ON $\mathrm{CH} + 2^{\aleph_1} \rightarrow (\alpha)^2_2$ FOR $\alpha < \omega_2$

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§1. Introduction.

We prove the consistency of

$$\mathrm{CH} + 2^{\aleph_1} \text{ is arbitrarily large} + 2^{\aleph_1} \not\rightarrow (\omega_1 \times \omega)^2_2$$

(Theorem 1). In fact, we can get $2^{\aleph_1} \not\rightarrow [\omega_1 \times \omega]^2_{\aleph_0}$, see 1A. In addition to this theorem, we give generalizations to other cardinals (Theorems 2 and 3). The $\omega_1 \times \omega$ is best possible as CH implies

$$\omega_3 \rightarrow (\omega \times n)^2_2.$$

We were motivated by a question of J. Baumgartner, in his talk in the MSRI meeting on set theory, October 1989, on whether $\omega_3 \rightarrow (\alpha)^2_2$ for $\alpha < \omega_2$ (if $2^{\aleph_1} = \aleph_2$, it follows from the Erdős–Rado theorem). Baumgartner proved the consistency of a positive answer with CH and $2^{\aleph_1}$ large. He has also proved [BH] in ZFC + CH a related polarized partition relation:

$$\binom{\aleph_3}{\aleph_2} \rightarrow \binom{\aleph_1}{\aleph_1}^{1,1}_{\aleph_0}$$

Note. The main proof here is that of Theorem 1. In that proof, in the way things are set up, the main point is proving the $\aleph_2$-c.c. The main idea in the proof is using $\mathbf{P}$ (defined in the proof). It turns out that we can use as elements of $\mathcal{P}$ (see the proof) just pairs (a, b). Not much would be changed if we used $\langle (a_n, \alpha_n) : n < \omega \rangle$, a_n a good approximation of the nth part of the suspected monochromatic set of order type $\omega_1 \times \omega$. In 1A, 2, and 3 we deal with generalizations and in Theorem 4 with complementary positive results.

§2. The main result.

THEOREM 1. *Suppose*

(a) CH

(b) $\lambda^{\aleph_1} = \lambda$.

Then there is an $\aleph_2$-c.c., $\aleph_1$-complete forcing notion $\mathbf{P}$ *such that*

(i) $|\mathbf{P}| = \lambda$

(ii) $\Vdash_{\mathbf{P}}$ "$2^{\aleph_1} = \lambda,\ \lambda \not\rightarrow (\omega_1 \times \omega)^2_2$"

(iii) $\Vdash_{\mathbf{P}}$ CH

(iv) Forcing with $\mathbf{P}$ *preserves cofinalities and cardinalities.*

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Proof. By Erdős and Hajnal [EH] there is an algebra $\mathbf{B}$ with $2^{\aleph_0} = \aleph_1$ ω-place functions, closed under composition (for simplicity only), such that

$$\otimes \qquad \text{If } \alpha_n < \lambda \text{ for } n < \omega, \text{ then for some } k$$
$$\alpha_k \in \mathrm{cl}_{\mathbf{B}}\{\,\alpha_l : k < l < \omega\,\}.$$

($\otimes$ implies that for every large enough k, for every m, $\alpha_k \in \mathrm{cl}_{\mathbf{B}}\{\,\alpha_l : m < l < \omega\,\}$.) Let

$$\mathcal{R}_\delta = \{\,b : b \subseteq \lambda,\ \mathrm{otp}(b) = \delta,\ \alpha \in b \Rightarrow b \subseteq \mathrm{cl}_{\mathbf{B}}(b \setminus \alpha)\,\}.$$

So by $\otimes$ we have

$$\oplus \qquad \text{If } \alpha \text{ is a limit ordinal, } b \subseteq \lambda,\ \mathrm{otp}(b) = \alpha,$$
$$\text{then for some } \alpha \in b,\quad b \setminus \alpha \in \textstyle\bigcup_\delta \mathcal{R}_\delta.$$

