Some Recent Results and Problems in the Theory of Value-Distribution

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Dedicated to Professor Wilhelm Stoll on the occasion of his inauguration as the Duncan Professor of Mathematics.

For meromorphic functions of one complex variable, the theory of value-distribution has tremendously developed already since the twenties of this century. Although it has a long history, there are still some interesting and remarkable results during the recent years. For instance, Drasin [6] proved that the F. Nevanlinna conjecture is correct; Lewis and Wu [13] made a significant step in proving the Arakelyan's conjecture [1]; Osgood [18] and Steinmetz [20] independently proved the defect relation for small functions, and so on.

In this lecture, I would like to mention some recent results and problems which are based on my own interests.

1. Precise estimate of total deficiency of meromorphic functions and their derivatives

Let $f(z)$ be a transcendental meromorphic function in the finite plane and a be a complex value (finite or infinite). According to R. Nevanlinna

$$\begin{aligned}\delta(a,f) &= \liminf_{r\to\infty} \frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)}\\ &= 1-\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)}\end{aligned}$$

It is clear that $0 \le \delta(a,f) \le 1$. If $\delta(a,f)$ is positive, then a is named a deficient value with respect to $f(z)$ and $\delta(a,f)$ is its deficiency. The most fundamental result of Nevanlinna theory can be stated as follows [12, 17, 23].

Any transcendental meromorphic function $f(z)$ in the finite plane has countable deficient values at most and the total deficiency does not exceed 2. i.e.

$$\sum_{a\in\bar{\mathbb{C}}} \delta(a,f) \leq 2.$$

It is the famous Defect Relation (or Deficient Relation). In the general case, the upper bound 2 is sharp. If the order λ or the lower order μ of $f(z)$ is assigned, then the following deficient problem can be introduced. (Edrei [8])

Problem 1. Let $\mathcal{F}_\mu$, be the set of all the meromorphic functions of finite lower order μ. Can we determine

$$\Omega(\mu) = \sup_{f\in\mathcal{F}_\mu} \left\{ \sum_{a\in\bar{\mathbb{C}}} \delta(a,f) \right\}?$$

Do extremal functions exist? If so, what other properties characterize extremal functions?

When μ is less than 1, Edrei [8] obtained a precise estimate for the total deficiency by using the spread relation proved by Baernstein [2]. In fact, he proved

$$\Omega(\mu) = \begin{cases} 1, & 0 \leq \mu \leq \frac{1}{2}, \\ 2 - \sin \mu\pi, & \frac{1}{2} < \mu \leq 1. \end{cases}$$

The Problem 1, however, is still open for meromorphic functions of lower order bigger than 1, although a suitable bound has been suggested by Drasin and Weitsman [7] as follows:

$$\Omega(\mu) = \max\{\Lambda_1(\mu), \Lambda_2(\mu)\},$$

where

$$\Lambda_1(\mu) = 2 - \frac{2\sin\frac{\pi}{2}(2\mu - [2\mu])}{[2\mu] + 2\sin\frac{\pi}{2}(2\mu - [2\mu])}$$

and

$$\wedge_2(\mu) = 2 - \frac{2\cos\frac{\pi}{2}(2\mu - [2\mu])}{[2\mu]+1}.$$

Now we consider the derivative $f^{(k)}(z)$ of order k, where k is a positive integer. Hayman [11] pointed out that

$$\sum_{a\in\mathbb{C}} \delta(a, f^{(k)}) \le \frac{k+2}{k+1}.$$

In 1971, Mues [15] improved this result to

$$\sum_{a\in\mathbb{C}} \delta(a, f^{(k)}) \le \frac{k^2+5k+4}{k^2+4k+2}.$$

Recently I proved [26]

Theorem 1. Let $f(z)$ be a transcendental meromorphic function in the finite plane and k be a positive integer. Then we have

$$\sum_{a\in\mathbb{C}} \delta(a, f^{(k)}) \le \frac{2k+2}{2k+1}.$$

It is clear that for any positive integer k, we always have

$$\frac{2k+2}{2k+1} < \frac{k^2+5k+4}{k^2+4k+2} < \frac{k+2}{k+1}$$

and

$$\frac{k^2+5k+4}{k^2+4k+2} - \frac{2k+2}{2k+1} > \frac{k+2}{k+1} - \frac{k^2+5k+4}{k^2+4k+2}.$$

Although Theorem 1 gives a much better estimate for $\sum\limits_{a\in\mathbb{C}} \delta(a, f^{(k)})$, it does not include $\delta(\infty, f^{(k)})$. For this reason, we have another estimate [26].

