Indeed, if we put $f_1 = \phi_1$, $f = \phi_2$ in Theorem 27 we get

$$\alpha\beta + \alpha\gamma = \alpha(\beta + \gamma),$$

and putting $f_1 = \phi_1$, $f_2 = \phi_1$, $f = \phi_2$, $f_3 = \phi_2$, Theorem 26 yields

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

Further, if we put $f_1 = \phi_2$, $f = \phi_3$, Theorem 27 yields

$$\alpha^{\beta} \cdot \alpha^{\gamma} = \alpha^{\beta+\gamma},$$

while putting $f_1 = \phi_2$, $f_2 = \phi_1$, $f = \phi_3$, $f_3 = \phi_2$ one obtains, according to Theorem 26,

$$(\alpha^{\beta})^{\gamma} = \alpha^{\beta\gamma}.$$

7. On the exponentiation of alephs

We have seen that an aleph is unchanged by elevation to a power with finite exponent. I shall add some remarks concerning the case of a transfinite exponent.

Since $2^{\aleph_0} > \aleph_0$, we have $(2^{\aleph_0})^{\aleph_0} \geqq \aleph_0{}^{\aleph_0}$, but $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\aleph_0} = 2^{\aleph_0}$. On the other hand $2^{\aleph_0} \leqq \aleph_0{}^{\aleph_0}$. Hence

$$2^{\aleph_0} = \aleph_0{}^{\aleph_0}.$$

Of course we then have for arbitrary finite n

$$2^{\aleph_0} = n^{\aleph_0} = \aleph_0{}^{\aleph_0},$$

and not only that. Let namely $\aleph_0 < \mathfrak{m} \leqq 2^{\aleph_0}$. Then

$$2^{\aleph_0} = \aleph_0{}^{\aleph_0} \leqq \mathfrak{m}^{\aleph_0} \leqq 2^{\aleph_0},$$

whence

$$\mathfrak{m}^{\aleph_0} = 2^{\aleph_0},$$

In a similar way we obtain for an arbitrary $\aleph_\alpha$

$$2^{\aleph_\alpha} = \mathfrak{m}^{\aleph_\alpha}$$

for all $\mathfrak{m} > 1$ and $\leqq 2^{\aleph_\alpha}$.

From our axioms, in particular the axiom of choice, we have derived that every cardinal is an aleph. Therefore $2^{\aleph_\alpha}$ is an aleph. We can also prove by the axiom of choice that $2^{\aleph_\alpha} > \aleph_{\alpha+1}$ or perhaps $= \aleph_{\alpha+1}$. One has never succeeded in proving one of these two alternatives and according to a result of Gödel such a decision is impossible. However, in many applications of set theory it has been convenient to introduce the so-called generalized continuum hypothesis or aleph hypothesis, namely

$$2^{\aleph_\alpha} = \aleph_{\alpha+1} .$$

In particular the equation $2^{\aleph_0} = \aleph_1$ is called the continuum hypothesis. Of course this assumption means that we introduce a new axiom, namely the following: Let M be a well-ordered set, UM as usual the set of its subsets, and N such a well-ordered set that every initial section of N is $\sim$ M, while N itself is not $\sim$M. Then there exist in our domain D a set ϕ of ordered pairs which yields a one-to-one correspondence between UM and N.

If we have the axiom of choice, we may say more simply that if M is infinite, then every subset of UM is either $\sim$ a subset of M or it is $\sim$ UM.

On the other hand there are a few aleph formulas which can be proved without the (generalized) continuum hypothesis. I shall give some of these.

A theorem of Kőnig says:

Theorem 28. *If γ runs through all ordinals $<\lambda$, where λ is a limit number, then*

$$\sum_{\gamma<\lambda} \aleph_\gamma < \prod_{\gamma<\lambda} \aleph_\gamma .$$

This follows from the general inequality theorem of Zermelo proved earlier. By the way, we have $\sum_{\gamma<\lambda} \aleph_\gamma = \aleph_\lambda$ of course. As a particular case we have $\aleph_\omega < \aleph_0\aleph_1\aleph_2 \ldots$. Since $\aleph_0\aleph_1\aleph_2 \ldots$ is $\leqq \aleph_\omega^{\aleph_0}$, we obtain the inequality

$$\aleph_\omega^{\aleph_0} > \aleph_\omega .$$

Similarly $\aleph_{\omega_1}^{\aleph_1}$ is $> \aleph_{\omega_1}$, etc.

An equation of Hausdorff is

Theorem 29. $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$,

where α and β are arbitrary ordinals.

