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# An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes

Robert J. Adler
*Technion–Israel Institute of Technology*

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Hayward, California

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ב"ה

To my family,

Nuclear and extended.

Happy is he who has inherited from his forbears,
Happy is he who has a niche in which to live.

(*Jerusalem Talmud*)

אשרי אדם שזכו לו אבותיו,
אשרי אדם שיש לו יתד במי להיתלות בה.

(ירושלמי, ברכות: פ"ד, ה"א)

# PREFACE

... on what these notes *are* meant to be, and what they are *not* meant to be.

They *are* meant to be an introduction to what I call the "modern" theory of sample path properties of Gaussian processes, where by "modern" I mean a theory based on concepts such as entropy and majorising measures. They are directed at an audience that has a reasonable probability background, at the level of any of the standard texts (Billingsley, Breiman, Chung, etc.). It also helps if the reader already knows something about Gaussian processes, since the modern treatment is very general and thus rather abstract, and it is a substantial help to one's understanding to have some concrete examples to hang the theory on. To help the novice get a feel for what we are talking about, Chapter 1 has a goodly collection of examples.

The main point of the modern theory is that the geometric structure of the parameter space of a Gaussian process has very little to do with its basic sample path properties. Thus, rather than having one literature treating Gaussian processes on the real line, another for multiparameter processes, yet another for function indexed processes, etc., there should be a way of treating all these processes at once. That this is in fact the case was noted by Dudley in the late sixties, and his development of the notion of entropy was meant to provide the right tool to handle the general theory.

While the concept of entropy turned out to be very useful, and in the hands of Dudley and Fernique lead to the development of necessary and sufficient conditions for the sample path continuity of stationary Gaussian processes, the general, non-stationary case remained beyond its reach. This case was finally solved when, in 1987, Talagrand showed how to use the notion of majorising measures to fully characterise the continuity problem for general Gaussian processes.

All of this would have been a topic of interest only for specialists, had it not been for the fact that on his way to solving the continuity problem Talagrand also showed us how to use many of our old tools in more efficient ways than we had been doing in the past.

It was in response to my desire to understand Talagrand's message clearly that these notes started to take form. The necessary prompting came from Holger Rootzén and Georg Lindgren, who asked me to deliver, in (a very cold) February 1988 in Lund, a short series of lectures on Gaussian processes. On the basis that the best way to understand something is to try to explain it to someone else, I decided to lecture on Talagrand's results, and, after further prompting, wrote the notes up. They have grown considerably over the past two years, one revision being finished in (an even colder) February in Ottawa, under the gracious hospitality of Don Dawson and Co., and the last major revision during a much more pleasant summer in Israel.

I rather hope that what is now before you will provide not only a generally accessible introduction to majorising measures and their ilk, but also to the general theory of continuity, boundedness, and suprema distributions for Gaussian processes.

Nevertheless, what these notes are *not* meant to provide is an encyclopædic and overly scholarly treatment of Gaussian sample path properties. I have chosen material on the basis of what interests me, in the hope that this will make it easier to pass on my interest to the reader. The choice of subject matter and of type of proof is therefore highly subjective. If what you want is an encyclopædic treatment, then you don't need notes with the title *An Introduction to* .... What you need, unfortunately, is to turn to the journal literature. The key papers, in terms of the breadth of their coverage, are those of Dudley (1973), Fernique (1975), Jain and Marcus (1978), Talagrand (1987, 1988a), Samorodnitsky (1987a,b, 1990), and Adler and Brown (1986), with the brand new monograph of Pisier (1989) being the best source for the Gaussian process/Banach space interface. Piterbarg (1988), which I received only recently, has a wealth of information related to the extremal results of Chapter 5. A large part of the material in my notes comes from these sources, so you now know where to turn to for further detail. What is new in these notes is not so much the results, as the organisation into what I hope is a readable and logical whole. Very often this has lead to much easier and shorter proofs than those in the journal literature.

I am grateful to Rootzén and Lindgren for convincing me to put the effort into giving the lectures and preparing the original set of notes. It has been an educational experience for me at least. I hope that others will also find this material interesting and entertaining. Ron Pyke suggested publishing them in the IMS Lecture Notes-Monograh Series, and, as he was a decade ago during the writing of *The Geometry of Random Fields*, has been a quiet pillar of support and constant source of encouragement throughout. Many readers – too many to list, even if I could remember them all – of earlier versions have made comments and/or corrections that have affected the version you are now reading. Special mention, however, must go to two of these. Patrik Albin, initially during the birth of these notes in Lund, later during a phrenetic two month visit to the Technion, and finally at the other end of what must have been the all time hottest e-mail connection between Israel and Sweden, not only found a seemingly uncountable number of mistakes, he also corrected every correctable one he found. In particular, the proof in Chapter 4 of the lower bound for the majorising measure characterisation of sample path boundedness is his streamlined version of the original Talagrand argument, and many of the other proofs of that chapter also contain heavy contributions on his part. Raya Epstein Feldman led a group of fellow sufferers at Berkeley through Version 2 of the notes. Hopefully their pains, along with those of my colleagues at the Technion Probability seminar

who partook of a similar experience, have lead to a better constructed final product. All the above have my heartfelt thanks.

Of course, there were other kinds of support as well, and my thanks go to the U.S. Air Force Office of Scientific Research (AFOSR 87-0298, 89-0261), the U.S.-Israel Binational Science Foundation (BSF 86-00285) and the Swedish Natural Sciences Research Council.

Finally, a technical note. Equations are numbered sequentially, beginning anew in each chapter. Theorems, lemmata, examples, etc. all share a separate, but common numbering system. The symbol ■ not only denotes the end of a proof, but has occasionally been used to denote the end of a discussion. The Exercises have a strange but self explanatory numbering system of their own that arose for historical reasons and that I was too lazy to change.

May, 1990
Haifa, Israel

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