ON THE GEOMETRY OF U-RANK 2 TYPES

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Abstract. Let T be a countable superstable theory with $< 2^{\aleph_0}$ countable models. We solve the algebraic problem from [Ne4, §4]. In particular, in some cases we complete the countable classification of skeletal p of U-rank 2 (cf. [Bu4]).

§0. **Introduction.** Throughout the paper we assume that T is a complete countable superstable theory with $< 2^{\aleph_0}$ countable models. For the background from stability theory see [Sh], [Ba], [Bu1], or [P]. The results in [Bu2] suggest that if T has infinite U-rank then every countable model M of T is determined by a subset A of M, called its skeleton (cf. [Bu4]). Hence in the course of proving Vaught's conjecture we have to determine possible isomorphism types of skeletons. The easiest non-trivial case we faced in [Bu4] and [Ne4] was as follows. Assume $p \in S(\emptyset)$ is stationary, non-isolated, has U-rank 2, and if b realizes p then for some $a \in \mathrm{acl}(b)$, $U(a) = 1$ and $\mathrm{tp}(b/a)$ is non-isolated. Let $I(p,\kappa)$ be the number of isomorphism types of sets $p(M)$ of power κ, where M is a model of T. We wanted to prove that $I(T,\aleph_0) < 2^{\aleph_0}$ implies $I(p,\aleph_0) \leq \aleph_0$. Anyway, considering $I(p,\aleph_0)$ seems to be a necessary step on a way to prove Vaught's conjecture for superstable T. Let us recall the main path of reasoning from [Bu4] and [Ne4] thus far.

For a, b as above let $q = \mathrm{tp}(a/\emptyset)$ and $p_a = \mathrm{tp}(b/a)$. We want to count, up to isomorphism of the monster model $\mathfrak{C}$, the number of sets $p(M)$, where M is countable. $p(M)$ is the union of sets $p_{a'}(M)$, where $a' \in M$ realizes q. q has finite multiplicity hence by adding an element of $\mathrm{acl}(\emptyset)$ to the signature we can assume that q is stationary. Throughout we assume $T = T^{\mathrm{eq}}$. Further on in determining the structure of $p(M)$ we can easily dispose with the cases when p_a is strongly minimal or trivial. Hence we can assume that p_a is properly minimal and non-trivial. Then [Ne1] implies that p_a has finite multiplicity, and [Bu1] gives that every stationarization of p_a is locally modular. Similarly we can assume that for b realizing p_a, $\mathrm{stp}(b/a)$ is not modular, non-orthogonal to $\emptyset$ and almost orthogonal to $\emptyset$. In particular, p_a is weakly orthogonal to $q \mid a$. Also, we can assume that all stationarizations of types p_a, $a \in q(\mathfrak{C})$, are non-orthogonal. If $q(M)$ has finite acl-dimension then $p(M)$ can be characterized up to isomorphism just as in [Bu2]. Hence we can assume that for every countable M we consider, $q(M)$ has dimension $\aleph_0$, and $Q = q(M)$ is fixed. As $p(M) = \bigcup\{p_a(M) : a \in Q\}$, classifying the structure of $p(M)$ amounts to describing how the weakly minimal sets $p_a(M)$, $a \in Q$, can be arranged together to form $p(M)$. The types p_a, $a \in Q$, are non-orthogonal, so the main difficulty lies in that we are not free in deciding whether p_a is realized in M or not: if $p_{a_1}, \dots, p_{a_n}$ are realized in M and $a \in Q$ then possibly p_a is realized in $\mathrm{acl}(p_{a_1}(M) \cup \dots \cup p_{a_n}(M))$. This determines a kind of dependence relation on types p_a, $a \in Q$. For various reasons it is easier to work with stationary types rather than with types of finite multiplicity. Thus instead of dependence on $\{p_a : a \in Q\}$ we define a dependence relation on the set of stationarizations of types p_a, $a \in Q$. Also, to make this dependence modular we have to consider some other weakly minimal types as well. The formal definition follows the idea from [Ne2]. Let P^* be the set of strong weakly minimal non-modular types r over Q (in T^{eq}) such that r is non-orthogonal to some (every) p_a, $a \in Q$, and for some finite set $A \subseteq Q$, r does not fork over A and has finitely many conjugates over A. Hence all types in P^* are non-orthogonal.

For $r \in P^*$ and $R \subseteq P^*$, $a \in \mathrm{ACL}(R)$ iff whenever A contains a realization of every type in R then r is realized in $\mathrm{acl}(A \cup Q)$. ACL is a modular dependence relation on P^* ([Bu4, 1.14] or [Ne3, 1.2]). For $A \subseteq Q$ let $P^*_A = \{r \in P^* : r$ is based on $A\}$, $P^0_A = \{\mathrm{stp}(b/a) \mid Q : a \in \mathrm{acl}(A) \cap Q$ and b realizes $p_a\}$, and $P_A = \mathrm{ACL}(P^0_A) \cap P^*_A$. By [Ne4, 1.1], P^*_A is essentially ACL-closed in P^*, meaning that every $r \in \mathrm{ACL}(P^*_A)$ is ACL-interdependent with some $r' \in P^*_A$. Let $P = P_Q$. Let us say that ACL-closed X, $Y \subseteq P$ are isomorphic if there is an automorphism f of $\mathfrak{C}$ with $f[Q] = Q$ and $f[X] = Y$. To compute $I(p, \aleph_0)$ it suffices to determine the isomorphism types of ACL-closed subsets of P.

For X, $Y \subseteq P^*$, $\mathrm{DIM}(X/Y)$ denotes the ACL-dimension of X over Y, and $\mathrm{DIM}(X)$ denotes the ACL-dimension of X. We say that X, $Y \subseteq P^*$ are independent over $Z \subseteq P^*$ ($X \perp Y(Z)$) iff any ACL basis of X over Z remains an ACL-basis of X over $Y \cup Z$. We have that for finite $A \subseteq Q$, $\mathrm{DIM}(P^*_A)$ is finite as well ([Ne4, 1.7], [Bu2, 5.2(a)], or [Bu4, 1.14]). [Bu4, §2] proves that for A, $B \subseteq Q$, $P^*_{A \cup B} \subseteq \mathrm{ACL}(P^*_A \cup P^*_B)$ (we call this a "local character of ACL"), and that q is locally modular. As a consequence we prove in [Ne4, 1.3] that if $C \subseteq A \cap B$, A, $B \subseteq Q$, and $A \perp B(C)$ then $P_A \perp P_B(P_C)$ (and $P^*_A \perp P^*_B(P^*_C)$ as well, also the assumption that $C \neq \emptyset$ is redundant there). We can assume that q is non-trivial. Also, by the local character of ACL, we can assume that $n_0 = \mathrm{DIM}(P_a) > 1$.

