RE in $\mathbf{a}$ such that neither $\mathbf{b} \leq \mathbf{c}$ nor $\mathbf{c} \leq \mathbf{b}$. The method used to define $\mathbf{0}'$, when relativized, shows that there is a largest degree RE in $\mathbf{a}$. It is called the jump of $\mathbf{a}$, and is designated by $\mathbf{a}'$. The relativized Limit Lemma shows that a real is a limit of a recursive in $\mathbf{a}$ sequence of reals iff it has degree $\leq \mathbf{a}'$.

16. Evaluation of Degrees

We shall now show how to evaluate the degrees of certain explicitly given relations.

Let Φ be a class of relations. We say a relation R is Φ complete if R is in Φ and every relation in Φ is reducible to R (where reducible is defined before 13.3). It follows that R has the largest degree of any relation in Φ; so any two Φ complete relations have the same degree. (Caution: Some authors use complete in a somewhat different way.)

EXAMPLE. If F is total, $W_e^F(x)$ is RE in F complete; its degree is the jump of dg F. Hence any RE in F complete relation has degree (dg F)$'$.

The degree obtained by applying the jump n times to $\mathbf{0}$ is designated by $\mathbf{0}^n$.

16.1. PROPOSITION. For every n, there is a Σ_n^0 complete set of degree $\mathbf{0}^n$ and a Π_n^0 complete set of degree $\mathbf{0}^n$.

Proof. We use induction on n. If $n = 1$, let P be a recursive set; if $n > 1$, let P be a Π_{n-1}^0 complete set of degree $\mathbf{0}^{n-1}$. Then $W_e^P(x)$ has degree $\mathbf{0}^n$ by the example. By Post's Theorem, Σ_n^0 is the class of relation RE in P; so $W_e^P(x)$ is Σ_n^0 complete. Then $\neg W_e^P(x)$ is of degree $\mathbf{0}^n$ and is Π_n^0 complete. □

16.2. COROLLARY. Every Σ_n^0 complete or Π_n^0 complete relation has degree $\mathbf{0}^n$. □

If Φ is a class of RE sets, then the set of indices of sets in Φ is called the index set of Φ.

16.3. PROPOSITION (RICE). If Φ is a non-empty class of RE sets which is not the class of all RE sets, then the index set of Φ is not recursive.

Proof. We may suppose the empty set ϕ is not in Φ; otherwise we replace Φ be the class of RE sets which are not in Φ. Let A be an RE set which is in Φ, and let B be a non–recursive RE set. By the Parameter Theorem, there is a recursive real S such that

$$W_{S(e)}(x) \leftrightarrow x \in A \ \& \ e \in B.$$

If $e \in B$, then $W_{S(e)} = A$, so $S(e)$ is in the index set of Φ; if $e \notin B$, $W_{S(e)} = \phi$, so $S(e)$ is not in the index set of Φ. Thus B is reducible to the index set of Φ; so this index set is not recursive. □

We are going to use 16.2 to evaluate the degrees of certain index sets.

Let TOT be the index set of the class of RE sets whose only member is ω. Then

$$e \in \text{TOT} \leftrightarrow \forall x W_e(x).$$

Since $W_e(x)$ is an RE relation of e and x, TOT is Π^0_2. We shall show that it is Π^0_2 complete and hence of degree $\mathbf{0}''$. Every Π^0_2 relation is reducible to its contraction, which is also Π^0_2; so it will suffice to show that every Π^0_2 set A is reducible to TOT. We have $A(x) \leftrightarrow \forall y P(x,y)$ where P is RE. By the RE Parameter Theorem, there is a recursive total S such that $W_{S(x)}(y) \leftrightarrow P(x,y)$. Hence

$$A(x) \leftrightarrow \forall y \ W_{S(x)}(y) \leftrightarrow S(x) \in \text{TOT}.$$

Thus A is reducible to TOT.