Theorem 2. Let $f(z)$ be a transcendental meromorphic function of finite order in the finite plane and k be a positive integer. Then we have

$$\sum_{a\in\bar{\mathbb{C}}} \delta(a, f^{(k)}) \leq 2 - \frac{2k(1-\Theta(\infty,f))}{1+k(1-\Theta(\infty,f))},$$

where $\Theta(\infty, f)$ is the ramification index of ∞ with respect to f, defined by

$$\Theta(\infty, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)}.$$

In particular, if $\Theta(\infty, f) < 1$, then we have

$$\lim_{k\to\infty} \left\{ \sum_{a\in\bar{\mathbb{C}}} \delta(a, f^{(k)}) \right\} = 0$$

It is natural to discuss the precise estimate of total deficiency of both the function itself and its derivative. For this subject, Drasin [5] posed the following questions.

Problem 2. Let $f(z)$ be meromorphic and of finite order in the finite plane. If $\sum_{a\in\bar{\mathbb{C}}} \delta(a, f) = 2$ and $\delta(\infty, f) = 0$, do we must have

$$\sum_{b\in\overline{\mathbb{C}}} \delta(b, f') = \delta(0, f') = 1?$$

Problem 3. Let $f(z)$ be meromorphic in the finite plane with $\delta(\infty, f) = 0$. Can we have

$$\sum_{a\in\bar{\mathbb{C}}} \delta(a, f) + \sum_{b\in\overline{\mathbb{C}}} \delta(b, f') = 4?$$

If not, what is best bound?

Quite recently, I proved the following theorem.

Theorem 3. Let $f(z)$ be a transcendental meromorphic function of finite order in the finite plane and k be a positive integer. Then we have

$$\sum_{a\in\mathbb{C}}\delta(a,f)+\sum_{b\in\overline{\mathbb{C}}}\delta(b,f^{(k)})\leq 3.$$

The equality holds if and only if either

(i) $\quad \Theta(\infty,f)=1,\quad \sum_{a\in\mathbb{C}}\delta(a,f)=1,\quad \sum_{b\in\overline{\mathbb{C}}}\delta(b,f^{(k)})=2;$

or

(ii) $k=1,\quad \Theta(\infty,f)=0,\quad \sum_{a\in\mathbb{C}}\delta(a,f)=2,\quad \sum_{b\in\overline{\mathbb{C}}}\delta(b,f')=1;$

Theorem 3 gives a positive answer to the Problem 2 by comparing Theorem 3 and the known fact

$$\sum_{a\in\mathbb{C}}\delta(a,f)\leq(2-\delta(\infty,f))\delta(o,f').$$

Theorem 3 answers also the Problem 3, when f is of finite order.

2. Conjectures of Frank, Goldberg and Mues

Mues [15] posed the following conjecture, when he improved the Hayman's estimate.

Problem 4. Let $f(z)$ be a transcendental meromorphic function in the finite plane and k be a positive integer. Then the following relation should be true.

$$\sum_{a\in\mathbb{C}}\delta(a,f^{(k)})\leq 1.$$

If $f(z)$ satisfies one of the following conditions, then the Mues conjecture can be easily verified.

(i) The order of f is finite and $\Theta(\infty,f)\leq 1-\frac{1}{k}$;

(ii) f has only poles with multiplicity $\leq k$;

(iii) The order of f is finite and

$$\delta(\infty,f^{(k)})+\sum_{a\in\mathbb{C}}\delta(a,f)\geq 2.$$

It seems that the Mues conjecture is true, when f has finite order.

Connecting the Problem 4, G. Frank and A. Goldberg recently raised two similar conjectures respectively.

Problem 5. Let $f(z)$ be a transcendental meromorphic function in the finite plane and k be a positive integer. If ε is an arbitrary small positive number, then the following estimate seems true.

$$k\overline{N}(r,f) < (1+\varepsilon)N(r,\frac{1}{f^{(k+1)}}) + S(r,f),$$

where

$$S(r,f) = O\{\log(rT(r,f))\},$$

except for r in a set with finite linear measure.