Let $\mathcal{R}_{<\omega_1} = \bigcup_{\alpha<\omega_1} \mathcal{R}_\alpha$. Let $\mathbf{P}$ be the set of forcing conditions

$$(w, c, \mathcal{P})$$

where w is a countable subset of λ, $c : [w]^2 \to \{\text{red}, \text{green}\} = \{0,1\}$ (but we write $c(\alpha,\beta)$ instead of $c(\{\alpha,\beta\})$), and $\mathcal{P}$ is a countable family of pairs (a,b) such that

(i) a, b are subsets of w
(ii) $b \in \mathcal{R}_{<\omega_1}$ and a is a finite union of members of $\mathcal{R}_{<\omega_1}$
(iii) $\sup(a) < \min(b)$
(iv) if $\sup(a) \le \gamma < \min(b)$, $\gamma \in w$, then $c(\gamma,\cdot)$ divides a or b into two infinite sets.

We use the notation

$$p = (w^p, c^p, \mathcal{P}^p)$$

for $p \in \mathbf{P}$. The ordering of the conditions is defined as follows:

$$p \le q \iff w^p \subseteq w^q \;\&\; c^p \subseteq c^q \;\&\; \mathcal{P}^p \subseteq \mathcal{P}^q.$$

Let

$$\underset{\sim}{c} = \bigcup\{\,c^p : p \in \underset{\sim}{G}_{\mathbf{P}}\,\}.$$

FACT A. *$\mathbf{P}$ is $\aleph_1$-complete.*

Proof. Trivial—take the union. □

FACT B. *For $\gamma < \lambda$, $\{\,q \in \mathbf{P} : \gamma \in w^q\,\}$ is open dense.*

Proof. Let $p \in \mathbf{P}$. If $\gamma \in w^p$, we are done. Otherwise we define q as follows: $w^q = w^p \cup \{\gamma\}$, $\mathcal{P}^q = \mathcal{P}^p$, $c^q \restriction w^p = c^p$ and $c^q(\gamma,\cdot)$ is defined so that if $(a,b) \in \mathcal{P}^q$, then $c^q(\gamma,\cdot)$ divides a and b into two infinite sets. □

FACT C. $\Vdash_{\mathbf{P}}$ "*$2^{\aleph_1} \ge \lambda$ and $\underset{\sim}{c} : [\lambda]^2 \to \{red,\ green\}$.*"

Proof. The second phrase follows from Fact B. For the first phrase, define $\underset{\sim}{\rho}_\alpha \in {}^{\omega_1}2$, for $\alpha < \lambda$, by: $\underset{\sim}{\rho}_\alpha(i) = \underset{\sim}{c}(0, \alpha + i)$. Easily

$$\Vdash_{\mathbf{P}} \text{“}\underset{\sim}{\rho}_\alpha \in {}^{\omega_1}2 \text{ and for } \alpha < \beta < \lambda,\ \underset{\sim}{\rho}_\alpha \neq \underset{\sim}{\rho}_\beta\text{”}$$

so $\Vdash_{\mathbf{P}}$ “$2^{\aleph_1} \geq \lambda$.” □

FACT D. **P** *satisfies the* $\aleph_2$*-c.c.*

Proof. Suppose $p_i \in \mathbf{P}$ for $i < \aleph_2$. For each i choose a countable family $\mathcal{A}^i$ of subsets of w^{p_i} such that $\mathcal{A}^i \subseteq \mathcal{R}_{<\omega_1}$ and $(a, b) \in \mathcal{P}^{p_i}$ implies $b \in \mathcal{A}^i$ and a is a finite union of members of $\mathcal{A}^i$. For each $\gamma \in c \in \mathcal{A}^i$ choose a function $F^i_{\gamma,c}$ (from those in the algebra **B**) such that $F^i_{\gamma,c}(c \setminus (\gamma + 1)) = \gamma$. Let v_i be the closure of w_i (in the order topology).

We may assume that $\langle v_i : i < \omega_2 \rangle$ is a Δ-system (we have CH) and that $\mathrm{otp}(v_i)$ is the same for all $i < \omega_2$. Without loss of generality (w.l.o.g.) for $i < j$ the unique order-preserving function $h_{i,j}$ from v_i onto v_j maps p_i onto p_j, $\mathcal{A}^i$ onto $\mathcal{A}^j$, $w^{p_i} \cap w^{p_j} = w^{p_0} \cap w^{p_1}$ onto itself, and

$$F^i_{\gamma,c} = F^j_{h_{i,j}(\gamma), h_{i,j}\text{“}c}$$

for $\gamma \in c \in \mathcal{A}^i$ (remember: **B** has $2^{\aleph_0} = \aleph_1$ functions only). Hence

$\otimes_1$ $\qquad h_{i,j}$ is the identity on $v_i \cap v_j$ for $i < j$.