Now applying [H] we can assume that q is the generic type of some connected weakly minimal type-definable (in $\mathfrak{C}^{\mathrm{eq}}$) group $(G, +)$, in particular that q is modular. By modularity we can associate with q a division ring K such that Q with acl may be regarded as a projective space over K. In fact [H] gives more. acl on $Q \cup \{0\}$ is just a K-vector space dependence (0 is the neutral element of G). Similarly we can associate with any stationarization r of p_a a division ring L. As indicated in [Ne4], P^* with ACL-dependence may be regarded as a projective space over the same L (after identifying ACL-interdependent types). We fix the meaning of K and L for the rest of the paper, unless indicated otherwise. We say that an $A \subseteq Q$ is closed if $\mathrm{acl}(A) \cap Q = A$. As in [Ne4], for $r \in P^*$ we define $A(r)$ as the minimal closed $A \subseteq Q$ such that for some $r_0 \in P^*_A$, $r \in \mathrm{ACL}(r_0)$. By local character of ACL, if for some closed A, $A' \subseteq Q$ and $r_0 \in P^*_A$, $r_1 \in P^*_{A'}$, $r \in \mathrm{ACL}(r_0) \cap \mathrm{ACL}(r_1)$, then for some $r_2 \in P^*_{A \cap A'}$, $r \in \mathrm{ACL}(r_2)$, hence the above definition is correct. Let $n(r) = \dim(A(r))$. In [Ne4, 1.13] and [Bu4] we prove that $n^* = \max\{n(r) : r \in P\}$ is finite (in [Ne4, 1.13] n^* is denoted by n_b). In [Ne4] we reduce the problem of counting isomorphism types of ACL-closed subsets of P to a problem from algebra in the following way. Suppose F_0 is a countable division ring, $n < \omega$, and $F_1 \subseteq M_{n\times n}(F_0)$ is a division subring of the ring of matrices $M_{n\times n}(F_0)$, meaning that addition and multiplication in F_1 are addition and multiplication of matrices, and 1_{F_1} is the identity matrix I. Let $\mathcal{K}(F_0, F_1)$ be the class of pairs (V, W) where V is an F_0-vector space and $W \subseteq V^n$ is an F_1-vector subspace of V^n. V^n is an F_1-space: regard elements of V^n as columns, and F_1 acts on them by matrix multiplication on the left. We say that (V, W), $(V', W') \in \mathcal{K}(F_0, F_1)$ are isomorphic if there is an F_0-linear isomorphism $f : V \to V'$ such that $\hat{f}[W] = W'$ for the induced mapping $\hat{f} : V^n \to (V')^n$. The elements of $\mathcal{K}(F_0, F_1)$ we call (F_0, F_1)-structures.

Assume C is a finite subset of Q, R is a basis of P_C, and E is a selector from $\{r(\mathfrak{C}) : r \in R\}$. In our reduction we need to add $C \cup E$ for some C and E to the signature. Then we replace p and q by $p \mid C \cup E$ and $q \mid C \cup E$, and make other changes accordingly. Notice that in doing so we do not need to change K and L. ACL on the new P corresponds to the old ACL on the old P, localized modulo R. Also, the new n^* equals the old one. Now we prove in [Ne4] that after adding this $C \cup E$ to the set of constants, there is an embedding of L into $M_{n^*\times n^*}(K)$ (so we can assume that $L \subseteq M_{n^*\times n^*}(K)$ is a division subring of $M_{n^*\times n^*}(K)$). Let us work in $T(C \cup E)$. We can regard $Q \cup \{0\}$ as a K-vector space V, acl-dependence in Q corresponding to K-linear dependence in V. We find a correspondence α between types in P and elements of V^{n^*} such that α is onto and translates ACL-dependence into L-linear dependence. We show that all stationarizations of a single p_a are ACL-interdependent, and if $r \in P$ is a stationarization of p_a then $\alpha(r) = (a, 0, 0, \dots) \in V^{n^*}$. This gives a full description of ACL on P. In particular, $p_a \in \mathrm{ACL}(p_{a_1}, \dots p_{a_n})$ iff $(a, 0, 0, \dots) \in$ L-span$((a_1, 0, 0, \dots), \dots, (a_n, 0, 0, \dots))$ and we get a 1-1 correspondence between ACL-closed subsets of P and L-closed $W \subseteq V^{n^*}$ such that non-isomorphic ACL-closed subsets of P correspond to non-isomorphic pairs (V, W) in $\mathcal{K}(K, L)$. Of course this is a translation of a localized version of the original problem. In many cases if there are $2^{\aleph_0}$-many non-isomorphic $(V, W) \in \mathcal{K}(K, L)$, then this still gives $I(T, \aleph_0) = 2^{\aleph_0}$ for the original T. In this paper we exhibit a solution of the problem of counting countable (K, L)-structures, and in some cases show how to apply this to compute $I(p, \aleph_0)$ for the original p.

Many conjectures in stability theory (like these of Zil'ber or Cherlin) indicate that "classifiable" stable structures correspond to a few general patterns, often appearing already in classical mathematics. One of the results in this direction was the work of Hrushovski [H] showing how group structures occur in the stable context. In particular he proved that any modular, stationary regular non-trivial type may be regarded as the generic type of some type-definable group, and forking dependence on it is just a linear dependence over some division ring. We used this result above. But to obtain this he needed some parameters. We may think of these parameters as needed to recover the original pattern in the regular type, which may be distorted due to some special features of the theory.