Problem 6. Let $f(z)$ be a transcendental meromorphic function in the finite plane. Then the following inequality should be correct.

$$\overline{N}(r,f) < N(r,\frac{1}{f''}) + S(r,f).$$

When $k \geq 2$, the Frank conjecture (Problem 5) is much stronger than the Goldberg conjecture (Problem 6). The Mues conjecture is a direct consequence of any one of them, when the order of $f(z)$ is finite.

3. Best coefficients of Hayman inequality

In 1959, Hayman [11] obtained a very interesting and remarkable inequality in which the characterstic function $T(r,f)$ can be bonnded by two counting functions. It is impossible without introducing the derivatives.

$$T(r,f) < (2+\frac{1}{k})N(r,\frac{1}{f})+ \\ (2+\frac{2}{k})N(r,\frac{1}{f^{(k)}-1}) + S(r,f). \tag{3.1}$$

Later on, Hayman [12] adopted this inequality as the principal result of Chapter 3 in his book. He also posed the following question.

Problem 7. What are the best coefficients of the inequality (3.1).

Concerning this problem, the following inequality was obtained by the author [25] about two years ago.

Theorem 4. Let $f(z)$ be a transcendental meromorphic function in the finite plane and k be a positive integer. If ε is an arbitrary small positive number, then we have

$$T(r,f) < (1+\frac{1}{k}+\varepsilon)N(r,\frac{1}{f})+(1+\frac{1}{k}+\varepsilon)N(r,\frac{1}{f^{(k)}-1})+S(r,f).$$

The proof of Theorem 4 is based on the following lemma.

Lemma. Under the conditions of Theorem 4, we have

$$\overline{N}(r,f) < (\frac{1}{k}+\varepsilon)N(r,\frac{1}{f})+(\frac{1}{k}+\varepsilon)N(r,\frac{1}{f^{(k)}-1})+S(r,f).$$

If the Goldberg conjecture is true, then we have

$$T(r,f) < N(r,\frac{1}{f})+N(r,\frac{1}{f^{(k)}-1})+S(r,f). \tag{3.2}$$

If the Frank conjecture is true, then we also have the inequality (3.2) in the case of $k \geq 2$ and

$$T(r,f) < (1+\varepsilon)N(r,\frac{1}{f})+(1+\varepsilon)N(r,\frac{1}{f'-1})+S(r,f)$$

in the ease of $k = 1$.

4. Normal families and fix-points of meromorphic functions

A theorem which makes the connection between the normality of a given family of meromorphic functions and the lack of fix-points of both these functions and their derivatives, was proved by the author [24] in 1986.

Theorem 5. Let $\mathcal{F}$ be a family of meromorphic functions in a region D and k be a positive integer. If, for every function $f(z)$ of $\mathcal{F}$, both $f(z)$ and $f^{(k)}(z)$ (the derivative of order k) have no fix-points in D, then $\mathcal{F}$ is normal there.

Since the iteration of $f(z)$ is very important and grows much faster than $f(z)$ and $f^{(k)}(z)$, it is natural to pose the following problem.

Problem 8. Let $\mathcal{F}$ be a family of entire functions, D be a region and k be a fixed positive integer. If, for every function $f(z)$ of $\mathcal{F}$, both $f(z)$ and $f^k(z)$ (the iteration of order k of $f(z)$) have no fix-points in D, is $\mathcal{F}$ normal there?

Schwick [19] proved several criteria for normality. Among others, he proved

Theorem 6. Let $\mathcal{F}$ be a family of meromorphic functions in a region D and n and k be two positive integers. If for every function $f(z)$ of $\mathcal{F}$,

$$(f^n)^{(k)} \neq 1$$

and

$$n \geq k + 3, \tag{4.1}$$

then $\mathcal{F}$ is normal in D. Moreover, if $\mathcal{F}$ is a family of holomorphic functions, then the condition (4.1) can be replaced as

$$n \geq k + 1. \tag{4.2}$$

It seems to me that the following assertion should be true.

Problem 9. Let $\mathcal{F}$ be a family of meromorphic functions in a region D and n and k be two positive integers with (4.1). If, for every function $f(z)$ of $\mathcal{F}$, $(f^n)^{(k)}$ has no fix-points, then $\mathcal{F}$ is normal in D. When $\mathcal{F}$ is a family of holomorphic functions, then (4.1) can also be replaced by (4.2).

