the unique maximal ideal of A, Spec(A) $- \mathfrak{m}$ is a scheme, but not affine (that it is a prescheme is seen by Spec(A) $- \mathfrak{m} = \bigcup_{t \in \mathfrak{m}} D(t)$.

We shall hence study the inner properties of local rings A. More specifically, we shall study:

1) Dimension theory. (Dimension, Depth, Regularity)
2) Behavior under local morphisms (Flatness, Ascent, and Descent)
3) Operations on a local ring (Completion, Normalization, Henselization)
4) Stability under the operations in 3. (Excellent rings)

Most of the topics covered will be found, under different treatments, in M. Nagata's book "Local Rings", or J.P. Serre's Algébre locale, Multiplicités, Springer-Verlag, 1965, or E.G.A., IV.

We again remind the reader that we shall limit ourselves to noetherian rings.

§1. DIMENSION THEORY - GENERAL NOTIONS

Let A be a ring. The prime ideals $(\mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ of A are said to form a chain of length n if $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n$.

Definition 1.1. (Krull) The dimension of A, dim(A) is equal to the l.u.b. of the lengths of the chains of prime ideals in A.

Clearly dim(A) need not be finite. For example, if

$A = k[X_1, X_2, \ldots, X_n \ldots]$ there are clearly chains of arbitrary length.

In fact, even when A is noetherian, an example of Nagata shows that dim(A) need not be finite. It is, however, if A is a local ring. (See theorem 2.3 ahead)

Definition 1.2. Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then we define

$\dim V(\mathfrak{p}) = \dim(A/\mathfrak{p})$

$\mathrm{Codim}\, V(\mathfrak{p}) = \dim(A_{\mathfrak{p}})$

Proposition 1.1. a) $\dim V(\mathfrak{p}) \leqq \dim(A)$; b) $\mathrm{Codim}\, V(\mathfrak{p}) \leqq \dim(A)$; c) $\dim V(\mathfrak{p}) + \mathrm{Codim}\, V(\mathfrak{p}) \leqq \dim(A)$.

Proof: We have two canonical morphisms

$$A \longrightarrow A/\mathfrak{p} \;;\; A \rightarrow A_{\mathfrak{p}}$$

and we immediately get a) from the first, b) from the second. Note that a) and b) hold also when the left-hand sides are ∞. Hence c) holds if either of the summands on the left is ∞. Now, any chain in $A/\mathfrak{p}$ gives rise to a chain of equal length in A, of prime ideals containing $\mathfrak{p}$, and any chain in $A_{\mathfrak{p}}$ gives rise to a chain of equal length in A, of prime ideals contained in $\mathfrak{p}$.

Furthermore, we may assume that the chain in $A/\mathfrak{p}$ of length $\dim(A/\mathfrak{p})$ start with (0), and the ones in $A_{\mathfrak{p}}$ of length $\dim(A_{\mathfrak{p}})$ ends with $\mathfrak{p}A_{\mathfrak{p}}$. Hence the corresponding combined chain in A consists of $(\dim V(\mathfrak{p}) + \mathrm{Codim}\, V(\mathfrak{p}) + 1)$ distinct prime ideals, which proves c).

Equally simple is the proof of the following two statements, proof which we leave to the reader.

1) If $\mathfrak{a}$ is any ideal of A, $\dim(A/\mathfrak{a}) \leq \dim(A)$.

2) If $\mathfrak{a}$ is not contained in any minimal prime ideal of A, then $\dim(A/\mathfrak{a}) < \dim(A)$.

Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A)$, $\mathfrak{p} \subset \mathfrak{q}$. A chain $\mathfrak{p} \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{q}$ is called a saturated chain connecting $\mathfrak{p}$ and $\mathfrak{q}$ if its length cannot be increased by insertion of some prime ideals.

Definition 1.3. If, for all pairs $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A)$, all saturated chains connecting $\mathfrak{p}$ and $\mathfrak{q}$ have the same length, A is said to be a catenary ring.

An example of Nagata shows that noetherian local rings need not be catenary.

Proposition 1.2. Let A be an integral local ring. Then

i) If A is catenary for all $\mathfrak{p} \in \operatorname{Spec}(A)$, $\dim(A) = \dim(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p})$.

ii) A is catenary if, and only if, for all $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A)$ with $\mathfrak{p} \subset \mathfrak{q}$, $\dim A_{\mathfrak{q}} = \dim A_{\mathfrak{p}} + \dim(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}})$.

