficult and very few results concerning it are known. Petkov [P] showed that $Z(s)$ admits an analytic continuation in the domain $\text{Re}(s) \geq s_0$. Moreover, under some additional assumptions about the geometry of the obstacle K and assuming rational independence of the primitive periods of periodic orbits, Petkov proved that $\sup_{t \in \mathbb{R}} |Z(s_0 + it)| = \infty$ (cf. [P] for some further results on the properties of $Z(s)$). Assuming that the broken geodesic flow has two periodic orbits γ_1 and γ_2 such that $T_{\gamma_1}/T_{\gamma_2}$ is a diophantine number, Naud [N] proved that $Z(s)$ has an analytic continuation to a domain of the form

$$s_0 - \frac{C}{|t|^{\rho}} < \text{Re}(s) \leq s_0 \quad , \quad |t| = |\text{Im}(s)| \geq 1 \quad ,$$

for some constants $C > 0$ and $\rho > 0$.

2. SPECTRUM OF THE RUELLE OPERATOR AND DYNAMICAL ZETA FUNCTION

In the case when the obstacle K has the form (1), it is sometimes more convenient to work with the *billiard ball map* B (i.e. the shift along the broken geodesic flow from boundary to boundary) on the *non-wandering set* M_0 . The latter is the set of all points $\sigma \in S_{\partial K}^*(\mathbb{R}^n)$ that generate trapped broken geodesics in both directions. As a dynamical system $B : M_0 \rightarrow M_0$ is naturally isomorphic to the Bernoulli shift on the symbolic space

$$\Sigma = \prod_{-\infty}^{\infty} \{1, 2, \dots, s\},$$

the isomorphism being given by the natural coding of the geodesics by sequences $\{i_j\}$, where $i_j = 1, \dots, s$ is the number of the connected component K_{i_j} containing the j th reflection point. Ikawa [I4] used the classical interpretation of the Ruelle operator as an operator acting on the space of Lipschitz functions on the symbol space Σ . However, having in mind some significant recent developments, it seems more convenient to use a different model which is more closely related to the dynamics of the flow.

In this section we describe some recent results of the author concerning the spectrum of the Ruelle operator and the dynamical zeta function related to the broken geodesic flow in the exterior of an obstacle of the form (1) in the \mathbb{R}^2 . The results are similar to these of Dolgopyat [D] in the case of Anosov flows on compact manifolds. One would expect that similar results could be obtained (by using similar techniques) for obstacles in \mathbb{R}^n , $n \geq 3$. Such results would very likely lead to partial solutions of the LPC for cases much more general than the one considered in Theorem 1 above.

From now on we assume that K is an obstacle of the form (1) in \mathbb{R}^2 . Let Λ be the *non-wandering set* of the broken geodesic flow in Ω . Clearly Λ is the union of all orbits generated by elements of the set M_0 defined above. For $x \in \Lambda \setminus S_{\partial K}^*(\Omega)$ and a sufficiently small $\epsilon > 0$ let

$$W_\epsilon^s(x) = \{y \in S^*(\Omega) : \rho(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \geq 0, \\ \rho(\phi_t(x), \phi_t(y)) \rightarrow_{t \rightarrow \infty} 0\},$$

$$W_\epsilon^u(x) = \{y \in S^*(\Omega) : \rho(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \leq 0, \\ \rho(\phi_t(x), \phi_t(y)) \rightarrow_{t \rightarrow -\infty} 0\}$$

be the (strong) *stable* and *unstable sets* of size ϵ . It is easy to show that for every $x \in \Lambda \setminus S_{\partial K}^*(\Omega)$ and every sufficiently small $\epsilon > 0$, $W_\epsilon^s(x)$ and

$W_\epsilon^u(x)$ are 1-dimensional submanifolds of $S^*(\Omega)$. It is worth mentioning that both $W_\epsilon^s(x) \cap \Lambda$ and $W_\epsilon^u(x) \cap \Lambda$ are Cantor sets.