5. Common Borel directions of a meromorphic function and its derivatives

Let $f(z)$ be meromorphic and of order λ in the finite plane, where $0 < \lambda < \infty$. Valiron [21] proved there exists a direction B : $\arg z = \theta_0 (0 \leq \theta_0 < 2\pi)$ such that, for any positive number ε and any complex value a, we always have

$$\limsup_{r\to\infty} \frac{\log n(r, \theta_0, \varepsilon, f = a)}{\log r} = \lambda,$$

except two values of a at most, where $n(r, \theta_0, \varepsilon, f = a)$ denotes the number of zeros of $f - a$ in the region $(|z| \leq r) \cap (|\arg z - \theta_0| \leq \varepsilon)$. Such direction is named a Borel direction of order λ of $f(z)$.

Since $f^{(k)}(z)$ is also meromorphic and of order λ in the finite plane, it has a Borel direction too. Valiron [21] asked the following question.

Problem 10. Let $f(z)$ be meromorphic and of order λ in the finite plane, where $0 < \lambda < \infty$. Is there a common Borel direction of $f(z)$ and all its derivatives?

Concerning this problem, the known result is

Theorem 7. Let $f(z)$ be meromorphic and of order λ in the finite plane, where $0 < \lambda < \infty$. If $f(z)$ has a Borel exceptional value (which is either a finite complex value or the infinity), then there exists a common Borel direction of $f(z)$ and all its derivatives.

The papers concerning Theorem 7 are due to Milloux, Zhang K. H., Zhang Q. D. and myself [14, 23, 27].

On the other hand, the following fact is also known.

There exists a meromorphic function $f_0(z)$ of order one in the finite plane such that its derivative $f_0'(z)$ has more Borel directions than $f_0(z)$.

For instance, the function

$$f_0(z) = \frac{e^{-iz}}{1 + e^z},$$

which was pointed out by Steinmetz, is such an example. $f_0(z)$ has the Borel directions $\arg z = \frac{\pi}{4}$, π and $\frac{3\pi}{2}$, but $f_0'(z)$ has these and in addition the direction $\arg z = \frac{\pi}{2}$.

6. Optimum condition to ensure the existence of Hayman direction

Corresponding to the Hayman's inequality, we may ask if there are some similar results in the angular distribution. My following theorem [22] aims at answer of this question.

Theorem 8. Let $f(z)$ be a meromorphic function in the finite plane. If

$$\limsup_{r\to\infty} \frac{T(r,f)}{(\log r)^3} = \infty, \tag{6.1}$$

then there exists a direction (H) : $\arg z = \theta_0$ such that, for any positive number ε, an arbitrary integer k and any two finite complex values a and $b (b \neq 0)$, we have

$$\lim_{r\to\infty} \{n(r,\theta_0,\varepsilon,f=a) + n(r,\theta_0,\varepsilon,f^{(k)}=b)\} = \infty.$$

It is convenient to name such kind of direction as Hayman direction.

For a meromorphic function, since the condition ensuring a Julia direction is

$$\limsup_{r\to\infty} \frac{T(r,f)}{(\log r)^2} = \infty, \tag{6.2}$$

Drasin [3] raised the following question in 1984.

Problem 11. Is Theorem 8 still true, if the condition (6.1) is replaced by (6.2)?

It seems that the answer of Problem 11 is positive, since Chen H. H. [4] proved recently the following fact.

Theorem 9. Let $f(z)$ be a meromorphic function in the finite plane. If the condition (6.2) is satisfied, then there exists a direction

(H) : $\arg z = \theta_0$ such that for any positive number ε, an arbitrary positive integer k and any two finite complex values a and $b (b \neq 0)$, we have

$$\limsup_{r\to\infty} \frac{n(r,\theta_0,\varepsilon,f=a)+n(r,\theta_0,\varepsilon,f^{(k)}=b)}{\log r} = \infty.$$

7. Picard type theorems and the existence of singular directions

Bloch principle says a family of holomorphic (or meromorphic) functions in a region satisfying a condition (or a set of conditions) uniformly which can only be possessed by the constant functions in the finite plane, must be normal there. In simple words, there usually is a criterion for normality to correspond a Picard type theorem. Similarly we have the following question.