Clearly by the definition of $\mathcal{R}_{<\omega_1}$ and the condition on $F^i_{\gamma,c}$:

$\otimes_2$ $\qquad$ If $a \in \mathcal{A}^i$, $i \neq j$ and $a \not\subseteq w^{p_i} \cap w^{p_j}$, then $a \setminus (w^{p_i} \cap w^{p_j})$ is infinite.

We define q as follows.

$w^q = w^{p_0} \cup w^{p_1}$.

$\mathcal{P}^q = \mathcal{P}^{p_0} \cup \mathcal{P}^{p_1}$.

c^q extends c^{p_0} and c^{p_1} in such a way that, for $e \in \{0, 1\}$,

(*) for every $\gamma \in w^{p_e} \setminus w^{p_{1-e}}$ and every $a \in \mathcal{A}^{1-e}$, $c^q(\gamma, \cdot)$ divides a into two infinite parts, provided that

(**) $a \setminus w^{p_e}$ is infinite.

This is easily done and $p_0 \leq q$, $p_1 \leq q$, ***provided that*** $q \in \mathbf{P}$. For this the problematic part is c^q and, in particular, part (iv) of the definition of **P**. So suppose $(a, b) \in \mathcal{P}^q$, e.g., $(a, b) \in \mathcal{P}^{p_0}$. Suppose also $\gamma^* \in w^q$ so that $\sup(a) \leq \gamma^* < \sup(b)$. If $\gamma^* \in w^{p_0}$, there is no problem, as $p_0 \in \mathbf{P}$. So let us assume $\gamma^* \in w^q \setminus w^{p_0} = w^{p_1} \setminus w^{p_0}$. If $a \setminus w^{p_1}$ or $b \setminus w^{p_1}$ is infinite, we are through in view of condition (*) in the definition of c^q. Let us finally assume $a \setminus w^{p_1}$ is finite. But $a \subseteq w^{p_0}$. Hence $a \setminus (w^{p_0} \cap w^{p_1})$ is finite and $\otimes_2$ implies it is empty, i.e., $a \subseteq w^{p_0} \cap w^{p_1}$. Similarly, $b \subseteq w^{p_0} \cap w^{p_1}$. So $h_{0,1} \restriction (a \cup b)$ is the identity. But $(a, b) \in \mathcal{P}^{p_0}$. But $h_{i,j}$ maps p_i onto p_j. Hence $(a, b) \in \mathcal{P}^{p_1}$. As $p_1 \in \mathbf{P}$, we get the desired conclusion. □

FACT E. $\Vdash_{\mathbf{P}}$ *"There is no $\underset{\sim}{c}$-monochromatic subset of λ of order-type $\omega_1 \times \omega$."*

Proof. Let p force the existence of a counterexample. Let G be $\mathbf{P}$-generic over V with $p \in G$. In $V[G]$ we can find $A \subseteq \lambda$ of order-type $\omega_1 \times \omega$ such that $\underset{\sim}{c}^G \restriction [A]^2$ is constant. Let $A = \bigcup_{n<\omega} A_n$ where $\mathrm{otp}(A_n) = \omega_1$ and $\sup(A_n) \le \min(A_{n+1})$. We can replace A_n by any $A'_n \subseteq A_n$ of the same cardinality. Hence we may assume w.l.o.g.

