ON SIMILARITIES OF COMPLETE THEORIES

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In classical model theory two objects of different nature correspond to every signature σ:

L — the first-order language of σ and
K — the class of all structures of σ.

But there is a well-known one-to-one correspondence between maximal consistent sets of sentences of L and minimal axiomatizable classes in L of structures from K. When we say that we study the complete theory T we usually mean the pair $\langle T, \mathrm{Mod}(T)\rangle$, where $\mathrm{Mod}(T)$ is the class of all models of T. In connection with this duality of the nature of complete theories I want to introduce two notions of similarity which play the role of isomorphisms and two notions of nearness of theories.

§1. Syntactical similarity.

Let $F_n(T)$, $n < \omega$, be the Boolean algebras of formulas of T with exactly n free variables $v_1, \dots, v_n$, and $F(T) = \bigcup_n F_n(T)$.

DEFINITION 1. Complete theories T_1 and T_2 are *syntactically similar* if and only if there exists a bijection $f : F(T_1) \to F(T_2)$ such that

(i) $f \restriction F_n(T_1)$ is an isomorphism of the Boolean algebras $F_n(T_1)$ and $F_n(T_2)$, $n < \omega$;
(ii) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in F_{n+1}(T)$, $n < \omega$;
(iii) $f(v_1 = v_2) = (v_1 = v_2)$.

EXAMPLE 1. The following theories T_1 and T_2 of the signature $\sigma = \langle \varphi, \psi \rangle$ are syntactically similar, where φ, ψ are binary functions:

$$T_1 = \mathrm{Th}(\langle \mathbf{Z}; +, \cdot \rangle), \qquad T_2 = \mathrm{Th}(\langle \mathbf{Z}; \cdot, + \rangle).$$

§2. Semantic similarity.

From the point of view of a model-theoretician, the object $\langle \mathrm{Mod}(T); \simeq, \preccurlyeq \rangle$ is important for the study of the class $\mathrm{Mod}(T)$. Properties of this object are more completely characterized by the triple $\langle \mathfrak{C}, \mathrm{Aut}(\mathfrak{C}), \mathcal{N}(\mathfrak{C}) \rangle$, where $\mathfrak{C}$ is the monster-model of T, $\mathrm{Aut}(\mathfrak{C})$ is the group of all automorphisms of $\mathfrak{C}$ and $\mathcal{N}(\mathfrak{C})$ is the class of all elementary substructures of $\mathfrak{C}$. Therefore the following definition of semantic similarity is justified.

I shall begin with some preliminary notions.

DEFINITION 2. (1) By a *pure triple* we mean $\langle A, \Gamma, \mathcal{M} \rangle$, where $A \neq \emptyset$, Γ is a permutation group on A, and $\mathcal{M}$ is a family of subsets of A such that

$$M \in \mathcal{M} \Rightarrow g(M) \in \mathcal{M} \quad \text{for every } g \in \Gamma.$$

(2) If $\langle A_1, \Gamma_1, \mathcal{M}_1 \rangle$ and $\langle A_2, \Gamma_2, \mathcal{M}_2 \rangle$ are pure triples, and $\psi : A_1 \to A_2$ is a bijection, then ψ is an *isomorphism*, if
(i) $\Gamma_2 = \{ \psi g \psi^{-1} : g \in \Gamma_1 \}$;
(ii) $\mathcal{M}_2 = \{ \psi(E) : E \in \mathcal{M}_1 \}$.

DEFINITION 3. The pure triple $\langle |\mathfrak{C}|, G, \mathcal{N} \rangle$ is called the *semantic triple* of T (abbreviated s.t.), where $|\mathfrak{C}|$ is the universe of $\mathfrak{C}$, $G = \mathrm{Aut}(\mathfrak{C})$, and $\mathcal{N}$ is the class of all subsets of $|\mathfrak{C}|$ which are universes of suitable elementary submodels of $\mathfrak{C}$.

