ON THE EXISTENCE OF ∅-DEFINABLE NORMAL SUBGROUPS OF A STABLE GROUP

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There is a family of results concerning the existence of (∅)-definable normal subgroups of a stable group. Namely:

(1) (Berline-Lascar [B-L]). If G is superstable, and

$$U(G) = \omega^{\alpha_1}n_1+\ldots+ \omega^{\alpha_k} n_k + \beta \ (\beta<\omega^{\alpha_k})$$

then G has a normal subgroup K with $U(K) = \omega^{\alpha_1}n_1 + \ldots + \omega^{\alpha_k}n_k$.

(2) (Hrushovski [H]). If G is stable and its generic type is nonorthogonal to a regular type p, then there is a definable normal subgroup K of G such that G/K is "p-internal" and infinite.

(3) (Pillay-Hrushovski [PH]). If G is 1-based and connected then every type q = stp(a/A) (a∈G) is the generic type of a coset of a normal acl(∅)-definable subgroup K of G.

In this expository paper we will prove these results and some variants.

We work throughout over acl (∅) (i.e. we assume acl (∅) = dcl (∅)). G is assumed throughout to be a saturated, stable, connected group. We prove:

Theorem A. Let g be a generic of G (over ∅), and let X⊂G be invariant and internally closed. Then there is a ∅-definable normal K⊂G such that (i) G/K⊂X and (ii) some stationarisation of tp(g/X) is the generic type of a generic coset of K° (the connected component of K).

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(We will see that the Berline-Lascar result (1) above follows from Theorem A when we take $X = \{a \in G^{eq}: U(tp(a/\emptyset)) < \omega^{\alpha_k}\}$).

Theorem B. Let g be a generic of G, and $A \subseteq G^{eq}$. Then $stp(g/A)$ is the generic type of a generic coset of a $\emptyset$-definable normal subgroup K of G if and only if for any generic a of G with $a \downarrow_{\emptyset} gA$, both $stp(g \cdot a/aA)$ and $stp\ (a \cdot g/aA)$ are 1-based types.

First some explanation of the notation:

Definition 1. (a) Let $X \subset M^{eq}$ (M stable saturated). We say X is <u>invariant</u> if X is fixed setwise by any automorphism of M. This clearly means that whether or not $a \in X$ depends on $tp(a / \emptyset)$.
(b) Let $X \subseteq M^{eq}$ be invariant. We say that X is <u>internally closed</u> if for any $a \in M^{eq}$, if for some $b \in M^{eq}$ $a \downarrow b$ and $a \in dcl(X \cup b)$ then also $a \in X$.
(c) $q = stp(a/A)$ is said to be <u>1-based</u> if $Cb(q) \subset acl(a)$.

The next Lemma also appears in [P].

Lemma 2. Let g be a generic of G, and $b \in G^{eq}$. Let $q = tp\ (g \wedge b/\emptyset)$. Let $K = \{a \in G$: for $g' \wedge b'$ realising $q \mid a$, $tp(a \cdot g' \wedge b'/a) = q \mid a.\}$ Then for generic a of G, a/K (the left coset aK), is in the internal closure of $tp(b)^G$.

Proof. Let G_1 be saturated, $G_1 \downarrow_{\emptyset} g \wedge b$, and let $Y \subset G_1$ be a large independent set of generics. Let X = the set of realisations of $tp(b)$ in G_1^{eq}. Let $a \in G_1$ be generic and independent with Y over $\emptyset$. Note that

$$tp(a \cdot g/G_1) \text{ is generic.} \qquad (*)$$

Claim. $tp(a \cdot g \wedge b/G_1)$ is definable over $Y \cup X$.
Proof of Claim. It is enough to show that $tp(a \cdot g \wedge b/G_1)$ is finitely satisfiable in $Y \cup X$. So suppose $\bar{m} \subset G$, and $\models \varphi(a \cdot g, b, \bar{m})$. As Y is large there is $c \in Y$ with $\bar{m} \downarrow_{\emptyset} c$. By (*) $tp(a \cdot g/\bar{m}) = tp(c/\bar{m})$. Thus $\models$

