is uniformly continuous. Further he proves that the values of such a function on a closed interval are bounded and that the upper and lower bounds are attained. He also proves that such a function takes every value between two of its values. If a quasi-primary function has a derivative for every (primary) real number, then this derivative is again a quasi-primary function.

He also develops a theo ry of integration, defining first the Riemann integral, later also Lebesque's. It might seem that a measure theory must be impossible in this system, because by ordinary concepts the measure should be = 0 for denumerable sets, and here all sets are denumerable in a sufficiently high layer. However, the distinction between primary and secondary sets makes a definition of measure possible in such a way that the primary sets all get the measure 0, but not the secondary sets.

This system has one great advantage in distinction to the previous ones, namely, that the objects we are dealing with are all definitely and explicitly given. It is true of course that the unsolvability or even undecidability of many problems remains as before, but we know what we are talking about. In the previous theories it was at any rate not required that our considerations should be restricted to the definable or constructible objects.

16. Some remarks on intuitionist mathematics

Of great interest is the so-called intuitionism which above all is due to the Dutch mathematician L. E. J. Brouwer. This theory is essentially characterized by the requirement that an assertion of the existence of a mathematical object must contain a means of finding or constructing such an object. Further, the use of such a formal logical principle as "tertium non datur" is only justified, if we have a decision procedure. The intuitionist critique of classical mathematics is similar to the critique of Kronecker who also declared that a great part of ordinary mathematics was only words. It would lead too far, however, if I should give in these lectures a detailed exposition of the intuitionist foundation of mathematics. I must confine my exposition here to a few remarks which I hope will give an idea of the intuitionist way of reasoning.

The conjunction $p \& q$ retains its usual meaning also in intuitionist logic. The disjunction $p \vee q$ can be asserted if and only if either p can be asserted or q can. The negation $\neg p$ shall mean that the assumption p leads to a contradiction. The implication $p \rightarrow q$ means that we are in possession of a certain construction which will furnish a proof of q as soon as a proof of p is available. The assertion $(x)p(x)$ is justified if we possess a schema showing the property $p(x)$ for an arbitrary x, and $(E(x)p(x)$ can be asserted if we know an x with the property p or at least have a method for constructing such an x.

Since we have no general method to prove either p or $\neg p$, the tertium non datur, $p \vee \neg p$, is not generally valid. It can be proved that $p \rightarrow \neg\neg p$ is generally true, but not the inverse implication. Such differences in the propositional logic cause differences in predicate logic of course. As an interesting example of the difference in the classical and the intuitionist way of stating a theorem, I will take an example mentioned in the book "Intuitionism" of Heyting.

Let us define a real number ρ by writing an infinite decimal fraction as follows. As long as no sequence of digits 0,1,2,3,4,5,6,7,8,9 has occured in the development of $\pi = 3.14...$ as a decimal fraction, there shall only be digits 3 in the development of ρ, however, if it should happen that the digits in the places $n - 9, \ldots\ldots, n$ should be just $0,1,\ldots,9$, then all digits after the n^{th} shall be 0 in the development of ρ. Then it is easy to prove that

$$\neg\left[(n)\left(\rho \neq \frac{10^n-1}{3.10^n}\right) \;\&\; \left(\rho \neq \frac{1}{3}\right)\right] \;.$$

This can, in classical mathematics, be expressed thus:

$$\left(\rho = \frac{1}{3}\right) \vee (En) \left(\rho = \frac{10^n-1}{3.10^n}\right) \;.$$

However, this is not correct intuitionistically, because the last statement would mean that we are able to prove either that $\rho = \frac{1}{3}$ or that $\rho = \frac{10^n-1}{3.10^n}$ for a certain n. But in order to do that we would have to decide whether a sequence $0,1,\ldots,9$ occurs in the development of π or not. This we are unable to do at (the) present. This is an example of the circumstance that the two statements

$$(Ex)p(x) \;, \qquad\qquad \neg(x)\neg p(x) \;,$$

which are equivalent in classical logic, are not generally equivalent in intuitionist logic.

Let a real number generator (abbreviated an rng) be any sequence of rational numbers a_n such that for every positive integer k we can find another positive integer n such that

$$|a_{n+p} - a_n| < \frac{1}{k}$$

for all p. We put

$$a = b$$

when for every k we can find n such that

$$|a_{n+p} - b_{n+p}| < \frac{1}{k}$$

for all p. Further, $a \neq b$ may mean

$$\neg(a = b),$$

that is, the assumption $a = b$ leads to a contradiction. On the other hand $a \# b$ shall mean that we know a k and an n such that, for all p,

$$|a_{n+p} - b_{n+p}| > \frac{1}{k} \,,$$

while $a < b$ shall mean that we know a k and an n such that

$$b_{n+p} - a_{n+p} > \frac{1}{k}$$

for all p.