Given $\epsilon > 0$, set

$$S_\epsilon^*(\Omega) = \{x = (q, v) \in S^*(\Omega) : \text{dist}(q, \partial K) > \epsilon\} \quad , \quad \Lambda_\epsilon = \Lambda \cap S_\epsilon^*(\Omega) .$$

In what follows in order to avoid ambiguity and unnecessary complications we will consider stable and unstable manifolds only for points x in $S_\epsilon^*(\Omega)$ or Λ_ϵ ; this will be enough for our purposes.

It follows from the general theory of horocycle foliations (cf. [S] or [KH]) that if $\epsilon > 0$ is sufficiently small, there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $\rho(x, y) < \delta$, then $W_\epsilon^s(x) \cap \Lambda$ and $\phi_{[-\epsilon, \epsilon]}(W_\epsilon^u(y) \cap \Lambda)$ intersect at exactly one point $[x, y]$. That is, there exists a unique $t \in [-\epsilon, \epsilon]$ such that $\phi_t([x, y]) \in W_\epsilon^u(x)$. Setting $\Delta(x, y) = t$, defines the so called *temporary distance function* Δ ([Ch1]). If $z \in \Lambda_\epsilon$ and $U \subset W_\epsilon^u(z)$ and $S \subset W_\epsilon^s(z)$ are closed (i.e. containing their end points) curves containing z and such that $U \cap \Lambda$ and $S \cap \Lambda$ have no isolated points, then

$$\Pi = [U \cap \Lambda, S \cap \Lambda] = \{[x, y] : x \in U \cap \Lambda, y \in S \cap \Lambda\}$$

is called a *rectangle* in Λ_ϵ . Notice that Π is "foliated" by leaves $[x, S \cap \Lambda]$ of stable manifolds.

Let $R = \{R_i\}_{i=1}^k$ be a family of rectangles such that each R_i is contained in a C^1 cross-section $D_i \subset S_\epsilon^*(\Omega)$ to the flow ϕ_t . Thus, for each i , $R_i = [U_i \cap \Lambda, S_i \cap \Lambda]$, where U_i and S_i are closed curves in $W_\epsilon^u(z_i)$ and $W_\epsilon^s(z_i)$, respectively, for some $z_i \in \Lambda_\epsilon$. Set

$$R = \bigcup_{i=1}^k R_i .$$

The family R is called *complete* if there exists $T > 0$ such that for every $x \in \Lambda$ there exist $t_1 \in [-T, 0)$ and $t_2 \in (0, T]$ with $\phi_{t_1}(x) \in R$ and $\phi_{t_2}(x) \in R$. Thus, $\tau(x) = t_2(x) > 0$ is the smallest positive time with $P(x) = \phi_{\tau(x)}(x) \in R$, and $P : R \rightarrow R$ is the *Poincaré map* related to the family R .

Following [B] and [Ra] we will say that a complete family $R = \{R_i\}_{i=1}^k$ of rectangles in $S_\epsilon^*(\Omega)$ is a *Markov family* of size $\chi \in (0, \epsilon/2)$ for the billiard flow ϕ_t if:

- (a) $R_i \cap R_j = \emptyset$ for $i \neq j$ and for each i the sets $U_i \cap \Lambda$ and $S_i \cap \Lambda$ are contained in the interior of the curves U_i and S_i , respectively;
- (b) $\text{diam}(R_i) \leq \chi$;
- (c) For any $i \neq j$ and any $x \in R_i \cap P^{-1}(R_j)$ we have $P([x, S_i \cap \Lambda]) \subset [P(x), S_j]$ and $P([U_i \cap \Lambda, x]) \supset [U_j, P(x)]$;

(d) For any $i = 1, \dots, k$ and $x \in R_i$ the function τ is constant on the set $[x, S_i \cap \Lambda]$.

The existence of a Markov family R of an arbitrarily small size $\chi > 0$ for ϕ_t follows from the construction of Bowen [B] (cf. also Ratner [Ra]).

It follows from a result of C. Robinson [Ro] that there exists an open neighbourhood V of Λ_ϵ in $S^*(\Omega)$ and C^1 *transverse foliations* F^u and F^s on V such that $W_\epsilon^u(x) \cap V = F^u(x)$ and $W_\epsilon^s(x) \cap V = F^s(x)$ for any $x \in \Lambda \cap V$. Fix a neighbourhood V and C^1 foliations F^u and F^s with these properties.