Problem 12. Corresponding to every Picard type theorem, is there a singular direction? To be precise, let P be a property (or a set of properties) such that any entire function (or a meromorphic function in the finite plane) satisfying P, must be a constant. Then for any transcendental entire function (or a meromorphic function with some suitable condition of growth), is there a ray $\arg z = \theta_0 (0 \leq \theta_o < 2\pi)$ such that $f(z)$ does not satisfy P in the angle $|\arg z - \theta_0| < \varepsilon$, for any small positive number ε.

The direction in the Problem 12 is a direction of Julia type. We can also pose a problem for a direction of Borel type.

For instance, the following fact is well known.

Let $f(z)$ be meromorphic in the finite plane. If $(f^n)^{(k)} \neq 1$ for two positive integers n and k with $n \geq k+3$, then $f(z)$ must reduce to a constant. When $f(z)$ is entire, the condition $n \geq k+1$ is sufficient.

Problem 13. Let $f(z)$ be meromorphic and of order λ in the finite plane, where $0 < \lambda < \infty$. Is there a direction $\arg z = \theta_0 (0 \leq \theta_0 < 2\pi)$ such that for any positive number ε two arbitrary positive integers n and k with $n \geq k+3$ and any finite, non-zero complex value a, we have

$$\limsup_{r\to\infty} \frac{\log n(r, \theta_0, \varepsilon, (f^n)^{(k)} = a)}{\log r} = \lambda?$$

When $f(z)$ is entire, the condition $n \geq k+1$ seems sufficient.

8. Growth, number of deficient values and Picard type theorem

Picard type theorems are not only involved in criteria for normality and singular directions, but also connect with the growth and number of deficient values.

Problem 14. Let P be a property (or a set of properties) given by the Problem 12. Suppose that $f(z)$ is entire (or meromorphic) and of finite lower order μ in the finite plane and that L_j : arg $z = \theta_j (j = 1, 2, \cdots, J; 0 \leq \theta_1 < \theta_2 < \cdots < \theta_J < 2\pi)$ are finite number of rays issued from the origin. If $f(z)$ satisfies P in $\mathbb{C}\backslash(\cup_{j=1}^{J} L_j)$ then the order λ of $f(z)$ seems to have the estimate

$$\lambda \leq \max_{1\leq j\leq J}\left\{\frac{\pi}{\theta_{j+1}-\theta_j}\right\}, \quad (\theta_{J+1} = \theta_1 + 2\pi)$$

and the number of finite non-zero deficient values does not exceed J.

The property P can be chosen as

(i) $f(z) \neq 0$ and $f^{(k)}(z) \neq 1 (k \in \mathbb{Z}^+)$;

(ii) $f'(z) - af(z)^n \neq b$, where a and b are two finite complex values with $a \neq 0$ and $n \geq 5$ is a positive integer;

(iii) $(f(z)^n)^{(k)} \neq 1$, where n and k are two positive integers with $n \geq k+3$, when f is meromorphic and $n \geq k+1$, when f is entire.

9. Value-distribution with respect to small functions

Let $f(z)$ be meromorphic in the finite plane and a be a complex value. The theory of value-distribntion investigates the distribution of zeros of $f(z) - a$. It is natural to instead of the complex value a by another meromorphic function $a(z)$ with the condition

$$T(r, a(z)) = o\{T(r, f)\}. \tag{9.1}$$

Therefore, corresponding to results in the theory of value-distribution, we can ask similar questions with respect to small functions. Some of them are very difficult and significant. For instance, R. Nevanlinna himself asked if his defect relation can be extended to small functions. Up to few years ago, Osgood [18] and Steinmetz [20] independently settled this problem with a positive answer.

Theorem 10. Let $f(z)$ be a transcendental meromorphic function in the finite plane and A be the set of all meromorphic functions $a(z)$ with the condition (9.1). Then we have

$$\sum_{a(z)\in A} \delta(a(z), f) \leq 2.$$

All the complex values including the infinity are contained in A.

There are still some problem. We mention only one of them here.

Recently, Lewis and Wu [13] proved

Theorem 11. If $f(z)$ is an entire function of finite lower order, then there exists a positive number τ_0 not depending on f such that the series $\sum_{a\in\overline{\mathbb{C}}}(\delta(a, f))^{\frac{1}{3}-\tau}$ is convergent for any τ with $0 < \tau < \tau_0$.

Problem 15. Is Lewis and Wu's result still true for small functions?

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