For example we can construct a stable structure in the following way. We may start with a stable group G, and then forget about a part of its structure, so that G will be stable, but will not be a group anymore. So the original pattern of G is distorted. Hrushovski's theorem says that sometimes we can recover a group structure, possibly in an imaginary extension G^{eq} of G. Returning to our context we think that this may be the role of the added parameters $C \cup E$. The question remains how much distorted the structure of the original p may be with respect to its regularized version.

An example. Now suppose K is any countable division ring, $n < \omega$, and $L \subseteq M_{n\times n}(K)$ is a division subring of $M_{n\times n}(K)$. We do not know too many complicated types of U-rank 2. This example is intended to fill this gap. We shall show that K and L give rise to a stationary type p of U-rank 2 so that ACL on P corresponds to L-dependence. That is, ACL-closed subsets of P correspond to (K, L)-structures.

Let V be a K-space, and V_1 be a subspace of V with $\dim(V_1) = \dim(V/V_1) = \aleph_0$. V^n and $(V/V_1)^n$ are (left) L-spaces. $(V/V_1)^n$ contains $(V/V_1, 0, 0, \dots) = V_2 \cong V/V_1$ as a K-subspace. Define $Q = V_2$. For $a \in Q$ let $P_a = a + V_1^n$. So P_a is an affine L-space (a translation of V_1^n).

If $\vec{\alpha} = (\alpha_1, \dots, \alpha_K) \subseteq L$, $a_1, \dots, a_k \in Q$, and $b_i \in a_i + V_1^n$ then $\sum_i \alpha_i b_i \in V^n$. If $\sum_i \alpha_i a_i = a$ for some $a \in V_2$ then $\sum_i \alpha_i B_i \in A + V_1^n$. Let $f_{\vec{\alpha}}$ be a k-ary partial function acting on $\bigcup_{a\in Q} P_a$ defined as follows. $f_{\vec{\alpha}}(b_1, \dots, b_k)$ is defined if $b_i \in P_{a_i}$ and $\sum_i \alpha_i a_i = a \in V_2$, and then $f_{\vec{\alpha}}(b_1, \dots, b_k) = \sum_i \alpha_i b_i$. Notice that whether $f_{\vec{\alpha}}$ is defined on $(b_1, \dots, b_k)$, with $b_i \in P_{a_i}$, depends only on the linear type of $a_1, \dots, a_k$.

Let $M = (Q \cup \bigcup_{a\in Q} P_a; Q(x), P(x,y), f_{\vec{\alpha}})_{\vec{\alpha}\subseteq L}$, where $Q(M) = Q$, $P(M^2) = \{(a,b) : a \in Q, b \in P_a\}$, equipped with the following additional structure: the structure of K-space on Q, the structure of L-space on P_0, and for every $a \in Q$ the structure of affine L-space on P_a, i.e., the binary subtraction function mapping $P_a \times P_a$ into P_0.

$T = \mathrm{Th}(M)$ is ω-stable; Q and every P_a is strongly minimal. Let p_a be the strongly minimal type isolated by $P_a(x)$ over a. Then p_a is locally modular, and ACL-dependence on $\{p_a : a \in Q\}$ is described by functions $f_{\vec{\alpha}}$, i.e., is just an L-dependence. If $b \in P_a$ for $a \neq 0$ then $p = \mathrm{tp}(b/\emptyset)$ is stationary, has U-rank 2 and is not almost orthogonal to $q = \mathrm{tp}(a/\emptyset)$.

Now we shall modify the construction to get properly weakly minimal p_a and a small superstable T. Then we need of course to assume that L is locally finite. For simplicity we assume that L is finite.

Let $W_0 = W^0 > W^1 > \dots > W^i \dots, i < \omega$, be a sequence of L-spaces such that $[W^i : W^{i+1}]$ is finite and $\bigcap_i W^i$ is $\aleph_0$-dimensional. We identify V_1^n with $\bigcap_i W^i$. Add an independent copy W_a of W_0 over every P_a, $a \neq 0$, i.e., form a formal affine space $a + W_0$ so that $a + V_1^n = P_a$. For $a \neq 0$ extend subtraction from P_a onto W_a so that for $x, y \in W_a$, $x - y \in W_0$ (if $a + x, a + y \in W_a$ then $(a + x) - (a + y) = x - y \in W_0$).

We have to extend also the functions $f_{\vec{\alpha}}$, $\vec{\alpha} \subseteq L$, onto the larger sets W_a, $a \in Q$. For $f_{\vec{\alpha}}$ and $\overline{a} = (a_1, \dots, a_k) \subseteq Q$ of suitable length with $\sum_i \alpha_i a_i = a \in Q$

let us define $f_{\vec{\alpha}}(b_1,\ldots,b_k)$ for $b_i \in W_{a_i}$ as follows: Take some $b_i' \in a_i + V_1^n$. Let $f_{\vec{\alpha}}(b_1,\ldots,b_k) = \sum_i \alpha_i b_i' + \sum_i \alpha_i(b_i - b_i')$. The first sum in this definition is taken in V^n, the second one in W_0. $\sum_i \alpha_i b_i' + \sum_i \alpha_i(b_i - b_i')$ is the only element y of W_a such that $y - \sum_i \alpha_i b_i' = \sum_i \alpha_i(b_i - b_i')$. It is easy to check that this definition does not depend on the choice of b_i'. Now let $M = (Q \cup \bigcup_{a\in Q} W_a; Q(x), W(x,y), W^i(x,y)\ (0 < i < \omega), f_{\vec{\alpha}}\ (\vec{\alpha} \subseteq L)$, the structure of K-space on Q, the structure of L-space on W_0, and the affine L-structure on each W_a given by subtraction), where $Q(M) = Q$, $W(M^2) = \{(a,b) : b \in W_a\}$, $W^i(M^2) = \{(a,b) : b \in a+W^i\}$. Then $T = \mathrm{Th}(M)$ is small and superstable, $p_a =$ the type over a generated by $W^i(a,x)$, $0 < i < \omega$, is properly weakly minimal, locally modular, non-isolated. ACL on $\{p_a : a \in Q\}$ is the L-dependence given by $f_{\vec{\alpha}}$'s. For $0 \neq a \in Q$ and b realizing p_a, $p = \mathrm{tp}(b/\emptyset)$ is stationary, of U-rank 2, not almost orthogonal to $q = \mathrm{tp}(a/\emptyset)$.

