18.2. SUBSTITUTION THEOREM. If G and H are recursive, there is a recursive F such that

$$(4) \qquad F(\vec{\alpha},\vec{x}) \simeq G(\lambda zH(z,\vec{\alpha},\vec{x}), \vec{\alpha}, \vec{x})$$

for all $\vec{\alpha},\vec{x}$ such that $\lambda zH(z,\vec{\alpha},\vec{x})$ is a real. In particular, if H is total, then the F defined by (4) is recursive.

Proof. Let g be an index of G. If $\lambda zH(z,\vec{\alpha},\vec{x})$ is a real, then the right side of (4) is, by (1),

$$U(\mu yT_{m,k}(g,\vec{x},y,\overline{H}(y,\vec{\alpha},\vec{x}),\overline{\vec{\alpha}}(y))).$$

We can use this as our definition of $F(\vec{\alpha},\vec{x})$. □

In particular, it follows that λy is a recursive expression when it is used in front of an expression defined for all values of y.

REMARK. If H is not total, there may be $\vec{\alpha},\vec{x}$ such that $F(\vec{\alpha},\vec{x})$ is defined, but such that $\lambda zH(y,\vec{\alpha},\vec{x})$ is not a real and hence such that the right side of (4) is not defined.

The results of §13 and §14 extend without difficulty. However, in §13 it is natural to consider a further extension in which we allow quantifiers on real variables. We investigate this in the next section.

19. The Analytical Hierarchy

A relation is <u>analytical</u> if it has an explicit definition with a prefix consisting of quantifiers, which may be either universal or existential and may be on either number variables or real variables, and a recursive matrix. The basic theory of analytical relations is due to Kleene.

We begin with some rules for simplifying prefixes. As before, these may change the matrix, but they leave it recursive.

Two quantifiers are of the <u>same kind</u> if they are both universal or both existential; they are of the <u>same type</u> if they are both on real variables or both on number variables.

I. Two adjacent quantifiers of the same kind and same type can be replaced by one quantifier of that kind and type.

For number quantifiers, this is just contraction of quantifiers as in §13. In order to treat real quantifiers, we need an analogue of $(x)_i$ for reals. We define $(\alpha)_x$ to be $\lambda y\ \alpha(<x,y>)$. Here we have a better result than for numbers: given an infinite sequence $\alpha_0,\alpha_1,\ldots$ of reals, there is an α such that $(\alpha)_i = \alpha_i$ for all i. In fact, we can define α by $\alpha(x) = \alpha_{(x)_0}((x)_1)$. Moreover, $(\alpha)_x$ is a recursive expression when used in contexts of the form $(\alpha)_x(\_\_\_\_)$; for we may replace this context by $\alpha(<x,\_\_\_\_>)$. We can then justify contraction of quantifiers for real variables just as we did for number variables in §13.

II. A number quantifier can be replaced by a real quantifier of the same kind.

This follows from the equivalences

$$\forall xP(x) \leftrightarrow \forall\alpha P(\alpha(0)),$$

$$\exists xP(x) \leftrightarrow \exists\alpha P(\alpha(0)).$$

III. If a number quantifier is immediately followed by a real quantifier, the real quantifier may be moved to the front of the number quantifier.

This follows from the equivalences

$$\forall x\forall\alpha P(\alpha,x) \leftrightarrow \forall\alpha\forall xP(\alpha,x),$$

$$\exists x\exists\alpha P(\alpha,x) \leftrightarrow \exists\alpha\exists xP(\alpha,x),$$

$$\forall x\exists\alpha P(\alpha,x) \leftrightarrow \exists\alpha\forall xP((\alpha)_x,x),$$

$$\exists x\forall\alpha P(\alpha,x) \leftrightarrow \forall\alpha\exists xP((\alpha)_x,x).$$

The first two of these are obvious. In the third, both sides say that there is an infinite sequence α_0, α_1, ... such that $P(\alpha_x,x)$ for all x. If we put $\neg P$ for P in the third, bring the quantifiers to the front by prenex rules, and drop the $\neg$ from both sides, we get the fourth.

We say that a prefix is $\underline{\Pi}^1_n$ ($\underline{\Sigma}^1_n$) if all of the real quantifiers precede all of the number quantifiers; there are exactly n real quantifiers; the real quantifiers alternate in kind; and the prefix begins with $\forall$ ($\exists$). A relation is $\underline{\Pi}^1_n$ if it has a definition with a Π^1_n prefix and a recursive matrix; similarly for Σ^1_n. A relation is $\underline{\Delta}^1_n$ if it is both Π^1_n and Σ^1_n.