Proof. i) Since A is an integral local ring, the following statements hold:

a) $A/\mathfrak{p}$, $A_{\mathfrak{p}}$ are integral local rings, hence all dimensions involved are finite.

b) Any chain in A of length equal to dim(A) is a saturated chain connecting (o) and $\mathfrak{m}_A$ ($\mathfrak{m}_A$ denotes the unique maximal ideal of A).

c) Statement b) above holds for $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$. Note that

$$\mathfrak{m}_{A_{\mathfrak{p}}} = \mathfrak{p}A_{\mathfrak{p}} \quad \text{and} \quad \mathfrak{m}_{A/\mathfrak{p}} = \mathfrak{m}_A(A/\mathfrak{p}).$$

Statement i) now follows immediately from a), b), c) above.

ii) We begin by observing that, if A is an arbitrary catenary ring, and $\mathfrak{p} \in \mathrm{Spec}(A)$, then $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are catenary. This is easily seen from the 1-1 onto correspondences that exist between the prime ideals of $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ respectively, and the appropriate prime ideals of A.

Let now $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$, $\mathfrak{p} \subset \mathfrak{q}$ and A an integral, local, catenary ring. Then $A_{\mathfrak{q}}$ is a local, integral catenary ring, and we may apply i) to the ideal $\mathfrak{p}A_{\mathfrak{q}}$. So

$$\dim(A_{\mathfrak{q}}) = \dim(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}) + \dim((A_{\mathfrak{q}})_{\mathfrak{p}A_{\mathfrak{q}}}).$$

The morphism $\varphi:(A_{\mathfrak{q}})_{\mathfrak{p}A_{\mathfrak{q}}} \to A_{\mathfrak{p}}$ given by $\varphi((a/s)/(b/t)) = at/bs$, $a \in A$, $s,t \notin \mathfrak{q}$, $b \notin \mathfrak{p}$ is well defined ($bs \notin \mathfrak{p}$) and easily seen to be an isomorphism. One part of ii) is proved.

To prove the converse, we observe first that any saturated chain, in A, connecting $\mathfrak{p}$ and $\mathfrak{q}$ gives rise to a saturated chain of equal length in $A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}$ connecting (0) and $\mathfrak{q}A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}$. Hence the length s of any saturated chain in A connecting $\mathfrak{p}$ and $\mathfrak{q}$ is at most $r = \dim(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}})$. We assert $s = r$. When $r = 0, 1$ the assertion is trivially true, and we proceed by induction on r. Let

$$\mathfrak{p} \subsetneq \mathfrak{p}_1 \subsetneq \cdots\cdots \subsetneq \mathfrak{p}_{s-1} \subsetneq \mathfrak{q}$$

be a saturated chain of length s in A connecting $\mathfrak{p}$ and $\mathfrak{q}$.

We have $\dim(A_{\mathfrak{q}} / \mathfrak{p}_{s-1} A_{\mathfrak{q}}) = 1$. Now

$$\dim(A_{\mathfrak{p}_{s-1}} / \mathfrak{p} A_{\mathfrak{p}_{s-1}}) = \dim(A_{\mathfrak{p}_{s-1}}) - \dim(A_{\mathfrak{p}}) =$$

$$\dim(A_{\mathfrak{q}}) - \dim(A_{\mathfrak{q}} / \mathfrak{p}_{s-1} A_{\mathfrak{q}}) - \dim(A_{\mathfrak{p}}) =$$

$$\dim(A_{\mathfrak{q}} / \mathfrak{p} A_{\mathfrak{q}}) - 1 = r - 1.$$

By induction $s - 1 = r - 1$ and we are done.

If $\varphi: A \to B$ is a homomorphism, B can be considered as an A-algebra by $a \cdot b = \varphi(a) \cdot b$. We say that B is integral over A if every $b \in B$ satisfies an equation of integral dependence over A, i.e. $b^n + a_{n-1} b^{n-1} + \ldots + a_0 = 0$, $a_i \in A$, $n > 0$.

Theorem 1.1. (Going-up theorem). Let $\varphi: A \to B$ be a homomorphism, B integral over A. Then

i) $\dim(B) \leqq \dim(A)$ (lame going-up theorem).

ii) If φ is mono, $\dim(A) = \dim(B)$.

Proof: i) Let $\mathfrak{q}$ be a proper prime ideal of B. We assert:

$$\text{a)} \quad \varphi^{-1}(\mathfrak{q}) \neq A$$

$$\text{b)} \quad \varphi^{-1}(\mathfrak{q}) \neq \ker(\varphi) \text{ if } \mathfrak{q} \neq (0),$$

and B is an integral domain. a) is trivial, since $\varphi(1) = 1$ and $\mathfrak{q}$ is proper.