One could wonder what description of ACL we obtain here if we apply the analysis from [Ne4] to this case. Notice that the set of first columns of elements of L is a right K-space, a subspace of K^n. It turns out that if K-span(first columns of L) $= K^n$ (equivalently: L-span$(K,0,\ldots,0)^t = K^n$, or there are $\alpha_1,\ldots,\alpha_n \in L$ with first columns K-independent), then through the construction from [Ne4] we recover the original embedding $L \subseteq M_{n\times n}(K)$ (compare [Ne4, 3.11]).

This example shows that the general pattern of a skeletal p of U-rank 2 obtained in [Ne4] occurs in reality.

§1. Counting (K,L)-structures. In this section we prove that there are either $2^{\aleph_0}$ or countably many countable (K,L)-structures. Also, we show that if K, L are finite (and by [Ne4], K being finite is equivalent to L being finite) and $n^* > 1$ then there are $2^{\aleph_0}$-many $(V,W) \in \mathcal{K}(K,L)$ with $\dim(V) = \aleph_0$. In case when $n^* = 1$ and K is a field, the number of countable (K,L)-structures depends on $[L : K]$. The proofs consist in applying in our context results and methods from algebra and from the "grey zone" between algebra and logic. The detailed analysis of $\mathcal{K}(K,L)$ was carried out in [DR]. Here we adapt their results.

Notice that there is a natural notion of direct sum in $\mathcal{K}(K,L)$: $(V,W) = (V_0,W_0) \oplus (V_1,W_1)$ iff $V = V_0 \oplus V_1$ (which determines V^{n^*}, and embeddings $V_0^{n^*}, V_1^{n^*} \subseteq V^{n^*}$, hence $W_0, W_1 \subseteq V^{n^*}$) and $W = W_0 + W_1$ (in V^{n^*}). It turns out that we can regard every $(K,\ L)$ structure as a left R-module for some matrix ring R. For any ring R, let $\mathcal{M}(R)$ denote the class of left R-modules. Let $R_0 = M^*_{1\times n}(K)$, and let $R = \begin{pmatrix} K & R_0 \\ 0 & L \end{pmatrix}$. For an ideal (possibly one-sided) J of R and an R-module M let $\mathrm{Ker}_M(J)$ denote $\{a \in M : Ja = 0\}$. $\mathrm{Ker}_M(J)$ is a subgroup of M, which is an R-module if J is a right-ideal. Let

$$I_L = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix} \qquad I_K = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \qquad I_{R_0} = \begin{pmatrix} 0 & R_0 \\ 0 & 0 \end{pmatrix}.$$

Then $(I_{R_0}+I_L)$ and $(I_K+I_{R_0})$ are two-sided ideals of R, I_K and I_{R_0} are left ideals and I_L is a right ideal of R. Now, with any $S = (V,W) \in \mathcal{K}(K,L)$ we associate the R-module $\begin{pmatrix} V \\ W \end{pmatrix}$ and call it $\underline{S}$. Notice that $(I_K + I_{R_0})\begin{pmatrix} V \\ W \end{pmatrix} = \begin{pmatrix} V \\ 0 \end{pmatrix}$. Hence we get

REMARK 1.1. *$S = (V, W) \in \mathcal{K}(K, L)$ is decomposable in $\mathcal{K}(K, L)$ iff $\underline{S}$ is decomposable in $\mathcal{M}(R)$. Also, for S, $S' \in \mathcal{K}(K, L)$, S and S' are isomorphic iff $\underline{S}$ and $\underline{S}'$ are isomorphic as R-modules.*

The model theory of modules is well developed. Unfortunately the mapping $\mathcal{K}(K, L) \ni S \mapsto \underline{S} \in \mathcal{M}(R)$ is not onto. We shall rely on the following results.

THEOREM 1.2. ([PR, 2.1]). *The following conditions on a ring are equivalent.*

(i) Every R-module is a direct sum of indecomposable modules.

(ii) Every R-module is totally transcendental.

If R satisfies these conditions then every indecomposable R-module is finitely generated, and there are at most $|R| + \aleph_0$ indecomposable R-modules.

THEOREM 1.3. ([BK, 8.7] or [Pr, 2.10]). *The following conditions are equivalent.*

(i) Up to isomorphism, there are countably many countable R-modules.

(ii) There are $< 2^{\aleph_0}$ countable R-modules.

(iii) R is of finite representation type (i.e., every R-module is a direct sum of indecomposable modules and there are finitely many indecomposable R-modules).

Let $\mathcal{M}'(R) = \{\underline{S} : S \in \mathcal{K}(K, L)\}$. We say that $\mathcal{K}(K, L)$ is of finite representation type if every (K, L)-structure is a direct sum of indecomposables, and there are finitely many indecomposable (K, L)-structures. Otherwise we say that $\mathcal{K}(K, L)$ is of infinite representation type. Theorems 1.2 and 1.3 deal with $\mathcal{M}(R)$. However, their proofs work as well for the smaller class $\mathcal{M}'(R)$ (this may be checked directly). Hence, modulo Remark 1.1 we get a proof of Theorem 1.4 below. We shall give also another, more direct proof of this theorem based on Theorems 1.2 and 1.3.

THEOREM 1.4. *The following conditions on K, L are equivalent.*

(i) Up to isomorphism, there are countably many countable (K, L)-structures.

(ii) There are $< 2^{\aleph_0}$ countable (K, L)-structures.

(iii) $\mathcal{K}(K, L)$ is of finite representation type.

If K, L satisfy these equivalent conditions, then every indecomposable (K, L)-structure is finitely generated.

Proof. We will show that every R-module N is a direct sum of M_0 and M_1, where $M_0 \in \mathcal{M}'(R)$, $M_1 \in \mathcal{M}(R)$, and $M_1 \subseteq \mathrm{Ker}_N(I_K + I_{R_0})$. Hence R acts on M_1 as I_L and M_1 is essentially an L-space. Modulo Theorems 1.2, 1.3, and Remark 1.1 this will prove Theorem 1.4.