Choosing $\chi > 0$ sufficiently small, we may assume that our Markov family R satisfies the following additional condition:

(e) For each i the cross-section D_i (and therefore the rectangle R_i) is contained in V .

In what follows we assume that $R = \{R_i\}_{i=1}^k$ is a fixed Markov family for ϕ_t satisfying the extra conditions (e) and (f). Then

$$U = \bigcup_{i=1}^k U_i,$$

is a finite disjoint union of compact curves in V .

Using the foliations F^u and F^s in V , assuming again that V is sufficiently small, we can extend the product $[x, y]$ over the whole $U_i \times S_i$ for any i as follows. Given $x \in U_i \subset W_\epsilon^u(z_i) = F^u(z_i) \cap V$ and $y \in S_i \subset W_\epsilon^s(z_i) = F^s(z_i) \cap V$, the sets $F^s(x)$ and $\phi_{[-\epsilon, \epsilon]}(F^u(y))$ intersect at exactly one point $[x, y]$.

The image of the C^1 map $U_i \times S_i \ni (x, y) \mapsto [x, y]$ is a 2-dimensional submanifold of $S_\epsilon^*(\Omega)$ which will be denoted by $\tilde{R}_i = [U_i, S_i]$. The *projection* $p : \tilde{R} = \bigcup_{i=1}^k \tilde{R}_i \rightarrow U$ along the leaves of F^s is C^1 . Then the Poincaré map $P : \tilde{R} \rightarrow \bigcup_{i=1}^k D_i$ and the corresponding *root function* $\tau : \tilde{R} \rightarrow [0, \infty)$ are well-defined and C^1 . Thus, σ can be extended to a C^1 map $\sigma : U \rightarrow U$ by the same formula: $\sigma = p \circ P$. In the same way one observes that $\Delta(x, y)$ is well-defined and C^1 for $(x, y) \in \bigcup_{i=1}^k \tilde{R}_i \times \tilde{R}_i$.

Let $C(U)$ be the space of *bounded continuous functions* on U . For $g \in C(U)$ denote $\|g\| = \|g\|_0 = \sup_{x \in U} |g(x)|$. Given g , the *Ruelle operator* $L_g : C(U) \rightarrow C(U)$ is defined in the usual way:

$$(L_g h)(u) = \sum_{\sigma(v)=u} e^{g(v)} h(v).$$

If $g \in C^1(U)$, then clearly L_g preserves the space $C^1(U)$. Here $C^1(U)$ denotes the *space of differentiable function* with bounded derivatives on U .

A conjugacy between $\Lambda \cap R$ and the Bernoulli shift on a symbolic space Σ_A is now defined in the usual way. Let $A = (A_{ij})_{i,j=1}^k$ be the matrix given by $A_{ij} = 1$ if $P(R_i) \cap R_j \neq \emptyset$ and $A_{ij} = 0$ otherwise. Then

$$\Sigma_A = \{ (i_j)_{j=-\infty}^{\infty} : 1 \leq i_j \leq k, A_{i_j i_{j+1}} = 1 \text{ for all } j \},$$

and the *Bernoulli shift* $\tilde{\sigma} : \Sigma_A \rightarrow \Sigma_A$ is given by $\tilde{\sigma}((i_j)) = ((i'_j))$, where $i'_j = i_{j+1}$ for all j . Define $\Phi : R \rightarrow \Sigma_A$ by $\Phi(x) = (i_j)_{j=-\infty}^{\infty}$, where $P^j(x) \in R_{i_j}$ for all $j \in \mathbf{Z}$. Notice that $P : R \rightarrow R$ is invertible, so P^j is well-defined for all integers j . The map Φ is a homeomorphism when Σ_A is considered with the product topology, and $\tilde{\sigma} \circ \Phi = \Phi \circ P$. The *projection* r of τ on Σ_A is given by $r \circ \Phi = \tau$.