19.1. PROPOSITION. Every analytical relation is either Π^1_n or Σ^1_n for some n.

Proof. By first using III, then applying contraction of quantifiers to the real variables. □

A Π^1_n or Σ^1_n prefix may be further simplified as follows. If there are any number quantifiers of the same kind as the last real quantifier, we change then to real quantifiers by II; move then to just after the last real quantifier by III; and then contract them with the last real quantifier. The remaining number quantifiers are of the opposite kind to the last real quantifier, and may be contracted to one number quantifier of this kind. If there are no number quantifiers left, we can add a superfluous one of the opposite kind to the last real quantifier. In summary: we may suppose that there is exactly one number quantifier, which is if the opposite kind to the last real quantifier.

We can still say some more. Consider, for example, a Π^1_1 relation P. By the above, we have $P(\vec{\alpha},\vec{x}) \leftrightarrow \forall\beta Q(\vec{\alpha},\vec{x},\beta)$ where Q is Σ^0_1. Then, using (3) of §14, $P(\vec{\alpha},\vec{x}) \leftrightarrow \forall\beta\exists y R(\overline{\vec{\alpha}}(y),\vec{x},\overline{\beta}(y),y)$ with R recursive. We can even omit the last y, since it may be replaced by $\mathrm{lh}(\overline{\beta}(y))$. Thus any Π^1_1 relation of $\vec{\alpha},\vec{x}$ can be written $\forall\beta\exists y R(\overline{\vec{\alpha}}(y),\vec{x},\overline{\beta}(y))$ with R recursive. Taking negations, any Σ^1_1 relation of $\vec{\alpha},\vec{x}$ can be written $\exists\beta\forall y R(\overline{\vec{\alpha}}(y),\vec{x},\overline{\beta}(y))$ with R recursive. Similar results hold for Π^1_k and Σ^1_k relations.

19.2. PROPOSITION. If R is Π^1_n or Σ^1_n, then R is Δ^1_k for all $k > n$. If R is arithmetical, it is Δ^1_n for all n.

Proof. By adding superfluous quantifiers. □

We shall now say that P is <u>reducible</u> to Q if

$$P(\vec{\alpha},\vec{x}) \leftrightarrow Q(\lambda y G_1(y,\vec{\alpha},\vec{x}),\dots,\lambda y G_m(y,\vec{\alpha},\vec{x}),F_1(\vec{\alpha},\vec{x}),\dots,F_k(\vec{\alpha},\vec{x}))$$

where $G_1,\dots,G_m,F_1,\dots,F_k$ are total and recursive.

19.3. Proposition. If P is Π^1_n and Q is reducible to P, then P is Π^1_n; and similarly with Σ^1_n or Δ^1_n in place of Π^1_n. □

The analogue of the table in §12 is the following table.

P,Q	$\neg P$	$P \vee Q$	$P \,\&\, Q$	$\forall\alpha P$	$\exists\alpha P$	QxP
Π^1_n	Σ^1_n	Π^1_n	Π^1_n	Π^1_n	Σ^1_{n+1}	Π^1_n
Σ^1_n	Π^1_n	Σ^1_n	Σ^1_n	Π^1_{n+1}	Σ^1_n	Σ^1_n
Δ^1_n	Δ^1_n	Δ^1_n	Δ^1_n	Π^1_n	Σ^1_n	Δ^1_n

It is proved and used in the same way as the earlier table.

The classification of analytical relations into the Π^1_n and Σ^1_n relations is called the <u>analytical hierarchy</u>.

19.4. Analytical Enumeration Theorem. For every n, m, and k, there is a Π^1_n $(m,k+1)$–ary function which enumerates the class of Π^1_n (m,k)–ary relations; and similarly with Σ^1_n for Π^1_n.

Proof. Suppose, for example,we want to enumerate the Π^1_2 $(1,1)$–ary relations. Every such relation R is of the form $\forall\alpha\exists\beta P$ where P is Π^0_1 by the remarks after 19.1. Thus if Q is Π^0_1 and enumerates the Π^0_1 $(3,1)$–ary relations, then $\forall\alpha\exists\beta Q(\alpha,\beta,\gamma,x,e)$ is the desired enumerating function. □

19.5. Analytical Hierarchy Theorem. For each n, there is a Π^1_n set which is not Σ^1_n, hence not Π^1_k or Σ^1_k for any $k < n$. The same holds with Π^1_n and Σ^1_n interchanged.

Proof. As in the arithmetical case. □

20. The Projective Hierarchy

The results of the last section can be relativized to a class Φ of total functions of number variables. A particularly interesting case is that in which Φ