Proof. 1) Let $\alpha < \beta$ so that $\alpha + 1 \leqq \beta$. Then, since $\aleph_{\alpha+1} \leqq \aleph_\beta < 2^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta}$,

$$\aleph_\alpha^{\aleph_\beta} = \aleph_{\alpha+1}^{\aleph_\beta} = 2^{\aleph_\beta} .$$

2) Let $\alpha \geqq \beta$. Then we can write

$$\aleph_{\alpha+1}^{\aleph_\beta} = \sum_{\mu<\omega_{\alpha+1}} \bar{\mu}^{\aleph_\beta} \leqq \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1} \leqq \aleph_{\alpha+1}^{\aleph_\beta} \aleph_{\alpha+1} = \aleph_{\alpha+1}^{\aleph_\beta} ,$$

whence the asserted equation.

A theorem of Tarski is:

Theorem 30. *If $\bar{\gamma} \leqq \aleph_\beta$, then* $\aleph_{\alpha+\gamma}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+\gamma}^{\bar{\gamma}}$.

The proof can be given by transfinite induction with respect to γ. The theorem is true for $\gamma = 0$. Let us assume its truth for γ. Then by Theorem 29

$$\aleph_{\alpha+\gamma+1}^{\aleph_\beta} = \aleph_{\alpha+\gamma}^{\aleph_\beta}\cdot\aleph_{\alpha+\gamma+1} = \aleph_{\alpha}^{\aleph_\beta}\aleph_{\alpha+\gamma}^{\overline{\gamma}}\aleph_{\alpha+\gamma+1} = \aleph_{\alpha}^{\aleph_\beta}\aleph_{\alpha+\gamma}^{\overline{\gamma+1}}\aleph_{\alpha+\gamma+1} = \aleph_{\alpha}^{\aleph_\beta}\aleph_{\alpha+\gamma+1}^{\overline{\gamma+1}}\,.$$

Now let λ be a limit number such that $\overline{\lambda} \leqq \aleph_\beta$, while the theorem is assumed valid for all $\gamma < \lambda$. Then

$$\aleph_{\alpha+\lambda} = \sum_{\gamma<\lambda}\aleph_{\alpha+\gamma} < \prod_{\gamma<\lambda}\aleph_{\alpha+\gamma}$$

according to the theorem of König. Hence

$$\aleph_{\alpha+\lambda}^{\aleph_\beta} \leqq \left(\prod_{\gamma<\lambda}\aleph_{\alpha+\gamma}\right)^{\aleph_\beta} = \prod_{\gamma<\lambda}\aleph_{\alpha+\gamma}^{\aleph_\beta} = \prod_{\gamma<\lambda}\aleph_{\alpha}^{\aleph_\beta}\aleph_{\alpha+\gamma}^{\overline{\gamma}} = \left(\aleph_{\alpha}^{\aleph_\beta}\right)^{\overline{\lambda}}\prod_{\gamma<\lambda}\aleph_{\alpha+\gamma}^{\overline{\gamma}}$$

$$\leqq \aleph_{\alpha}^{\aleph_\beta\overline{\lambda}}\cdot\aleph_{\alpha+\lambda}^{\overline{\lambda}\overline{\lambda}} = \aleph_{\alpha}^{\aleph_\beta}\cdot\aleph_{\alpha+\lambda}^{\overline{\lambda}}$$

while on the other hand

$$\aleph_{\alpha}^{\aleph_\beta}\aleph_{\alpha+\lambda}^{\overline{\lambda}} \leqq \aleph_{\alpha+\lambda}^{\aleph_\beta}\aleph_{\alpha+\lambda}^{\aleph_\beta} = \aleph_{\alpha+\lambda}^{\aleph_\beta}$$

Therefore the theorem is valid for λ and is proved.

I shall further mention without proof the following two theorems:

1) In order that $2^{\aleph_\alpha} = \aleph_\beta$ it is necessary and sufficient that β is the least ordinal number ξ such that $\aleph_{\xi}^{\aleph_\alpha} < \aleph_{\xi+1}^{\aleph_\alpha}$.

2) We have $2^{\aleph_\alpha} = \aleph_\beta$ if and only if β is the least ordinal number ξ such that $\aleph_{\xi}^{\aleph_\alpha} = \aleph_\xi$.

A further question concerning the cardinal numbers is whether the so-called inaccessible cardinals exist. An aleph $\aleph_\Omega$ would be called inaccessible if $\omega_\Omega = \Omega$, or if one prefers, $\overline{\Omega} = \aleph_\Omega$. This question may again be undecidable so that the introduction of further axioms might be desirable. However, I will not pursue this subject further here.