$\exists y(\varphi(c,y,\bar{m}) \wedge r(y))$ where $r = tp(b/\emptyset)$. As G_1 is saturated, we can find such a y in G_1, i.e. in X. This proves the claim.
Let $Y_o \cup X_o \subset Y \cup X$ be small such that $tp(a{\cdot}g\wedge b/G_1)$ is definable over $Y_o \cup X_o$. As $tp(g\wedge b/G_1)$ is definable over $\emptyset$ we see that any automorphism f of G_1 takes $tp(a{\cdot}g\wedge b/G_1)$ to $tp(f(a){\cdot}g\wedge b/G_1)$. Thus, if f is a $Y_o \cup X_o$ automorphism of G_1, then $tp(a{\cdot}g\wedge b/G_1) = tp(f(a){\cdot}g\wedge b/G_1)$, i.e. $tp(g\wedge b/G_1) = tp(a^{-1}f(a){\cdot}g\wedge b/G_1)$, that is $a^{-1}{\cdot}f(a) \in K$. Thus $a^{-1}{\cdot}f(a) \in K$. Thus $a/K \in dcl(Y_o \cup X_o)$. But K is $\emptyset$-definable and $a \downarrow Y_o$. Thus a/K is in the internal closure of X as required. □

Lemma 3. Let $X \subseteq G^{eq}$ be invariant and internally closed. Let K be $\emptyset$-definable such that for generic a of G, a/K (left coset) is in X. Let L = intersection of conjugates K^g of K, $g \in G$. Then $G/L \subset X$ (and L is normal $\emptyset$-definable).

Proof. First $L = K \cap K^{g_1} \cap \ldots \cap K^{g_n}$. Clearly L is normal and $\emptyset$-definable. Note that if a is a generic of G over g_i then a/K^{g_i} is interdefinable with a^{g_i}/K over g_i, and moreover a^{g_i}/K is a generic coset of K, and thus is in X. Thus, if a is a generic of G over $g_1,\ldots,g_n$ then $a/L \in dcl(a/K, \ldots, a^{g_i}/K, g_1,\ldots,g_n)$. So a/L is in A (as X is internally closed) As every element of G/L is a product of generics of $G/L \subset X$. □

Proof of Theorem A

Let K be the intersection of all $\emptyset$-definable subgroups L of G such that for generic a of G, $a/L \in X$. By Lemma 3, K is normal, clearly $\emptyset$-definable, and $G/K \subset X$. (K need not be connected). Let a realise the generic of K° over G.

So by definition, $tp(a{\cdot}g/G)$ is a generic of the coset g/K° ($=a{\cdot}g/K^\circ$) of K° and so is definable over g/K°. On the other hand $g/_K \in X$ and $g/K^\circ \in acl(g/K)$. Thus $tp(a{\cdot}d/G)$ does not fork over X.

But by Lemma 2, $tp(a{\cdot}g/X) = tp(g/X)$. Thus $tp(a{\cdot}g/G)$ is a stationarisation of $tp(g/X)$ and we finish.

Before proving the Berline-Lascar theorem, we recall some facts about U-rank.

Fact 4. $U(a/bA) + U(b/A) \leq U(a^\wedge b / A) \leq U(a/bA) \oplus U(b/A)$, and of course $U(a^\wedge b/A) = U(b^\wedge a/A)$.

Lemma 5. Let $U(a/A) = \omega^{\alpha_1}n_1 + \ldots + \omega^{\alpha_k}n_1 + \beta$ where $\alpha_1 > \alpha_2 > \ldots > \alpha_k$ and $\beta<\omega^{\alpha_k}$. Let $B \supset A$, and $U(a/B) = \omega^{\alpha_1}n_1 + \ldots + \omega^{\alpha_k}n_k$; let $c \in Cb(stp(a/B))$. Then $U(c/A)<\omega^{\alpha_k}$.