Let INF be the index set of the class of infinite RE sets. Then

$$e \in \text{INF} \leftrightarrow \forall x \exists y (y > x \ \& \ W_e(y)).$$

Hence INF is Π^0_2. We shall show that it is Π^0_2 complete. Let A be a Π^0_2 set. Then, writing $\mathrm{I}z$ for *for infinitely many z*,

$$A(x) \leftrightarrow \forall y P(x,y) \leftrightarrow \mathrm{I}z (\forall y < z) P(x,y)$$

where P is RE. By the table, $(\forall y < z)P(x,y)$ is an RE relation of z and x. Hence by the RE Parameter Theorem, there is a recursive total S such that

$W_{S(x)}(z) \leftrightarrow (\forall y < z)P(x,y)$. Then

$$A(x) \leftrightarrow \text{I}z\, W_{S(x)}(z) \leftrightarrow S(x) \in \text{INF}.$$

We say that A is reducible to B,C if there is a recursive real F such that for all x, $x \in A \to F(x) \in B$ and $x \notin A \to F(x) \notin C$. Then A is reducible to every set D such that $B \subseteq D \subseteq C$.

Let COF be the index set of the class of cofinite sets. (A set is cofinite if its complement is finite.) Since

$$e \in \text{COF} \leftrightarrow \exists x \forall y(\neg W_e(y) \to y \leq x),$$

COF is Σ^0_3.

Let REC be the index set of the class of recursive sets. By 14.6,

$$e \in \text{REC} \leftrightarrow \exists f(W_f = W_e{}^{\text{c}})$$

(where a superscript c indicates a complement). Now

$$W_f = W_e{}^{\text{c}} \leftrightarrow \forall x(W_f(x) \vee W_e(x)) \;\&\; \neg\exists x(W_f(x) \;\&\; W_e(x)).$$

The right side is Π^0_2; so REC is Σ^0_3.

Since COF $\subseteq$ REC, the following result shows that both COF and REC are Σ^0_3 complete.

16.4. PROPOSITION (ROGERS). Every Σ^0_3 set is reducible to COF,REC.

Proof. Let A be Σ^0_3. For each z, we give an RE construction of a set B_z so that B_z is cofinite if $z \in A$ and B_z is non–recursive if $z \notin A$. Moreover, we will insure that x is put into B_z at step s is a recursive relation of x,s, and z. Since

$$x \in B_z \leftrightarrow \exists s(x \text{ is put into } B_z \text{ at step } s),$$

it follows from the Parameter Theorem that there is a recursive real S such that $W_{S(z)} = B_z$ for all z. The proposition will clearly follow.

Since $z \in A \leftrightarrow \exists y P(z,y)$ where P is Π^0_2 and hence reducible to INF, there is a total recursive function F such that

$$z \in A \leftrightarrow \exists y(W_{F(z,y)} \text{ is infinite}).$$

Using 14.7, choose a one–one recursive real G such that the range of G is not recursive.

Now we describe step s in the construction of B_z. Let B_z^s be the finite set of numbers put into B_z before step s, and let x_0^s, x_1^s, ... be the members of the complement of B_z^s in increasing order. We put $x_{G(s)}^s$ into B_z. For each $y < s$ such that $W_{F(z,y),s+1}$ contains a number not in $W_{F(z,y),s}$, we put x_y^s into B_z.

Suppose that $z \in A$. Choose y so that $W_{F(z,y)}$ is infinite. Since each $W_{F(z,y),s}$ is finite, there are infinitely many s at which x_y^s is put into B_z. It follows easily that the complement of B_z has at most y members; so B_z is cofinite.

Now suppose that $z \notin A$, so that each $W_{F(z,y)}$ is finite. Since G is one–one, we see that for each y, there are only finitely many steps s at which x_y^s is put into B_z. It follows that for each y, there is an x_y such that $x_y^s = x_y$ for all sufficiently large s. Hence x_0, x_1, ... are the members of the complements of B_z in increasing order. We show that B_z is not recursive by showing that the range of G is recursive in B_z. Let w be given; it is sufficient to find, using an oracle for B_z, a bound on the numbers s for which $G(s) = w$. Now if $G(s) = w$, then x_w^s is put into B_z at step w; so $x_w^s \neq x_w$. Since x_w^s is increasing in s, it suffices to find a stage s at which $x_w^s = x_w$. We can do this with an oracle for B_z, since the oracle enables us to compute x_w. □

The <u>index set</u> of a degree **a** is defined to be the index set of the class of RE sets having degree **a**. Thus REC is the index set of **0**. The result we have just proved is a special case of the following theorem of Yates: if **a** is RE, then the index set of **a** is Σ_3^0 in **a** complete. We shall not prove this result, which requires extensive use of the priority method.