It is then possible to prove a lot of theorems about these rng. A real number is the set of all rng which are = a certain rng. The intuitionist notion set will soon be explained below. I shall mention a few of the most important theorems about the rng. One proves that $a = b$ is equivalent $\neg\neg(a = b)$, or, in other words, if the assumption $a \neq b$ leads to a contradiction, then $a = b$. Further, if $a \# b$, then for every c we have $a \# c . \vee b \# c$. It is clear that $a \# b \rightarrow a \neq b$. Further, $a \# b$ is equivalent to $a < b . \vee . b < a$. Instead of $\neg(a < b)$ one writes $a \not< b$. Then we have that $a \not< b \,\&\, b > c \rightarrow a > c$.

Addition, subtraction, multiplication of the rng's a and b is defined by taking the rng with the general term

$$a_n + b_n, \quad a_n - b_n, \quad a_n b_n ,$$

whereas the quotient $\frac{a}{b}$ is defined as $a \cdot \frac{1}{b}$ under the assumption $b \# 0$, where $\frac{1}{b}$ is the rng c whose general term is $c_n = \frac{1}{b_n}$ whenever $b_n \neq 0$ and $c_n = 0$, if $b_n = 0$. It is then trivial to prove the associative, commutative and distributive laws. It may be noticed that $a + b \# 0 \rightarrow a \# 0 \cdot \vee \cdot b \# 0$. For a more thorough study of this subject I recommend Heyting's book.

As an introduction to the intuitionist set theory it is convenient to define the notion ips, that is, infinitely proceeding sequence of natural numbers. We are dealing with an ips, if we first choose a natural number a_1 and, for every n, as soon as $a_1, \ldots, a_n$ have been chosen, we choose a_{n+1}. What determines these choices, whether they obey a certain law or are made at random or more or less arbitrarily, is irrelevant. We are justified in saying that an ips is something that becomes, not that is. If we let a mathematical entity correspond to every finite initial sequence $a_1, \ldots, a_n$ of an ips, we obtain an infinitely proceeding sequence of such entities.

A set can be built in two ways: 1) There may be a common way of generating its elements, 2) one considers all elements having a common property. The sets which are obtained in the first manner are called spreads. The sets obtained according to the second point of view are called species.

The definition of a spread is as follows: One has two rules, a spread rule and a complementary rule. The spread rule Λ determines a process for the generation of ips in the following way. 1) Λ determines for every natural number k, whether it is allowed to be the first element of an ips or not. 2) Every allowed sequence $a_1, \ldots, a_{n+1}$ shall be generated from an earlier allowed sequence $a_1, \ldots, a_n$. 3) Whenever an allowed sequence $a_1, \ldots, a_n$ is given, the rule Λ determines, for any natural number k, whether $a_1, \ldots, a_n, k$ is an allowed sequence or not. 4) To every admitted sequence $a_1, \ldots, a_n$ at least one natural number k can be found such that $a_1, \ldots, a_n, k$ is an admitted sequence.

The complementary rule Γ determines for every allowed sequence $a_1, \ldots, a_n$ a corresponding mathematical object b_n.

Some elucidating examples, taken from Heyting's book, may be suitably mentioned here.

1) Let $r_1, r_2, \ldots$ be an enumeration of all rational numbers. We build a spread M by letting the rule Λ_M be this: Every natural number is admitted as a_1. Whenever $a_1, \ldots, a_n$ is an admitted sequence, $a_1, \ldots, a_n, a_{n+1}$ shall be admitted if and only if

$$|r_{a_n} - r_{a_{n+1}}| < 2^{-n}.$$

The rule Γ_M shall be: To every admitted sequence $a_1,\ldots, a_n$ we let correspond the rational number r_{a_n}.

It is easy to see that the elements of M are rng, and indeed, if c is an arbitrary rng, we can find an element m of M such that m = c. Thus M is simply the continuum consisting of all rng.

2) If the rule Λ_M in example 1 is restricted by adding the requirement $0 < r_{a_n} < 1$ for every n, then M is the spread consisting of all rng x such that

$$0 \not> x \not> 1.$$

3) If the rule Λ_M in example 2 is further restricted by the requirement that for each $n > 1$ we shall have

$$\left|\frac{1}{2} - r_{a_n}\right| \mathrel{\overset{=}{<}} \left|\frac{1}{2} - r_{a_1}\right| ,$$

then M will consist of all rng y such that $0 < y < 1$.

It is evident that by changing the rules Λ_M and Γ_M one can obtain the most varied spreads of rng.