We have $R + I_K + I_{R_0} + I_L$, $I_L I_K I_K I_L = I_{R_0} I_{R_0} = I_L I_{R_0} = 0$. Let $N_1 = \mathrm{Ker}_N(I_K + I_{R_0})$, $N_2 = \mathrm{Ker}_N(I_{R_0} + I_L)$, $N_3 = I_L N$, $N_4 = (I_{R_0}\text{-span})N_3$, where $(I_{R_0}\text{-span})N_3$ is the subgroup of N generated by $I_{R_0} N_3$. Notice that N_3 is a subgroup of N. N_1, N_2, $N_3 + N_4$, and N_4 are submodules of N. $N_1 \cap N_2 = 0$.

The action of R on N_2 is that of I_K, hence N_2 is essentially a K-space. As $(I_{R_0}+I_L)(I_K+I_{R_0})=0$, $(I_K+I_{R_0})N\subseteq N_2$, hence $N=RN=I_1N+(I_K+I_{R_0})N=N_3+N_2=(N_3+N_4)+N_2$. Let $N_2'=(N_3+N_4)\cap N_2$. As N_2 is essentially a K-space, we can find N_2'' such that $N_2=N_2'\oplus N_2''$. Hence $N=(N_3+N_4)\oplus N_2''$. $N_1\cap(N_2''+N_4)=0$, because $N_2''+N_4\subseteq N_2$. Let $N_1'=N_1\cap N_3$. N_1' is a submodule of N_3+N_4, and $(I_K+I_{R_0})N_1'=0$. Also, N_1' is an I_L-space. Choose a subgroup $N_3'<N_3$ so that $N_3=N_1'+N_3'$, $I_LN_3'\subseteq N_3'$ and $N_1'\cap N_3'=0$. Then still $N_4=(I_{R_0}\text{-span})N_3'$, and $N_3'+N_4$ is an R-module. We show that $N_3+N_4=N_1'\oplus(N_3'+N_4)$.

Indeed, if $a\in N_3'$, $b\in N_4$, and $a+b\in N_1'$, then as $I_LN_4=0$, $I_La=I_L(a+b)\subseteq N_1'$, Also, $I_La\subseteq N_3'$, hence $I_La=I_L(a+b)=0$. But N_1' is an I_L-space, so $a+b=0$. As $I_KI_L=0$, $I_KN_3'=0$. $N_4\subseteq \operatorname{Ker}(I_{R_0}+I_L)$, hence $N_3'\cap N_4'=0$. This implies $a=b=0$. Hence we get

$$N=N_1'\oplus N_2''\oplus(N_3'+N_4)\quad\text{and}\quad N_3'\cap N_4=0.$$

Now let $M_1=N_1'$, $M_0=N_2''\oplus(N_3'+N_4)$. It suffices to show $M_0\in\mathcal{M}'(R)$. Obviously, $N_2''\in\mathcal{M}'(R)$ ($N_2''\cong\begin{pmatrix}V\\0\end{pmatrix}$ for some V), so it suffices to show $N_3'+N_4\in\mathcal{M}'(R)$. Consider the mapping

$$\varphi: N_3'+N_4\to\begin{pmatrix}N_4\\(N_4)^{n^*}\end{pmatrix}$$

defined by $\varphi(x)=\begin{pmatrix}x\\0\end{pmatrix}$ for $x\in N_4$, and $\varphi(x)=\begin{pmatrix}0\\(A_kx)_{1\leq k\leq n^*}\end{pmatrix}$ for $x\in N_3'$, where $A_k=(a^k_{ij})\in R$ is defined by $a^k_{ij}=1$ if $i=0$, $j=k+1$ and $a^k_{ij}=0$ otherwise. If $x\in N_3'$, $y\in N_4$, let $\varphi(x+y)=\varphi(x)+\varphi(y)$. We check that φ is 1-1. It suffices to see that $\varphi\restriction N_3'$ is 1-1. If $x\in N_3'$ and $\varphi(x)=0$, then $I_{R_0}x=0$. Also, $I_Kx=0$ as $x\in I_LN$ and $I_KI_L=0$, hence $X\in N_1$, contradicting the choice of N_3'. Let $\begin{pmatrix}V\\W\end{pmatrix}$ be the image of φ. V is a K-space, and by direct checking we see that φ translates the action of I_L on N_3' into the action of $L\cong I_L$ on $\begin{pmatrix}0\\W\end{pmatrix}$, hence W is an L-subspace of V^{n^*}, and $\begin{pmatrix}V\\W\end{pmatrix}\in\mathcal{M}'(R)$. It is easy to see that φ is an isomorphism of R-modules.

In Theorem 1.4 we reduced the problem of counting countable (K,L)-structures to determining representation type of $\mathcal{K}(K,L)$. If this representation type is infinite then there are $2^{\aleph_0}$ countable (K,L)-structures, and we get $I(T,\aleph_0)=2^{\aleph_0}$ as well (at least when K, L are finite). If $\mathcal{K}(K,L)$ has finite representation type then there are countably many countable (K,L)-structures, also there are finitely many finite dimensional indecomposable (K,L)-structures and every (K,L)-structure can be presented as a direct sum of indecomposables. As in the proof of [Pr, 2.10], this decomposition is unique up to isomorphism, that is if $\oplus_{i\in I}(K_i, L_i) = \oplus_{j\in J}(K_j, L_j)$ and (K_i, L_i), (K_j, L_j) are indecomposable then there is a bijection $f : I \to J$ such that (K_i, L_i) is isomorphic to $(K_{f(i)}, L_{f(i)})$. Still in the case of finite representation type of $\mathcal{K}(K, L)$ we can not determine $I(p, \aleph_0)$. We need a more detailed information, furnished in [DR]. $\mathcal{K}(K, L)$ is a special case of the structures considered in [DR]. Unfortunately, Dlab and Ringel assume throughout that there is a central field F contained in $K \cap L$ such that $[K : F]$ and $[L : F]$ are finite. In our case, in general probably we can not hope for that much. However, this assumption is obviously true if both K and L are finite, and as we indicate below, is true also when $n^* = 1$ and K is a field.