$$(*)_1 \qquad A_n \in \mathcal{R}_{\omega_1} \qquad \text{for } n < \omega.$$

Let $\delta_n = \sup(A_n)$ and

$$\beta_n = \min\{\, \beta : \delta_n \le \beta < \lambda,\ d(\beta, \cdot) \text{ does not divide } \bigcup_{l \le n} A_l \text{ into two infinite sets} \,\},$$

where $d = \underset{\sim}{c}^G$. Clearly $\beta_n \le \min(A_{n+1})$. Hence $\beta_n < \beta_{n+1}$. Let $d_n \in \{0, 1\}$ be such that $d(\beta_n, \gamma) = d_n$ for all but finitely many $\gamma \in \bigcup_{l \le n} A_l$. Let u be an infinite subset of ω such that d_n is constant for $n \in u$ and $\{\beta_n : n \in u\} \in \mathcal{R}_\omega$. Let $A_l = \{\, \alpha^l_i : i < \omega_1 \,\}$ in increasing order. So p forces all this on suitable names

$$\langle \underset{\sim}{\beta}_n : n < \omega \rangle,\ \langle \underset{\sim}{\alpha}^l_i : i < \omega_1 \rangle,\ \langle \underset{\sim}{\delta}_n : n < \omega \rangle.$$

As $\mathbf{P}$ is $\aleph_1$-complete, we can find $p_0 \in \mathbf{P}$ with $p \le p_0$ so that p_0 forces $\underset{\sim}{\beta}_l = \beta_l$ and $\underset{\sim}{\delta}_n = \delta_n$ for some β_l and δ_n. We can choose inductively conditions $p_k \in \mathbf{P}$ such that $p_k \le p_{k+1}$ and there are $i_k < j_k$ and α^l_i (for $i < j_k$) with

$$\begin{aligned} p_{k+1} \Vdash {}``&\alpha^l_{i_k} > \sup(w^{p_k} \cap \delta_l), \\ &\alpha^l_i \in w^{p_{k+1}} \text{ for } i < j_k, \\ &\{\, \alpha^l_i : i < i_k \,\} \subseteq \mathrm{cl}_{\mathbf{B}}\{\, \alpha^l_i : i_k < i < j_k \,\}, \\ &\underset{\sim}{\alpha}^l_i = \alpha^l_i \text{ for } i < j_k, \\ &\underset{\sim}{c}(\beta_n, \alpha^l_i) = d_n \text{ for } l \le n, i > i_0, \text{ and} \\ &\gamma \in [\delta_m, \beta_m) \cap w^{p_k} \text{ implies } \underset{\sim}{c}(\gamma, \cdot) \text{ divides} \\ &\quad \{\, \alpha^l_i : i < j_k,\ l \le m \,\} \text{ into two infinite sets"} \end{aligned}$$

(remember our choice of β_m). Let

$$\begin{aligned} l(*) &= \min(u) \\ a &= \{\, \alpha^l_i : l \le l(*),\ i < \bigcup_k j_k \,\} \\ b &= \{\, \beta_l : l \in u \,\} \\ q &= (\bigcup_k w^{p_k}, \bigcup_k c^{p_k}, \bigcup_k \mathcal{P}^{p_k} \cup \{(a, b)\}). \end{aligned}$$

Now $q \in \mathbf{P}$. To see that q satisfies condition (iv) of the definition of $\mathbf{P}$, let $\sup(a) \le \gamma < \min(b)$. Then $\sup\{\, \alpha^{l(*)}_{i_k} : k < \omega \,\} \le \gamma < \beta_{l(*)}$. But $\gamma \in w^q =$

$\bigcup_k w^{p_k}$, so for some k, $\gamma \in w^{p_k}$. This implies

$$\gamma \notin \left(\alpha^{l(*)}_{i_k+1}, \delta_{l(*)}\right),$$

whence $\gamma \geq \delta_{l(*)}$ and

$$\{\, \alpha^l_i : l \leq l(*),\ i < j_k \,\} \subseteq a,$$

which implies the needed conclusion.

Also $q \geq p_k \geq p$. But now, if $r \geq q$ forces a value to $\alpha^{l(*)}_{\bigcup_k j_k}$; we get a contradiction. □

Remark 1A. Note that the proof of Theorem 1 also gives the consistency of $\lambda \not\to [\omega_1 \times \omega]^2_{\aleph_0}$: replace "$c(\gamma,\cdot)$ divides a set x into two infinite parts" by "$c(\gamma,\cdot)$ gets all values on a set x."

§3. Generalizations to other cardinals.

How much does the proof of Theorem 1 depend on $\aleph_1$? Suppose we replace $\aleph_0$ by μ.