The subset $\Lambda \cap U$ of R can be naturally identified with

$$\Sigma_A^+ = \{ (i_j)_{j=0}^{\infty} : 1 \leq i_j \leq k, A_{i_j i_{j+1}} = 1 \text{ for all } j \geq 0 \}.$$

Namely, if $\pi : \Sigma_A \rightarrow \Sigma_A^+$ is the *natural projection*, then $\Phi^+ = \pi \circ \Phi|_{\Lambda \cap U} : \Lambda \cap U \rightarrow \Sigma_A^+$ is a bijection with $\tilde{\sigma} \circ \Phi^+ = \Phi^+ \circ \sigma$, where $\tilde{\sigma}$ denotes the corresponding Bernoulli shift on Σ_A^+ .

The *dynamical zeta function* of the broken geodesic flow ϕ_t is defined by

$$\zeta(s) = \prod_{\gamma} (1 - e^{-s\ell(\gamma)})^{-1},$$

where γ runs over the set of closed orbits of ϕ_t and $\ell(\gamma)$ is the least period of γ . One can easily see that

$$\zeta(s) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{\sigma^n(x)=x \\ x \in U \cap \Lambda}} e^{-s\tau_n(x)} \right),$$

where

$$\tau_n(x) = \tau(x) + \tau(\sigma(x)) + \dots + \tau(\sigma^n(x))$$

is the period (length) of the closed orbit generated by x . That is, we have

$$\zeta(s) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} Z_n(-s\tau) \right),$$

where

$$Z_n(-s\tau) = \sum_{\substack{\sigma^n(x)=x \\ x \in U \cap \Lambda}} e^{-s\tau_n(x)}.$$

The following lemma¹ of Ruelle ([R2], cf. also [PoS]) partially explains the relationship between the Ruelle operator and the behaviour of the dynamical zeta function $\zeta(s)$.

Lemma 1. (Ruelle) *Let x_j be an arbitrary point in $U_j \cap \Lambda$ for every $j = 1, \dots, k$. There exist constants $C > 0$, $\epsilon > 0$, $\rho \in (0, 1)$ such that*

$$\left| Z_n(-s\tau) - \sum_{j=1}^k (L_{-s\tau}^n \chi_{U_j})(x_j) \right| \leq C |\operatorname{Im}(s)| n \rho^n$$

for all $n \geq 1$ and all $s \in \mathbf{C}$ with $\operatorname{Re}(s) \geq h_T - \epsilon$, h_T being the topological entropy of the flow ϕ_t , and χ_{U_j} is the characteristic function of U_j in U .

The *suspended flow* $\tilde{\sigma}^r$ over $\Sigma_A^r = \{(\xi, t) : \xi \in \Sigma_A, 0 \leq t \leq r(\xi)\}$ is defined by $\tilde{\sigma}_s^r(\xi, t) = (\xi, t + s)$, where we use the identification $(\xi, r(\xi)) = (\sigma(\xi), 0)$ in $\Sigma_A \times \mathbb{R}$. Then $\Sigma_A^{r,+} = \{(\xi, t) \in \Sigma_A^r : \xi \in \Sigma_A^+\}$ is a closed subset of Σ_A^r which is invariant under the semiflow $\tilde{\sigma}_t^r$, $t \geq 0$ (cf. [PP2] for details). A conjugacy between $\tilde{\sigma}_t^r$ and ϕ_t on Λ is defined by $\Phi^r(\phi_s(x)) = (\Phi(x), s)$ for $x \in R$ and $0 \leq s \leq \tau(x)$.

Given $\theta \in (0, 1)$, consider the *metric* d_θ on Σ_A given by $d_\theta(\xi, \eta) = 0$ if $\xi = \eta$ and $d_\theta(\xi, \eta) = \theta^n$ if $\xi_i = \eta_i$ for $|i| \leq n$ and n is maximal with this property. In a similar way one defines a metric d_θ on Σ_A^+ . Using it, we get a metric on $\Sigma_A^{r,+}$ by setting $d_\theta^r((\xi, t), (\eta, s)) = d_\theta(\xi, \eta) + |t - s|^\theta$. The *spaces of Lipschitz functions* on Σ_A^+ and $\Sigma_A^{r,+}$ with respect to the metrics just defined will be denoted by $F_\theta(\Sigma_A^+)$ and $F_\theta(\Sigma_A^{r,+})$, respectively.