Our (K, L)-structures correspond to representations of "F-species" $\mathcal{S} = (L, K, {}_K M_L)$, where $M = R_0$ is a K, L-bimodule: K acts on M in the natural way, L acts on M by matrix multiplication on the right. *Representation* of $\mathcal{S}$ is a tuple $({}_L W, {}_K V, \varphi)$, where $\varphi : {}_K(M \oplus {}_L W) \to {}_K V$. Let $\mathfrak{R}$ be the category of representations of $\mathcal{S}$. Let $\mathfrak{Rm}$ be those representations of $\mathcal{S}$ for which the adjoint mapping $\varphi^* : {}_L W \to \mathrm{Hom}_K({}_K M_L, {}_K V)$ of φ is monomorphism. Elements of $\mathfrak{Rm}$ correspond precisely to R-modules of the form $\begin{pmatrix} V \\ W \end{pmatrix}$. As mentioned before Proposition 5.2 in [DR], $\mathfrak{R}$ is of finite representation type iff $\mathfrak{Rm}$ is of finite representation type. Our feeling is that $n^* > 1$ should imply $I(T, \aleph_0) = 2^{\aleph_0}$. We were only able to prove this for finite K, L. In fact by [Ne4, 3.11], K is finite iff L is finite (in [Ne4, §3], F_q, F_p stand for K, L respectively).

PROPOSITION 1.5. *If $n^* > 1$ and K, L are finite then $\mathcal{K}(K, L)$ has infinite representation type. In particular, in this case $I(T, \aleph_0) = 2^{\aleph_0}$.*

Proof. Let ${}_L M' = M^*_{n\times 1}(K)$, L acts on M' by matrix multiplication on the left. By [Ne4, 3.11], $\dim({}_L M') \geq 2$. Now, K, L being finite implies that also $\dim(M_L) \geq 2$. Thus $\dim({}_K M) \times \dim(M_L) \geq 4$, hence the assumptions of [DR, 5.2] are satisfied. This proposition (particularly part (ii) of its proof) gives that $\mathfrak{Rm}$, hence also $\mathcal{K}(K, L)$, has infinite representation type.

Now let us discuss the case $n^* = 1$. Then the situation is much simpler; we have just $L \subseteq K$. [Ne4] gives us in this case that (in $T(C \cup E)$) every $r \in R$ is ACL-interdependent with a stationarization of some p_a and all stationarizations of a single p_a are ACL-interdependent (hence we can consider ACL as a dependence on Q: $a \in \mathrm{ACL}(B)$ iff $p_a \in \mathrm{ACL}(\{p_b : b \in B\})$. $Q \cup \{0\}$ is a K-vector space, hence also an L-space, and ACL-dependence on Q is just L-linear dependence.

By [Ne1] or [Bu3] we know that L is a locally finite field. By [Ne4, 0.3], $\dim({}_L K)$ is finite. By [Bu4], K is also a locally finite field. The elements of $\mathcal{K}(K, L)$ are what Dlab and Ringel call in [DR] representations of L-structures. From [DR, Theorem A] we get the following.

COROLLARY 1.6. *If $n^* = 1$, K is a field, and $[K : L] \geq 4$ then $\mathcal{K}(K, L)$ has infinite representation type. If $[K : L] \leq 3$ then $\mathcal{K}(K, L)$ has finite representation type. Hence if $n^* = 1$, K is a field, and $[K : L] \geq 4$ then $I(T, \aleph_0) = 2^{\aleph_0}$.*

In case $n^* = 1$, that $[K : L] \geq 4$ implies $I(T, \aleph_0) = 2^{\aleph_0}$ was proved also in [Bu4, §4]. In case $n^* = 1$, and $[K : L] \leq 3$ we get that there are countably many countable (K, L)-structures. This still does not automatically yield the value of $I(p.\aleph_0)$, as we have added parameters $C \cup E$ to the signature. In case when $[K : L] = 2$ or 1, we proved in [Ne4] that $I(T, \aleph_0) < 2^{\aleph_0}$ implies $I(p, \aleph_0) = \aleph_0$. In case when $n^* = 1$ and $[K : L] = 3$ we shall prove this in the next section. We shall rely on the special form of decomposition of a (K, L)-structure into indecomposables, implied by the proof of Proposition 4.2 in [DR]. From now on in this section we assume that $n^* = 1$, K is a field, and $[K : L] = 3$. Let $\{1, e, f\}$ be a basis of K as an L-vector space. In [DR, 4.2] it is proved that there are exactly five indecomposable (K, L)-structures: $\alpha^0 = (K, K)$, $\alpha^1 = (K, L + eL)$, $\alpha^2 = (K \times K, (L \times L) + (e, f)L)$, $\alpha^3 = (K, L)$, and $\alpha^4 = (K, 0)$.

For (K, L)-structures $(V, W), (V', W')$ we say that (V', W') is a strong substructure of (V, W) $((V', W') < (V, W))$ if V' is a K-subspace of V and $W' = W \cap V'$ (that is we regard W in (V, W) as a predicate). For $\alpha \in \mathcal{K}(K, L)$ we stipulate $\alpha = (V_\alpha, W_\alpha)$. Let $S = (V, W)$ be a (K, L)-structure. We will indicate an algorithm of decomposing S. We define by induction sets X_0, X_1, X_2, X_3, X_4. Let $X_i = \{\alpha \in \mathcal{K}(K, L) : \alpha < S, \alpha \cong \alpha^i$ and $V_\alpha \cap K\text{-span}(X_{<i}) = 0\}$. Here K-span$(X_{<i})$ is the K-subspace of V generated by $\bigcup\{V_\alpha : \alpha \in X_j, j < i\}$. If $\alpha, \beta \in \mathcal{K}(K, L)$, $\alpha, \beta < S$, then let $\alpha + \beta = (V_\alpha + V_\beta, W_\alpha + W_\beta)$, if $X \subseteq \mathcal{K}(K, L)$ is a family of $\alpha < S$, then let $\Sigma X = (\Sigma\{V_\alpha : \alpha \in X\}, \Sigma\{W_\alpha : \alpha \in X\})$.

CLAIM 1.7.