Proof. Let $a_1,a_2,\ldots$ be a Morley sequence in $stp(a/B)$. So $c \in acl(a_1,\ldots,a_n)$ for some $n<\omega$. By Fact (4) we have: ($\bar{a}$ denotes $(a_1,\ldots,a_n)$)
$U(\bar{a}/cA) + U(c/A) \leq U(\bar{a}c/A) \leq U(c/\bar{a}A) \oplus U(\bar{a}/A)$.
But $\omega^{\alpha_1}n_1 \cdot n + \ldots + \omega^{\alpha_k}n_k \cdot n \leq U(\bar{a}/cA)$, $U(c/\bar{a}A) = 0$ and $U(\bar{a}/A) < \omega^{\alpha_1}n_1 \cdot n + \ldots + \omega^{\alpha_k}(n_k \cdot n + 1)$.
Thus $\omega^{\alpha_1}n_1 \cdot n + \ldots + \omega^{\alpha_k}n_k \cdot n + U(c/A) < \omega^{\alpha_1}n_1 \cdot n + \ldots + \omega^{\alpha_k}(n_k \cdot n + 1)$, whereby $U(c/A) < \omega^{\alpha_k}$. □

Corollary 6. Let $U(G) = \omega^{\alpha_1}n_1 + \ldots + \omega^{\alpha_k}n_k + \beta$ $(\beta<\omega^{\alpha_k})$.
Then there is a normal $\varnothing$-definable subgroup K of G, with $U(K) = \omega^{\alpha_1}n_1 + \ldots + \omega^{\alpha_k}n_k$.

Proof. By Lemma 5, there is $c \in G^{eq}$ with $U(c/\varnothing) <\omega^{\alpha_k}$ and $U(g/c) = \omega^{\alpha_1}n_1 + \ldots + \omega^{\alpha_k}n_k$ (where g is a generic of G over $\varnothing$). On the other hand, if $X = \{a \in G^{eq}: U(a) < \omega^{\alpha}\}$, then by Fact 4 $U(g/X) \geq \omega^{\alpha_1}n_1 + \ldots + \omega^{\alpha_k}n_k$. Thus $U(g/X) = \omega^{\alpha_1}n_1 + \ldots + \omega^{\alpha_k}n_k$. Clearly X is invariant and internally closed. So by, Theorem A we find a subgroup K of G which is normal and $\varnothing$-definable, with $U(K) = \omega^{\alpha_1}n_1 + \ldots + \omega^{\alpha_k}n_k$ ($tp(g/X)$ is the generic type of a coset of K, so $U(g/X) = U(K)$). □

We now consider Hrushovski's result. Let $p \in S(A)$ be a regular type, not orthogonal to $\emptyset$. We say c is p-internal over $\emptyset$ if $\exists B\ c \downarrow_{\emptyset} B$ and realisations $d_1 \ldots d_k$ of extension of conjugates of p over B such that $c \in dcl(B, d_1, \ldots, d_k)$. Note that $\{c : c$ is p-internal over $\emptyset\}$ is invariant and internally closed.

Lemma 7. (T stable). Let $tp(a/\emptyset)$ be nonorthogonal to p (p regular). Then there is c which is p-internal over $\emptyset$ such that $a \not\downarrow c$. (In fact we can choose $c \in dcl(a)$, and c having "nonzero p-weight").

Proof Let $a \downarrow B$ and let e realise $p|B$, such that $a \not\downarrow_{B} e$. Let $D = Cb(stp(eB/a))$. So $D \not\subseteq acl(\emptyset)$. Let $\{e_iB_i : i < \omega\}$ be a Morley sequence in $stp(eB/a)$. Then $D \subseteq dcl(\{e_i, B_i : i < \omega\})$, and $D \subseteq acl(a)$.
A nonalgebraic member of D will then satisfy the requirements (as $D \subset acl(a)$, $D \downarrow_{\emptyset} \{B_i : i < \omega\}$). □

Now we can obtain Hrushovski's result (2) mentioned in the introduction: for suppose the generic type of G is nonorthogonal to regular p. By Lemma 7, we can find generic g of G and c p-internal over $\emptyset$ such that $g \not\downarrow c$. Let $X = \{c \in G^{eq} : c$ is p-internal over $\emptyset\}$. By Theorem A, there is $\emptyset$-definable normal $K < G$ with $G/K \subset X$. (In fact it turns out that the generic of G/K again has "nonzero p-weight").