In the present setting the well-known Perron-Ruelle-Frobenius theorem reads as follows.

Perron-Ruelle-Frobenius Theorem. *Let $f \in F_\theta(\Sigma_A^+)$ be a real-valued function.*

(a) *The Ruelle operator $L_f : F_\theta(\Sigma_A^+) \rightarrow F_\theta(\Sigma_A^+)$ has a simple eigenvalue $\lambda = e^{\operatorname{Pr}(f)}$, where $\operatorname{Pr}(f)$ is the topological pressure of f , and a strictly positive eigenfunction $h \in F_\theta(\Sigma_A^+)$.*

(b) *$\operatorname{spec}(L_f) \setminus \{\lambda\}$ is contained in a disk of radius strictly less than λ .*

(c) *There exists a unique probability measure $\nu = \nu_f$ on Σ_A^+ such that*

$$\int L_f(g) d\nu = \lambda \int g d\nu$$

for all $g \in F_\theta(\Sigma_A^+)$.

¹Its statement is slightly modified to suit the present context.

The measure ν_f is the so called *Gibbs measure* related to the potential f ([R1], [Si]).

The complex case is more complicated. It was established by Pollicott [Po] that for any $f = u + iv \in F_\theta(\Sigma_A^+)$ the spectral radius of L_f as an operator on $F_\theta(\Sigma_A^+)$ is not greater than $e^{\text{Pr}(u)}$. Moreover, if L_f has an eigenvalue λ with $|\lambda| = e^{\text{Pr}(u)}$, then λ is simple and unique and the rest of the spectrum is contained in a disk of strictly smaller radius. If L_f has no eigenvalues λ with $|\lambda| = e^{\text{Pr}(u)}$, then the whole spectrum of L_f is contained in a disk of radius less than $e^{\text{Pr}(u)}$.

The following result was obtained by using a modification of the technique developed by Dolgopyat [D] in order to prove a similar result in the case of Anosov flows on compact manifolds², in particular for geodesic flows on surfaces of negative curvature.

Theorem 2. ([St2]) *There exist constant $c_0 < h_T$ and $\rho \in (0, 1)$ such that for $\text{Re}(s) \geq c_0$ and $|\text{Im}(s)| \gg 0$ the spectral radius of the Ruelle operator $L_{-s\tau}$ does not exceed ρ .*

Using the above theorem and applying the argument of Pollicott and Sharp [PoS] in the case of geodesic flows on compact surfaces of negative curvature, one derives that the *zeta function*

$$\zeta(s) = \prod_{\gamma} (1 - e^{-s\ell(\gamma)})^{-1}$$

of the billiard flow ϕ_t has an analytic continuation in a half-plane $\text{Re}(s) > c_0$ for some $c_0 < h_T$ except for a simple pole at $s = h_T$. Moreover, following [PoS] again, one derives that there exists $c \in (0, h_T)$ such that

$$\pi(\lambda) = \#\{\gamma : \ell(\gamma) \leq \lambda\} = \text{li}(e^{h_T\lambda}) + O(e^{c\lambda})$$

as $\lambda \rightarrow \infty$, where $\text{li}(x) = \int_2^x du/\log u$. The latter is a much stronger result than the standard Prime Orbit Theorem for open planar billiards in [Mor] (cf. [St1] for the higher dimensional case) derived by means of a result of Parry and Pollicott [PP1].

3. EXPONENTIAL DECAY OF CORRELATIONS

In this section we continue to consider the case when K is an obstacle of the form (1) in \mathbb{R}^2 , and we also use most of the notation from Sect.2.

²In fact the primary aim of Dolgopyat was to establish exponential decay of correlations for such flows. See Sect.3 for more information.