THEOREM 2. *Assume $2^\mu = \mu^+ < \lambda = \lambda^\mu$ and $2 \leq \kappa \leq \mu$. Then for some μ^+-complete μ^{++}-c.c. forcing notion* $\mathbf{P}$ *of cardinality 2^μ:*

$$\Vdash_{\mathbf{P}} 2^\mu = \lambda, \qquad \lambda \not\to [\mu^+ \times \mu]^2_\kappa.$$

Proof. Let $\mathbf{B}$ and $\mathcal{R}_\delta$ be defined as above (for $\delta \leq \mu^+$). Clearly

⊕ If $a \subseteq \lambda$ has no last element, then for some $\alpha \in a$, $a \setminus \alpha \in \bigcup_\delta \mathcal{R}_\delta$.

Hence, if $\delta = \mathrm{otp}(a)$ is additively indecomposable, then $a \setminus \alpha \in \mathcal{R}_\delta$ for some $\alpha \in a$.

Let $\mathbf{P}_\mu$ be the set of forcing conditions

$$(w, c, \mathcal{P})$$

where $w \subseteq \lambda$, $|w| \leq \mu$, $c : [w]^2 \to \kappa$, and $\mathcal{P}$ is a set of $\leq \mu$ pairs (a, b) such that

(i) a, b are subsets of w
(ii) $b \in \mathcal{R}_\mu$, and a is a finite union of members of $\bigcup_{\mu\leq\delta<\mu^+} \mathcal{R}_\delta$
(iii) $\sup(a) < \min(b)$
(iv) if $\sup(a) \leq \gamma < \min(b)$, $\gamma \in w$, then the function $c(\gamma,\cdot)$ gets all values ($< \kappa$) on a or on b.

With the same proof as above we get

$$\mathbf{P}_\mu \text{ satisfies the } \mu^{++}\text{-c.c.},$$
$$\mathbf{P}_\mu \text{ is } \mu^+\text{-complete},$$

(so cardinal arithmetic is clear) and

$$\Vdash_{\mathbf{P}_\mu} \lambda \not\to [\mu^+ \times \mu]^2_\kappa.$$

□

What about replacing μ^+ by an inaccessible θ? We can manage by demanding

$$\{ a \cap (\alpha,\beta) : (a,b) \in \mathcal{P},\ \bigcup_n \mathrm{otp}(a \cap (\alpha,\beta)) \times n = \mathrm{otp}(a)$$
$$(\alpha,\beta) \text{ maximal under these conditions} \}$$

is free (meaning there are pairwise disjoint end segments) and by taking care in defining the order. Hence the completeness drops to θ-strategical completeness. This is carried out in Theorem 3 below.

THEOREM 3. *Assume $\theta = \theta^{<\theta} > \aleph_0$ and $\lambda = \lambda^{<\theta}$. Then for some θ^+-c.c. θ-strategically complete forcing $\mathbf{P}$, $|\mathbf{P}| = \lambda$ and*

$$\Vdash_{\mathbf{P}} 2^\theta = \lambda,\ \lambda \not\to (\theta \times \theta)^2_2.$$

Proof. For W a family of subsets of λ, each with no last element, let

$$\mathrm{Fr}(W) = \{ f : f \text{ is a choice function on } W \text{ such that } \{ a \setminus f(a) : a \in W \} \text{ are pairwise disjoint} \}.$$

If $\mathrm{Fr}(W) \neq \emptyset$, W is called *free*.

Let $\mathbf{P}_{<\theta}$ be the set of forcing conditions

$$(w, c, \mathcal{P}, W)$$

where $w \subseteq \lambda$, $|w| < \theta$, $c : [w]^2 \to \{\mathrm{red}, \mathrm{green}\}$, W is a free family of $< \theta$ subsets of w, each of which is in $\bigcup_{\delta<\theta} \mathcal{R}_\delta$, and $\mathcal{P}$ is a set of $< \theta$ pairs (a,b) such that

(i) a, b are subsets of w
(ii) $b \in \mathcal{R}_\omega$
(iii) $\sup(a) < \min(b)$ and for some $\delta_0 < \delta_1 < \cdots < \delta_n$, $\delta_0 < \min(a)$, $\sup(a) \leq \delta_n$, $a \cap [\delta_l, \delta_{l+1}) \in W$
(iv) if $\sup(a) \leq \gamma < \min(b)$, $\gamma \in w$, then $c(\gamma, \cdot)$ divides a or b into two infinite sets.