DEFINITION 4. Complete theories T_1 and T_2 are *semantically similar* if and only if their semantic triples are isomorphic.

EXAMPLE 2. The following theories T_1 and T_2 are semantically similar, where

$$\begin{aligned} T_1 &= \mathrm{Th}(\langle \mathcal{M}_1; P_n, n < \omega; a_{nm}, n, m < \omega \rangle), \\ \mathcal{M}_1 &= \{ a_{nm} : n, m < \omega \}, \\ P_n(\mathcal{M}_1) &= \{ a_{nm} : m < \omega \}, \end{aligned}$$

and

$$\begin{aligned} T_2 &= \mathrm{Th}(\langle \mathcal{M}_2; Q_n, n < \omega; Q_{nm}, n, m < \omega; b_{nmk}, n, m, k < \omega \rangle), \\ \mathcal{M}_2 &= \{ b_{nmk} : n, m, k < \omega \}, \\ Q_n(\mathcal{M}_2) &= \{ b_{nmk} : m, k < \omega \}, \\ Q_{nm}(\mathcal{M}_2) &= \{ b_{nmk} : k < \omega \}. \end{aligned}$$

§3. Criteria of syntactical and semantical similarities.

It turns out that the notions of syntactical and semantical similarity may be defined in a common language, namely, in terms of so-called semisystems.

DEFINITION 5. (1) By a *semisystem* we mean a pair $\langle A, \mathcal{F} \rangle$, $A \neq \emptyset$, $\mathcal{F} \subseteq \bigcup_n \mathcal{P}(A^n)$, where $\mathcal{P}(x)$ denotes the set of all subsets of X.

(2) If $\langle A_1, \mathcal{F}_1 \rangle$ and $\langle A_2, \mathcal{F}_2 \rangle$ are semisystems, $\psi : A_1 \to A_2$ is a bijection, then ψ is called an *isomorphism* if and only if $\mathcal{F}_2 = \{ \psi(E) : E \in \mathcal{F}_1 \}$, where $\psi(E) = \{ \langle \psi(e_1), \dots, \psi(e_n) \rangle : \langle e_1, \dots, e_n \rangle \in E \}$.

DEFINITION 6. $X \in \mathsf{F}(\mathfrak{C}) \iff \exists n < \omega, \varphi \in F_n(T)$ such that

$$X = \{ \langle a_1, \dots, a_n \rangle \in |\mathfrak{C}^n| : \mathfrak{C} \models \varphi(a_1, \dots, a_n) \}.$$

THEOREM 1. *The following are equivalent:*
(i) T_1 *and* T_2 *are syntactically similar.*
(ii) The semisystems $\langle |\mathfrak{C}_1|, \mathsf{F}(\mathfrak{C}_1) \rangle$ *and* $\langle |\mathfrak{C}_2|, \mathsf{F}(\mathfrak{C}_2) \rangle$ *are isomorphic.*

DEFINITION 7. $X \in \mathrm{TV}(\mathfrak{C})$ (i.e., X is a Tarski–Vaught set) if and only if there is $n < \omega$ such that

(i) $X \subseteq |\mathfrak{C}|^n$;
(ii) $\forall g \in \mathrm{Aut}(\mathfrak{C})(X = g(X))$;
(iii) $\forall M \prec \mathfrak{C}$, $\forall m, 1 \leq m < n$, $\forall b_1, \ldots, b_m, b_{m+2}, \ldots, b_n \in M$

$$\exists y \in |\mathfrak{C}|(\langle b_1, \ldots, b_m, y, b_{m+2}, \ldots, b_n\rangle \in X) \Rightarrow$$
$$\exists y \in M(\langle b_1, \ldots, b_m, y, b_{m+2}, \ldots, b_n\rangle \in X).$$

THEOREM 2. *The following are equivalent:*
(i) T_1 *and* T_2 *are semantically similar.*
(ii) The semisystems $\langle|\mathfrak{C}_1|, TV(\mathfrak{C}_1)\rangle$ *and* $\langle|\mathfrak{C}_2|, TV(\mathfrak{C}_2)\rangle$ *are isomorphic.*

PROPOSITION 1. *If* T_1 *and* T_2 *are syntactically similar, then* T_1 *and* T_2 *are semantically similar. The converse implication fails.*

Proof. "$\Rightarrow$" is easy; "$\not\Leftarrow$" follows from Example 2.