A simple example of a species is the notion real number. A real number is the species whose elements are all rng equal to a given one. A general remark is that the definition of an element of a species must always precede the definition of the species in order to avoid circular definitions.

Also in the intuitionist theory we have the operations of union and intersection of two species. If ϵ as usual means the membership relation we have the definitions

$$S \subseteq T \text{ stands for } (x)\,(x \epsilon S \rightarrow x \epsilon T)$$

$$S = T \text{ means } (S \subseteq T)\ \&\ (T \subseteq S).$$

Further we have for arbitrary x the equivalences

$$(x \epsilon S \cap T) \leftrightarrow (x \epsilon S)\ \&\ (x \epsilon T), \qquad (x \epsilon S \cup T) \leftrightarrow (x \epsilon S) \vee (x \epsilon T).$$

Letting $\notin$ mean the negation of ϵ in the intuitionist sence, we have the following definition of the difference species S - T:

$$(x \epsilon S - T) \leftrightarrow (x \epsilon S)\ \&\ (x \notin T).$$

It must then be noticed that we don't always have $S = T \cup (S - T)$. That is only the case if $T \subseteq S$ and we are able, for every $x \epsilon S$, to prove either $x \epsilon T$ or $x \notin T$. A subspecies T of S is called detachable when we possess such a decision method to decide for any $x \epsilon S$ whether it is ϵT or not.

A characteristic notion is "S is congruent to T". That means

$$\neg(Ex)(x \epsilon S\ \&\ x \notin T \cdot \vee \cdot x \notin S\ \&\ x \epsilon T),$$

which can also be written

$$\neg(Ex)(x \epsilon S\ \&\ x \notin T) \cdot \& \cdot \neg(Ex)(x \notin S\ \&\ x \epsilon T).$$

As an example of the use of this notion I shall mention the theorem:

Let $T \subseteq S$ and $S' = T \cup (S - T)$. Then S and S' are congruent.

Proof. First we have $S' \subseteq S$ because $T \subseteq S$ and $S - T \subseteq S$. Hence $\neg(Ex)(x \in S \ \& \ x \notin S')$. Therefore it remains only to prove that $\neg(Ex)(x \notin S \ \& \ x \in S')$. But this is equivalent to $\neg(Ex)(x \notin S \ \& \ (x \in T \cdot v \cdot x \in S \ \& \ x \notin T))$ which again is equivalent to $\neg(Ex)((x \notin S \ \& \ x \in T) \ v \ (x \notin S \ \& \ x \in S \ \& \ x \in T))$ which is equivalent to $(Ex)(x \notin S \ \& \ x \in T)$ which follows from $(x)(x \in T \rightarrow x \in S)$.

Simple examples of detachable subspecies of the natural number sequence are given by the even or the odd numbers. The linear continuum can be shown to have no other detachable subspecies than itself and the null species.

A species is said to be finite if there is a 1-to-1 correspondence between it and an initial part 1,..., n of the natural number series. It is called denumerable if there is such a correspondence between the species and the whole number series. A species is called numerable if it can be mapped onto a detachable subspecies of the sequence of natural numbers.

An important notion is "finitary spread" or, more briefly, "fan". A fan is a spread with such a spread law that there are only finitely many allowed first terms, and for every n every admitted sequence with n terms has only a finite number of sequences with n + 1 terms as admitted continuations. Above all the so-called fan theorem is important here. It says that if $\phi(\sigma)$ is an integral-valued function of σ, σ varying through the different elements of the fan, then the value of ϕ is already determined by a finite initial sequence of σ. Therefore, if $\phi(\sigma_1) = m$, there exists an n such that $\phi(\sigma_2) = m$ as often as σ_2 has the same first n terms as σ_1. An important application of the fan theorem is the proof of the statement that every function which is continuous on a bounded and closed point species is uniformly continuous on the point species. Further, such covering theorems as that of Heine-Borel can be proved. However, not all of the theorems of classical analysis can be proved in intuitionist mathematics.

I must confine my exposition of intuitionism to these scattered remarks A more thorough exposition would require a more complete treatment of intuitionist logic, and that would take more space than I have at my disposal here.

17. Mathematics without quantifiers

In all the theories we have treated above we have made use of the logical quantifiers, the universal one and the existential one. We have used them without scruples even in the case of an infinite number of objects. There is now a way of developing mathematics, in particular arithmetic, without the use of these operations which, in the case of an infinite number of objects, may be considered as an extension or extrapolation of conjunction and disjunction in the finite case. If we shall really consider the infinite as something becoming, something not finished or finishable, one might argue that we ought to avoid the quantifiers extended over an infinite range. Such a theory is possible. I myself published in 1923 a first beginning of such a strict finitist mathematics. I treated arithmetic, showing that by the use of