We order $\mathbf{P}_{<\theta}$ as follows:

$$p \leq q \text{ iff } w^p \subseteq w^q,\ c^p \subseteq c^q,\ \mathcal{P}^p \subseteq \mathcal{P}^q,\ W^p \subseteq W^q \text{ and every } f \in \mathrm{Fr}(W^p) \text{ can be extended to a member of } \mathrm{Fr}(W^q).$$

□

§4. A provable partition relation.

CLAIM 4. *Suppose $\theta > \aleph_0$, $n, r < \omega$, and $\lambda = \lambda^{<\theta}$. Then*

$$(\lambda^+)^r \times n \to (\theta \times n, \theta \times r)^2_2.$$

Proof. We prove this by induction on r. Clearly the claim holds for $r = 0, 1$. So w.l.o.g. we assume $r \geq 2$. Let c be a 2-place function from $(\lambda^+)^r \times n$ to $\{\text{red}, \text{green}\}$. Let $\chi = \beth_2(\lambda)^+$. Choose by induction on l a model N_l such that

$$N_l \prec (H(\chi), \in, <^*),$$

$|N_l| = \lambda$, $\lambda + 1 \subseteq N_l$, $N_l^{<\theta} \subseteq N_l$, $c \in N_l$ and $N_l \in N_{l+1}$. Here $<^*$ is a well-ordering of $H(\chi)$. Let

$$A_l = \left[(\lambda^+)^r \times l,\ (\lambda^+)^r \times (l+1)\right),$$

and let $\delta_l \in A_l \setminus N_l$ be such that $\delta_l \notin x$ whenever $x \in N_l$ is a subset of A_l and $\mathrm{otp}(x) < (\lambda^+)^r$. W.l.o.g. we have $\delta_l \in N_{l+1}$. Now we shall show

(*) If $Y \in N_0$, $Y \subseteq A_m$, $|Y| = \lambda^+$ and $\delta_m \in Y$,
then we can find $\beta \in Y$ such that $c(\beta, \delta_l) = \text{red}$ for all $l < n$.

Why does (*) suffice? Assume (*) holds. We can construct by induction on $i < \theta$ and for each i by induction on $l < n$ an ordinal $\alpha_{i,l}$ such that

(a) $\alpha_{i,l} \in A_l$ and $j < i \Rightarrow \alpha_{j,l} < \alpha_{i,l}$
(b) $\alpha_{i,l} \in N_0$
(c) $c(\alpha_{i,l}, \delta_m) = \text{red}$ for $m < n$
(d) $c(\alpha_{i,l}, \alpha_{i_1,l_1}) = \text{red}$ when $i_1 < i$ or $i_1 = i \,\&\, l_1 < l$.

Accomplishing this suffices as $\alpha_{i,l} \in A_l$ and

$$l < m \Rightarrow \sup A_l \leq \min A_m.$$

Arriving in the inductive process at (i, l), let

$$Y = \{\, \beta \in A_l : c(\beta, \alpha_{j,m}) = \text{red} \quad \text{if } j < i,\ m < n, \text{ or } j = i,\ m < l \,\}.$$

Now clearly $Y \subseteq A_l$. Also $Y \in N_0$ as all parameters are from N_0, their number is $< \theta$ and $N_0^{<\theta} \subseteq N_0$. Also $\delta_l \in Y$ by the induction hypothesis (and $\delta_l \in A_l$). So by (*) we can find $\alpha_{i,l}$ as required.