§4. A list of semantic properties of theories.

DEFINITION 8. A property (or a notion) of theories (or models, or elements of models) is called *semantic* if and only if it is invariant relative to semantic similarity.

PROPOSITION 2. *The following properties and notions are semantic:*
(1) type,
(2) forking,
(3) λ*-stability,*
(4) Lascar rank,
(5) strong type,
(6) Morley sequence,
(7) orthogonality, regularity of types,
(8) $I(\aleph_\alpha, T)$*—the spectrum function.*

§5. Quasisimilarity of theories.

DEFINITION 9. Let $\langle A, \Gamma, \mathcal{M}\rangle$ be an arbitrary pure triple, $\sim$ an equivalence relation on A. Then $\sim$ is *congruence* if and only if

(i) $a_1 \sim a_2 \Rightarrow g(a_1) \sim g(a_2)$, $\forall g \in \Gamma$, $\forall a_1, a_2 \in A$;
(ii) $a_1 \in M \;\&\; M \in \mathcal{M} \;\&\; a_1 \sim a_2 \Rightarrow a_2 \in M$, $\forall a_1, a_2 \in A$.

Remark. (1) If $\sim$ is a congruence, then $\sim$ induces a group congruence $\approx$ on the group Γ in the following way:

$$g_1 \approx g_2 \iff \forall a \in A(g_1(a) \sim g_2(a)), \text{ where } g_1, g_2 \in \Gamma.$$

(2) If $\sim$ is a congruence, then the triple $\langle A/\sim, \Gamma/\sim, \mathcal{M}/\sim\rangle$ will be a pure triple too which is called the *quotient pure triple*. Here

$$\mathcal{M}/\sim = \{ M/\sim : M \in \mathcal{M} \}.$$

(3) The following relation ε on the semantic triple of any theory is a congruence:

$$a \,\varepsilon\, b \iff \begin{cases} a = b, & \text{if } a, b \in \operatorname{acl}(\emptyset); \\ \operatorname{acl}(a) = \operatorname{acl}(b) & \text{in the other case.} \end{cases}$$

DEFINITION 10. (1) T_1 and T_2 are *ε-similar* if and only if the semantic quotient triples $\langle |\mathfrak{C}_1|/\varepsilon, G_1/\varepsilon, \mathcal{N}_1/\varepsilon\rangle$ and $\langle |\mathfrak{C}_2|/\varepsilon, G_2/\varepsilon, \mathcal{N}_2/\varepsilon\rangle$ are isomorphic.

(2) T_1 and T_2 are *quasisimilar* if and only if there are $M_1 \models T_1$, $M_2 \models T_2$ such that $\operatorname{Th}((M_1, m)_{m \in M_1})$ and $\operatorname{Th}((M_2, m)_{m \in M_2})$ are ε-similar.

EXAMPLE 3. $\operatorname{Th}(\langle \mathbf{Z}; {}'\rangle)$, where $x' = x + 1$, is quasisimilar to the theory of infinite sets without any structure.

The following question is natural: What kind of theories are quasisimilar to theories with only unary predicates? To answer this question we need some notions.

DEFINITION 11. We say that a theory T *admits* a closure operator J if and only if J is a closure operator on $|\mathfrak{C}|$ which satisfies the condition

$$J(g(B)) = g(J(B)), \ \forall g \in \operatorname{Aut}(\mathfrak{C}), \ \forall B \subset |\mathfrak{C}|.$$

Notations.