Let $F \in F_\alpha(\Lambda)$ for some $\alpha > 0$. Denote by \tilde{F} the function on Σ_A^r such that $\tilde{F} \circ \Phi^r = \Phi^r \circ F$. There exists $\theta = \theta(\alpha) \in (0, 1)$ such that $\tilde{F} \in F_\theta(\Sigma_A^r)$. Let $\tilde{\mu}$ be the *Gibbs measure* related to \tilde{F} with respect to the suspended flow $\tilde{\sigma}^r$, and let $P = \text{Pr}_{\tilde{\sigma}^r}(\tilde{F})$ be the *topological pressure* of \tilde{F} with respect to $\tilde{\sigma}^r$. A function $\tilde{f} \in F_\theta(\Sigma_A^r)$ related to \tilde{F} is defined by

$$\tilde{f}(\xi) = \int_0^{r(\xi)} \tilde{F}(\xi, s) ds .$$

Let $\tilde{\nu}$ be the *Gibbs measure* on Σ_A determined by the function $\tilde{f} - Pr$. Then (cf. [PP2])

$$d\tilde{\mu}(\xi, s) = \frac{1}{\nu(r)} d\tilde{\nu}(\xi) ds , \text{ where } \nu(\tilde{g}) = \int_{\Sigma_A} \tilde{g}(\xi) d\tilde{\nu}(\xi) .$$

Moreover, we have $\text{Pr}_{\tilde{\sigma}}(\tilde{f} - Pr) = 0$. The Gibbs measures $\tilde{\mu}$ and $\tilde{\nu}$ give rise to measures μ and ν on R and $\Lambda \cap U$, respectively, via the conjugacies Φ^r and Φ . If f is the function on $\Lambda \cap U$ such that $\tilde{f} \circ \Phi = \Phi \circ f$, then $f(x) = \int_0^{\tau(x)} F(\phi_s(x)) ds$, so $f \in F_\alpha(\Lambda \cap U)$. The measures μ and ν are called the *Gibbs measures* related to F and $f - P\tau$, respectively. It follows from above that $\text{Pr}_{\phi_t}(F) = P$ and $\text{Pr}_\sigma(f - P\tau) = 0$.

Given a Hölder continuous potential F on Λ and arbitrary $A, B \in F_\alpha(\Lambda)$, the *correlation function* of A and B is defined by

$$\rho_{A,B}(t) = \int_\Lambda A(x)B(\phi_t(x)) d\nu_F(x) - \left(\int_\Lambda A(x) d\nu_F(x) \right) \left(\int_\Lambda B(x) d\nu_F(x) \right) .$$

It is an important problem in smooth ergodic theory (and also in various areas in physics) to know whether such a function decays exponentially fast as $t \rightarrow \infty$.

Using Theorem 2 above and the technique developed by Dolgopyat [D], one immediately gets the following.

Theorem 3. *Let F be a Hölder continuous function on Λ in $S^*(\Omega)$ and let ν_F be the Gibbs measure determined by F on Λ . For every $\alpha > 0$ there exist constants $C = C(\alpha) > 0$ and $c = c(\alpha) > 0$ such that*

$$|\rho_{A,B}(t)| \leq Ce^{-ct} \|A\|_\alpha \|B\|_\alpha$$

for any two functions $A, B \in F_\alpha(\Lambda)$.

Here $\|h\|_\alpha$ denotes the Hölder constant of $h \in F_\alpha(\Lambda)$.

We refer the reader to the recent survey article of Baladi [Ba] for general information and historical remarks on decay of correlations. Amongst the most recent achievements one should mention the important articles of Liverani [L], Young [Y], Chernov [Ch1] and Dolgopyat [D] answering long standing questions. It appears that for billiards the only results of this type that have been known so far concern the corresponding discrete dynamical system (generated by the billiard ball map from boundary to boundary). To my knowledge, these results are: the subexponential decay of correlations established for a very large class of dispersing billiards by Bunimovich, Sinai and Chernov [BSC] and the exponential decay of correlations for some classes of dispersing billiards in the plane and on the two-dimensional torus established by Young [Y] and Chernov [Ch2] as consequences of their more general arguments. Theorem 3 above describes a non-trivial class of billiard (broken geodesic) flows with exponential decay of correlations for any Hölder continuous potential.

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