To prove b) assume $\varphi^{-1}(\mathfrak{q}) = \ker \varphi$. Then $\operatorname{Im} A \cap \mathfrak{q} = (0)$. Let $b \in \mathfrak{q}$, $b \neq 0$. Let

$$b^n + c_{n-1} b^{n-1} + \ldots + c_0 = 0$$

be an equation of integral dependence of minimal degree. Now $c_0 \in \operatorname{Im}(A)$ and clearly $c_0 \in \mathfrak{q}$. Hence $c_0 = 0$, and

$$b(b^{n-1} + c_{n-1}\, b^{n-2} + \ldots + c_1) = 0.$$

this is a contradiction, since B is an integral domain.

To prove i) from a) and b), let $\mathfrak{p} \subsetneq \mathfrak{q}$ be prime ideals of B. From $A \xrightarrow{\varphi} B \xrightarrow{c} B/\mathfrak{p}$ we see that $B/\mathfrak{p}$ is an integral domain, integral over A, and that

$$\varphi^{-1}(\mathfrak{p}) = \ker(c \circ \varphi)$$

$$\varphi^{-1}(\mathfrak{q}) = (c \circ \varphi)^{-1} (\mathfrak{q} \cdot B/\mathfrak{p}) \text{ and } \mathfrak{q} B/\mathfrak{p} \neq (0).$$

Hence, from b) above $\varphi^{-1}(\mathfrak{q}) \subsetneq \varphi^{-1}(\mathfrak{p})$, and i) follows.

<u>Note</u>: i) holds under the weaker assumption that B is algebraic over A.

ii) Let $\mathfrak{p} \subsetneq \mathfrak{q}$ be prime ideals of A. By theorem 1 of Chapter V, 2 of B.C.A., there exists a prime ideal $\mathfrak{p}'$ in B such that $\varphi^{-1}(\mathfrak{p}') = \mathfrak{p}$. Then $\varphi(\mathfrak{p}) \subset \mathfrak{p}'$, the morphism

$$\varphi' : A/\mathfrak{p} \to B/\mathfrak{p}'$$

is mono, and $B/\mathfrak{p}'$ is integral over $A/\mathfrak{p}$. Now $\mathfrak{q}(A/\mathfrak{p}) \neq (0)$ is a prime ideal of $A/\mathfrak{p}$, and hence there exists a prime ideal $\mathfrak{q}''$ of $B/\mathfrak{p}'$ such that $\varphi'^{-1}(\mathfrak{q}'') = \mathfrak{q}(A/\mathfrak{p})$. We have $\mathfrak{q}'' = \mathfrak{q}' \cdot B/\mathfrak{p}'$, where $\mathfrak{q}'$ is a prime ideal of B, and clearly $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$. Since $\mathfrak{q}(A/\mathfrak{p}) \neq (0)$ and φ' is mono, we have $\mathfrak{q}'' \neq (0)$, whence $\mathfrak{q}' \supsetneq \mathfrak{p}'$. This implies

$$\dim(A) \leqq \dim(B) \quad \text{whence ii) follows.}$$

Definition 1.2. gives the notion of dimension for an irreducible closed subset of Spec(A). We extend this notion to arbitrary closed subsets by the formula

$$\dim(V(\mathfrak{a})) = \dim(A/\mathfrak{a})$$

where $\mathfrak{a}$ is an arbitrary ideal of A.

If M is a finitely generated A-module we define

$$\dim(M) = \dim(\mathrm{Supp}(M)) = \dim(A/\mathrm{ann}(M)).$$

Here we use the fact, mentioned in the preliminaries, that Supp(M) is the closure in Spec(A) of Ass(M), and Ass(M) consists of the prime ideals associated to ann(M).

If $N \subset M$ is another A-module we see trivially that

$$\dim(N) \leqq \dim(M)$$

$$\dim(M/_N) \leqq \dim(M)$$

In fact $\mathrm{ann}(N) \supset \mathrm{ann}(M)$, $\mathrm{ann}(M/_N) \supset \mathrm{ann}(M)$.
A non-trivial statement, proved in Bourbaki's, chapter IV, §2, is the following:

Theorem 1.2. $\dim(M) = 0$ if, and only if, M has finite length, in the composition series sense.

§2. HILBERT-SAMUEL POLYNOMIAL

Let H be a graded ring, i.e.

$$H = \bigoplus_{n \geqq 0} H_n$$

where H_n are (additive) groups and $h_n \cdot h_m \in H_{n+m}$, for $h_n \in H_n$, $h_m \in H_m$. Clearly H_n is an H_0-module. We assume:

a) H_0 is an artinian ring

b) H is generated (as an H_0-algebra) by finitely many elements of H_1.