Finally, by a slight refinement of [PH] we give necessary and sufficient conditions for $stp(g/A)$ (g generic of G) to be a generic type of a (generic) coset of an $\emptyset$-definable normal subgroup.

Proof of Theorem B.

Suppose first $stp(b/A)$ to be generic type of a generic coset of K, where K is $\emptyset$-definable, normal and of course connected. Let a be generic of G over g^A. So $stp(g/aA)$ does not fork over A. Let $G_1 \supset a\,A$, $stp(g/G_1)$ dnf over A. Let q = generic type of K over G_1. By assumption,

$tp(g/G_1) = q \cdot b = b \cdot q$ for some $b \in G$. But then $tp(g \cdot a/G_1) = tp\ (q \cdot (b \cdot a)/ G_1)$ and $tp(a \cdot g/G_1) = tp\ ((a \cdot b) \cdot q/G_1)$. But then $tp\ (g \cdot a/G_1)$ is definable over $b \cdot a/K = g \cdot a/K \in dcl(g \cdot a)$, so is 1-based. Similarly $tp(a \cdot g/G_1)$ is 1-based.

Conversely, suppose the right hand side conditions hold. Again, let a be generic of G over gA, and let $G_1 \supset A \cup a$ with $tp(g/G_1)$ not forking over A.

Let K_1 = left stabiliser of $tp(g/G_1)$ (= $\{b \in G_1 : tp(b \cdot g/G_1) = tp\ (g/G_1\}$), and let K_2 = right stabiliser of $tp(g/G_1)$. So K_1, K_2 are both acl(A)-definable.

Then any automorphism of G_1 which fixes acl(A) and $tp(a \cdot g/G_1)$ fixes a/K_1 (=left coset aK_1). Thus as $tp(a \cdot g/G_1)$ is based, $a/K_1 \in acl(A \cup a \cdot g)$. On the other hand $stp(g/a \cdot g \cup a/K_1 \cup A)$ does not fork over A. But the right coset of g mod K_1 is definable over $a \cdot g \cup a/K_1 \cup acl(A)$.

Thus the right coset $g/K_1 \in stp(g/a \cdot g \cup a/K_1)$. It easily follows that $stp(g/A)$ is the generic type of the right coset $K_1 \cdot g$ of K_1. (1)

We can do the same thing for K_2, deducing that $stp(g/A)$ is the generic type of the left coset $g \cdot K_2$ of K_2. (2)

Note also that K_1 = left stabiliser of $tp(g \cdot a/G_1)$, so as the latter type is by assumption 1-based, K_1 is $acl(g \cdot a)$-definable. (3)

Similarly K_2 is $acl(a \cdot g)$-definable. (4)

As $g \cdot a \downarrow A$, $a \cdot g \downarrow A$, it follows from (3),(4) and the acl(A)-definability of K_1, K_2, that both K_1, K_2 are $acl(\emptyset)$-definable.

It easily follows from (1) and (2) that $K_1{}^g = K_2$. As both K_1, K_2 are acl ($\emptyset$)-definable and g is generic over $\emptyset$ it follows that for generic independent g_1, g_2 of G $K_1^{g_1} = K_2 = K_1^{g_2}$, and $K_1^{g_1 g_2^{-1}} = K_1$.

But $g_1 \cdot g_2^{-1}$ is also generic, so $K_1^g = K_1$ for generic g. Thus K_1 is normal and $K_1 = K_2$, proving the Theorem.

References

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