(1) For $\alpha < S$, $\alpha \in X_i$ iff $\alpha \cong \alpha^i$ and $V_\alpha \not\subseteq K$-span$(X_{<i})$.

(2) $\Sigma X_{<i} < S$.

(3) Assume $\beta_0, \ldots, \beta_n \in X_i$, $V_{\beta_0} \cap K$-span$(X_{<i} \cup \{\beta_1, \ldots, \beta_n\}) \neq 0$. Then $V_{\beta_0} \subseteq K$-span$(X_{<i} \cup \{\beta_1, \ldots, \beta_n\})$.

(4) Assume $\beta_1, \ldots, \beta_n \in X_i$. Then $\Sigma(X_{<i} \cup \{\beta_1, \ldots, \beta_n\}) < S$.

Proof. The proof is a modification of [DR, 4.2]. $\{1, e, f\}$ is a basis of K over L. We proceed by induction on i. For $i = 0$ the claim is easy. Also, (2) follows always from the induction hypothesis, and except for $i = 2$, $\dim(V_{\alpha^i}) = 1$, hence for $i \neq 2$, (1) and (3) are trivial. Let $i = 1$. We need to prove (4). We proceed by induction on n. Without loss of generality (wlog) K-span$(X_{<i})$ has dimension $k < \omega$. Let $\Sigma X_{<i} = (V_0, V_0)$, $\beta_j = (V_j, W_j)$. By (3) and the inductive hypothesis we can assume that $V_0, V_1, \ldots, V_n$ are independent. Let $W' = W \cap V'$ where $V' = V_0 + \cdots + V_n$. Suppose $W' \neq V_0 + W_1 + \cdots + W_n$. We know that $\dim_L(V_0 + W_1 + \cdots + W_n) = 3k + 2n$, hence $\dim_L(W') > 3k + 2n$, Thus $e^{-1}W' \cap f^{-1}W' \cap W'$ properly extends V_0. Let $u \in W' \cap e^{-1}W' \cap f^{-1}W' \setminus V_0$. Then u, eu, $fu \in W$, hence $(Ku, Ku) \in X_0$, contradicting $u \notin V_0$.

Now let $i = 2$.

(1) $\leftarrow$. Suppose $\Sigma X_0 = (V_0, V_0)$. Wlog K-span$(X_{<2})$ has finite dimension. Assume $\beta_1 = (V_1, W_1), \ldots, \beta_n = (V_n, W_n) \in X_1$ are independent (i.e., $V_t \cap (V_0 + \cdots + V_{t-1}) = 0$), $\alpha \in X_2$, $V_\alpha \cap (V_0 + \cdots + V_n) \neq 0$. Let $W' = W \cap V'$ where $V' = V_0 + \cdots + V_n + V_\alpha$, $\dim_K(V_0) = k$. So $\dim_L(W') \geq 3k + 2(n+1)$, and $\dim_L(V') = 3k + 3(n+1)$. $V_0 \subseteq W' \cap e^{-1}W'$ and $\dim_L(W' \cap e^{-1}W') \geq 3k + n + 1$. Let $u_t \in W_t \cap e^{-1}W'$, $t = 1, \ldots, n$, and $u_\alpha \in W' \cap e^{-1}W' \setminus (V_0 + L\text{-span}(u_1, \ldots, u_n))$.

If $u_\alpha \in V_0+\cdots+V_n$, we get a contradiction as in case 1. Hence $u_\alpha \notin V_0+\cdots+V_n$. Also, $u_\alpha, eu_\alpha \in W$, hence $(Ku_\alpha, Lu_\alpha + Leu_\alpha) \in X_1$, a contradiction.

We prove (3) and (4) simultaneously, by induction on n. As above, wlog K-span$(X_{<i})$ has finite dimension. For $n = 0$ we are done by (1). So we can assume that K-span$(X_{<i}), V_1, \ldots, V_n$ are independent, where $\beta_t = (V_t, W_t)$. Now if $V_0 \cap (K\text{-span}(X_{<i}) + V_1 + \cdots + V_n) \neq 0$ or $\Sigma(X_{<i} \cup \{\beta_1, \ldots, \beta_n\}) \not< S$ then as in the proof of (1) we get a $u \in (K\text{-span}(X_{<i}) + \cdots + V_n) \setminus K\text{-span}(X_{<i})$ such that $(Ku, Lu + Leu) \in X_1$, a contradiction.

Let $i = 3$. We need to prove (4). We can assume K-span$(X_{<i})$ has finite dimension. Let $\beta_n = (V_n, W_n), \Sigma X_{<i} = (V_0, W_0), V' = V_0 + \cdots + V_n$, $W' = W \cap V'$. Let $X = W_0 + W_1 + \ldots W_n$. We have $X + eX + fX = V'$. Suppose $X \neq W'$. Then we can choose $v \in W' \setminus X$, and $v = v_0 + ev_1 + fv_2$ for some $v_0, v_1, v_2 \in X$. Replacing v by $v - v_0$ we can assume $v_0 = 0$, and $v + ev_1 + fv_2$. If $v_1, v_2 \in W_0$, we get a contradiction with the inductive hypothesis (Claim 1.7(4)). If $v_1, v_2 \notin W_0$, then v, v_1, v_2 give rise to an $\alpha < S$ with $V_\alpha \cap V_0 = 0$ and $\alpha \in X_{<i}$, a contradiction. If, say, $v_1 \in W_0$ and $v_2 \notin W_0$ then $(Kv_1 + Kv_2, Lv_1 + Lv_2 + Lv) \in X_2$, a contradiction.

The case when $i = 4$ is trivial.

REMARK. The above claim is true as well when K, L are only division rings, and there is a central subfield F of both K and L such that $[K : F]$ and $[L : F]$ are finite.