Proof of (*). $Y \not\subseteq N_0$, because $\delta_m \in Y$ and $Y \in N_0$. As $|Y| = \lambda^+$, we have $\mathrm{otp}(Y) \geq \lambda^+$. But $\lambda^+ \to (\lambda^+, \theta)^2$, so there is $B \subseteq Y$ such that $|B| = \lambda^+$ and $c \restriction B \times B$ is constantly red or there is $B \subseteq Y$ such that $|B| = \theta$ and $c \restriction B \times B$ is constantly green. In the former case we get the conclusion of the claim. In the latter case we may assume $B \in N_0$, hence $B \subseteq N_0$, and let $k \leq n$ be maximal such that

$$B' = \{\, \xi \in B : \bigwedge_{l<k} c(\delta_l, \xi) = \text{red} \,\}$$

has cardinality θ. If $k = n$, any member of B' is as required in (*). So assume $k < n$. Now $B' \in N_k$, since $B \in N_0 \prec N_k$ and $\{N_l, A_l\} \in N_k$ and $\delta_l \in N_k$ for $l < k$. Also

$$\{\, \xi \in B' : c(\delta_k, \xi) = \text{red} \,\}$$

is a subset of B' of cardinality $< \theta$ by the choice of k. So for some $B'' \in N_0$, $c \restriction \{\delta_k\} \times (B' \setminus B'')$ is constantly green (e.g., as $B' \subseteq N_0$, and $N_0^{<\theta} \subseteq N_0$). Let

$$Z = \{\, \delta \in A_k : c \restriction \{\delta\} \times (B' \setminus B'') \text{ is constantly green} \,\}$$

and

$$Z' = \{\, \delta \in Z : (\forall \alpha \in B' \setminus B'')(\delta < \alpha \Leftrightarrow \delta_k < \alpha) \,\}.$$

So $Z \subseteq A_k$, $Z \in N_k$, $\delta_k \in Z$ and therefore $\mathrm{otp}(Z) = \mathrm{otp}(A_k) = (\lambda^+)^r$. Note that $k \neq m \Rightarrow Z' = Z$ and $k = m \Rightarrow Z' = Z \setminus \sup(B' \setminus B'')$, so Z' has the same properties. Now we apply the induction hypothesis; one of the following holds (note that we can interchange the colours): (a) There is $Z'' \subseteq Z'$, $\mathrm{otp}(Z'') = \theta \times n$, $c \restriction Z'' \times Z''$ is constantly red, w.l.o.g. $Z'' \in N_k$, or (b) there is $Z'' \subseteq Z'$, $\mathrm{otp}(Z'') = \theta \times (r-1)$, $c \restriction Z'' \times Z''$ green and w.l.o.g. $Z'' \in N_k$. If (a), we are done; if (b), $Z'' \cup (B' \setminus B'')$ is as required. $\square$

Remark 4A. So $(\lambda^+)^{n+1} \rightarrow (\theta \times n)^2$ for $\lambda = \lambda^{<\theta}$, $\theta = \mathrm{cf}(\theta) > \aleph_0$ (e.g., $\lambda = 2^{<\theta}$).

Remark 4B. Suppose $\lambda = \lambda^{<\theta}$, $\theta > \aleph_0$. If c is a 2-colouring of $(\lambda^{+r})^s \times n$ by k colours and every subset of it of order type $(\lambda^{+(r-1)})^s \times n$ has a monochromatic subset of order type θ for each of the colours, one of the colours being red, then by the last proof we get

(a) There is a monochromatic subset of order type $\theta \times n$ and of colour red *or*

(b) There is a colour d and a set Z of order type $(\lambda^{+r})^s$ and a set B of order type θ such that $B < Z$ or $Z < B$ and

$$\{\, (\alpha, \beta) : \alpha \in B,\ \beta \in Z \text{ or } \alpha \neq \beta \in B \,\}$$

are all coloured with d.

So we can prove that for 2-colourings by k colours c

$$(\lambda^{+r})^s \times n \rightarrow (\theta \times n_1, \ldots, \theta \times n_k)^2$$

when r, s, n are sufficiently large (e.g., $n \geq \min\{\, n_l : l = 1, \ldots, k,\ s \geq \sum_{l=1}^k n_l \,\}$) by induction on $\sum_{l=1}^k n_l$.

Note that if c is a 2-colouring of λ^{+2k}, then for some $l < k$ and $A \subseteq \lambda^{+2k}$ of order type $\lambda^{+(2l+2)}$ we have

(*) If $A' \subseteq A$, $\mathrm{otp}(A') = \lambda^{+2l}$, and d is a colour which appears in A, then there is $B \subseteq A'$ of order type θ such that B is monochromatic of colour d.

We can conclude $\lambda^{+2k} \rightarrow (\theta \times n)^2_k$.

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