INTRODUCTION

The statements of analysis can be grouped into three classes according to the depth to which the limit concept is used in their formulation and proof.

A first class consists of theorems which are entirely independent of the concept of limit, and deal with approximations. To this group belong graphical and numerical differentiation and integration as well as statements concerning the reciprocity of these two approximative operations.

A second class consists of formulae in whose proofs the concept of limit is used in a mild, so to speak, algebraic, way. This group comprises the bulk of formulae of calculus concerning elementary functions and some formulae concerning all differentiable functions: the rules for the formation of the derivatives of elementary functions, the determination of antiderivatives by substitution and by parts, etc. (Not included in this group is the theorem that each two antiderivatives of the same function differ at most by a constant).

A third group of statements is based on the assumption that in each closed interval each continuous function assumes its maximum. To this group belong the mean value theorem and its applications, of which I mention the Taylor development and its implications concerning maxima and minima, indeterminate forms, and the theorem about the antiderivatives of the same function.

In this book, we shall develop the second group of statements from a few assumptions concerning three operations (addition, multiplication, substitution) and two operators (derivation and antiderivation). A first part is devoted to the three operations. A second part deals with the Algebra of Derivation and Antiderivation. A third part contains a sketch of the theory of functions of several variables.

In developing the Algebra of Analysis, we shall make use of the notation of the operator theory. Furthermore, we shall completely avoid variables in our formulae. These principles necessitate changes of the current notation most of which result in formal as well as conceptual simplification and systematization.

In initiating students into calculus, at the present time one may find it hard to take full advantage of all these simplifications - not on account of any specific difficulties inherent to the proposed set-up or the proposed notation, but because a student of calculus must be enabled to read books on differential equations, theoretical physics, mathematical economics, etc., all of which at present are written in the classical notation. It will take a long time till these applied fields will be presented in a more modern way. In the meantime, students must acquire not only the knowledge but an operative grasp of the traditional notation with all its shortcomings from which, in fact, some applied fields, as physical chemistry, suffer more than mathematics proper. However, a gradual change of our obsolete notation probably is not only desirable but unavoidable. The first step in this direction is undoubtedly an uncompromising exposition of the new ideas for professional mathematicians and especially teachers of mathematics. It is one of the aims of this book to provide the reader with such an exposition.

Several sections of this publication may be helpful in simplifying the current presentation of calculus even when our treatment is translated into the classical notation. In this connection, we mention the Algebra of Antiderivation, the development of the entire differential calculus from a few formulae, our treatment of the exponential and tangential functions on the basis of the functional equations and the formulae

$$D \exp 0 = 1 \quad \text{and} \quad D \tan 0 = 1,$$

our introduction of the power functions, and the treatment of antiderivation by substitution.

Apart from these pedagogical aspects, our Algebra of Analysis seems to open an extended field of research. Many additional results will be published in more technical papers. Perhaps one will find the general idea of Algebra of Analysis related to that of our Algebra of Geometry whose development the author has outlined in the second lecture in the Rice Institute Pamphlets, Vol. 27, 1940, p. 41-80.