(1) If $a, b \in |\mathfrak{C}| \setminus B$, then $b \in C_B^J(a)$ if and only if there are $n < \omega$ and $b_0, \dots, b_n \in |\mathfrak{C}| \setminus B$ such that $b_0 = a$, $b_n = b$ and $b_i \in J(b_{i+1})$ or $b_{i+1} \in J(b_i)$, for every $i < n$.
(2) $C_B^J(A) = \bigcup\{ C_B^J(a) : a \in A\}$.
(3) $\hat{B} = \bigcup\{ J(b) : b \in B\}$.
(4) If $M \models T$, and $a, b \in |\mathfrak{C}| \setminus M$, then $aE_M^J b \iff (M \cap J(a) = M \cap J(b))$.
(5) $\chi_M^J(a) = \left|C_M^J(a)/E_M^J\right|$.
(6) $\chi^J(T) = \min\{ \mu : \chi_M^J(a) < \mu, \ \forall M \models T, \ \forall a \in |\mathfrak{C}| \setminus M\}$, if such a cardinal exists. Otherwise $\chi^J(T) = \infty$.

THEOREM 3. *The following are equivalent:*

(i) T is quasisimilar to some theory of unary predicates.
(ii) T is a bounded dimensional superstable theory which admits a closure operator J for which the following conditions are satisfied:
  (1) $M \models T \Rightarrow \hat{M} = M$,
  (2) $B = \hat{B}$ & $C_B^J(\bar{a}) = C_B^J(\bar{b}) \Rightarrow \bar{a} \downarrow_B \bar{b}$,
  (3) $\chi^J(T) < \infty$.

As an application of this theorem we have

PROPOSITION 3. *$\operatorname{Th}(\langle M; f\rangle)$, where f is unary function (i.e., the theory of so called unars), is ω_1-categorical if and only if it is quasisimilar to the theory of infinite sets without any structure.*

Remark. In connection with the last proposition it is necessary to note the following. For the description of some classes of concrete algebraic systems defined in model theory language there may not exist a characterisation in an appropriate algebraic language. For example, Shishmarev [5] gave in 1972 a criterion of ω_1-categoricity of unars in a complex mixed language (model theoretical language with algebraic one). But until now no appropriate criterion in a purely algebraic language was found. In these cases, as the above proposition shows, the language of quasisimilarity can become useful and understandable.

§6. Envelope and almost envelope.

If $Q \in F_1(T)$ and $M \models T$, then $M^Q = \langle Q(M); R_\varphi \rangle_{\varphi \in F(T)}$, where $R_\varphi = (Q(M))^n \cap \varphi(M)$, for $\varphi \in F_n(T)$. $T^Q = \mathrm{Th}(M^Q)$.

DEFINITION 12. A theory T_1 is the *envelope* of T_2 if and only if there is $Q \in F_1(T_1)$ such that

(i) T_2 is syntactically similar to T_1^Q;
(ii) For all $N \models T_1^Q$ there is $M \models T_1$ such that $N = M^Q$;
(iii) $M = \mathrm{dcl}(Q(M))$, $\forall M \models T_1$.

EXAMPLE 4. Let T_2 be the theory of infinite sets without any structure, $T_1 = \mathrm{Th}(\langle M_1; Q, f \rangle)$, where Q is a unary relation, f is a unary function such that (i) $f^2 = \mathrm{id}$, (ii) $f \restriction Q$ is a bijection between Q and $M \setminus Q$. (See Figure 1.)

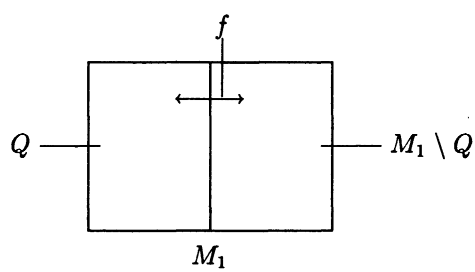

Figure 1.

Then T_2 is syntactically similar to T_1^Q and T_1 is an envelope of T_2.

DEFINITION 13. By a *polygon* over a monoid S we mean a structure with only unary functions $\langle A; f_\alpha : \alpha \in S \rangle$ such that

(i) $f_e(a) = a$, $\forall a \in A$, where e is the unit of S;
(ii) $f_{\alpha\beta}(a) = f_\alpha(f_\beta(a))$, $\forall \alpha, \beta \in S$, $\forall a \in A$.

THEOREM 4. *For every theory T_2 in a finite signature there is a theory T_1 of polygons such that some inessential extension of T_1 is an envelope of T_2.*

For the case when the signature is infinite we have a weak variant of this theorem. I shall introduce new notions for the formulation of the next theorem.

DEFINITION 14. A type $p \in S_1(T)$ is called *neutral* if and only if

(i) $M \models T \Rightarrow M_p \models T$, where $M_p = M \setminus p(M)$;
(ii) $p(M) \setminus A$ is indiscernable over A, $\forall M \models T$, $\forall A \subset M$.

DEFINITION 15. T_1 is an *almost envelope* of T_2 if and only if there are $Q \in F_1(T)$ and a neutral type $p \in S_1(T)$ such that
(i) T_2 is syntactically similar to T_1^Q;
(ii) $\forall N \models T_1^Q \; \exists M \models T_1 \; (N = M^Q)$;
(iii) $M_p = \mathrm{dcl}(Q(M))$, $\forall M \models T_1$.

THEOREM 5. *For every theory T_2 in an infinite signature there is a theory T_1 of polygons such that some inessential extension of T_1 is an almost envelope of T_2.*

Remark. The notions of envelope and almost envelope express the close connection between theories. The following shows it:

PROPOSITION 4. *If T_1 is an envelope (or almost envelope) of T_2, then*
(i) T_1 is ω-stable $\iff$ T_2 is λ-stable, $\forall\lambda$;
(ii) $I(\aleph_\alpha, T_1) = I(\aleph_\alpha, T_2)$ (or $I(\aleph_\alpha, T_1) = I(\aleph_\alpha, T_2) + |\aleph_0 + \alpha|$), $\forall\alpha$.

From this it is clear that many problems of model theory can be reduced to the analogous problems of polygon theory in an exact way. In particular, this is true for Vaught's conjecture about the number of countable models.

§7. Polygons.

It is known that the polygons over an any cyclic monoid have a superstable theory. The question of describing all such monoids is natural.

DEFINITION 16. (1) Monoid S is called a ***stabilisator*** (or ***superstabilisator***, or ***ω-stabilisator***) if for every polygon A over S, $\mathrm{Th}(A)$ is stable (or superstable, or ω-stable).
(2) If $\alpha, \beta \in S$ then

$$\alpha \trianglelefteq \beta \iff S_\alpha \supseteq S_\beta,$$
$$\alpha \sim \beta \iff S_\alpha = S_\beta,$$
$$I_S = |S/{\sim}|.$$

(3) If $\langle S/{\sim}; \trianglelefteq\rangle$ is linearly ordered (or well ordered) then S is called ***LO-monoid*** (or ***WO-monoid***).

THEOREM 6. *(i) S is a stabilisator if and only if S is LO-monoid.*
(ii) S is a superstabilisator if and only if S is WO-monoid.

Problem: When is S is an ω-stabilisator?

We have the following information on the problem.

THEOREM 7. *(i) If S is a stabilisator then $I_S \le 2$.*
(ii) If $I_S = 1$ (i.e., S is a group), then S is an ω-stabilisator if and only if S has at most countably many subgroups.
(iii) If $I_S = 2$ (in which case S may be represented by $S = G \cup J$, where J is the unique proper left ideal, $G = S \setminus J$ and G is a group) and S is an ω-stabilisator, then
(1) G has at most countably many subgroups;
(2) $|G| < \omega \Rightarrow |S| < \omega$.

Conjecture. If $I_S = 2$ then S is an ω-stabilisator $\iff$ $|S| < \omega$.

The proofs of the results are given in [1]–[4].

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