Claim 1.7 justifies the following algorithm for decomposing $S = (V, W) \in \mathcal{K}(K, L)$ into a direct sum of (K, L)-structures of type $\alpha^0, \ldots, \alpha^4$. Suppose $Y_i = \{\beta_i^j : j < n_i\} \subseteq X_i$, $i < 5$, satisfy the condition: $V_{\beta_i^j} \not\subseteq K\text{-span}(X_{<i} \cup \{\beta_i^t : t < j\})$. Then $(V, W) = \oplus\{\beta_i^j : i < 5, j < n_i\}$. This algorithm enables us in the next section to get rid of the parameters $C \cup E$ added to the signature and determine $I(p, \aleph_0)$. More generally, if K, L are arbitrary, $R = \begin{pmatrix} K & R_0 \\ 0 & L \end{pmatrix}$ is of finite representation type, $\alpha_0, \ldots, \alpha_n$ are the indecomposable R-modules, and the counterpart of 1.7 holds then we can also get rid of the parameters $C \cup E$ and determine $I(p, \aleph_0)$. The reason for that is that the algorithm shown above is "cumulative." It is not clear to us if such an algorithm may be found for any ring R of finite type.

§2. Getting rid of the parameters. In this section we show how to prove that $I(p, \aleph_0)$ is countable in the case when K, L are fields, $n^* = 1$, and $[K : L] = 3$. So from now on as far as Theorem 2.5 we assume that $n^* = 1$, K is a field, and $[K : L] = 3$. By the discussion in the previous section we know that for some finite $C \cup E$, $I(p \restriction C \cup E, \aleph_0)$ is countable, and isomorphism types of sets of realizations of $p \restriction C \cup E$ in countable models of T correspond to isomorphism types of (K, L)-structures. The algorithm of decomposition of any (K, L)-structure into indecomposables from §1 enables us to find a decomposition of $p(M)$. More precisely we find essentially finitely many indiscernible sets $I_1, \ldots, I_k$ such that $p(M)$ is prime over $I_1 \cup \cdots \cup I_k$. The proof we give here is a variant of the reasoning from [Ne4, §4] (see Fact 4.8, Lemmas 4.9, 4.11, and Theorem 4.13 there).

We return now to the original meaning of Q, i.e., $Q = q(M)$ for some M, $\dim(Q) = \aleph_0$, q is a stationary, modular type over $\emptyset$, generic of a connected, type-definable over $\emptyset$ weakly minimal (w.m.) group $G = (G, +)$. K is the division ring of pseudo-endomorphisms of G, so that $Q \cup \{0\}$ is a vector space over K.

Let $\{E_n : n < \omega\}$ be an enumeration of $FE(\emptyset)$. Let $\mathrm{acl}_n(\emptyset) = \{a/E_k : a \in \mathfrak{C}, k \leq n\}$. Notice that $\mathrm{acl}_n(\emptyset)$ is finite. Assume A is a finite subset of Q, $R \subseteq P_A$ is finite, ACL-independent and such that $R \perp P_b$ for some (any) $b \in Q \setminus \mathrm{acl}(A)$ (by the local character of ACL it implies that $R \perp P_B$ for any $B \subseteq Q$ with $B \perp A$). Let B be a selector from $\{s(\mathfrak{C}) : s \in R\}$ and $r = \mathrm{tp}(B/A)$. By the transitivity of finite multiplicity ([PS] or [Sa, 1.5]), $\mathrm{Mlt}(BA/\emptyset)$ is finite. We say that r and (R, A) are n-determined if $\mathrm{tp}(BA/\mathrm{acl}_n(\emptyset))$ is stationary. Also, we say that (R, A) corresponds to $\mathrm{tp}(BA/\mathrm{acl}_n(\emptyset))$. This definition tacitly assumes an enumeration of A and R.

FACT 2.1. *(1) n-determined implies k-determined for $k > n$.*
(2) Every r is n-determined for some n.
(3) If $C \perp BA$ then every completion of r over $A \cup acl_n(\emptyset)$ is weakly orthogonal to $tp(C/A \cup \mathrm{acl}_n(\emptyset))$.

Proof. Easy.

Fix a finite $E \subseteq Q$ large enough, so that if R is a basis of P_E and C is a selector from $\{s(\mathfrak{C}) : s \in R\}$, then over $C \cup E$ the translation Φ of ACL-dependence on P_Q into L-dependence on Q works. Assume A is a finite subset of Q independent from E, $R \subseteq P_{AE} \setminus P_E$ is finite, ACL-independent, with $R \perp P_E$, and B is a selector from $\{s(\mathfrak{C}) : s \in\}$. We say that (R, A) and (B, A) are of type α^i if the (K, L)-structure (V, W) corresponding to $\mathrm{ACL}(R \cup P_E)$ (through Φ) is isomorphic to α^i (α^i, $i < 5$, are defined before Claim 1.7). This implies of course $\dim(A/E) \leq 2$. Let M be a countable model of T with $Q = q(M)$. For $A \subseteq Q$ let $P_A^M = \{r \in P_A : r \text{ is realized in } M\}$. Notice that P_A^M is ACL-closed in P_A. We define by induction sets X_i^M, $i < 5$, corresponding to sets X_i defined before Claim 1.7. Let $X_i^M = \{A \subseteq Q : \text{for some } R \subseteq P_{AE}^M, (R, A) \text{ is of type } \alpha^i \text{ and } A \perp X_{<i}^M(E)\}$, where $X_{<i}^M = \bigcup_{j<i} X_j^M$. Applying Claim 1.7 we get

CLAIM 2.2.
(1) For $A \subseteq Q$, $A \in X_i^M$ iff for some $R \subseteq P_{AE}^m$, (R, A) is of type α^i and $A \not\subseteq acl(\bigcup X_{<i}^M \cup E)$.
(2) Choose for any j and $A \in X_j^M$ an $R_A \subseteq P_{AE}^M$ witnessing $A \in X_j^m$. Then $P_{EX_{<i}^M}^M = ACL(P_E^M \cup \bigcup\{R_A : A \in X_{<i}^M\}) \cap P_{EX_{<i}^M}$.
(3) Assume $A_0, \dots, A_n \in X_i^M$, $A_0 \not\perp (X_{<i}^M \cup A_1 \cup \dots \cup A_n)(E)$. Then $A_0 \subseteq acl(X_{<i}^M \cup A_1 \cup \dots \cup A_n \cup E)$.
(4) Assume $A_1, \dots, A_n \in X_i^M$. Then $P_{EX_{<i}^M \cup A_1 \cup \dots \cup A_n}^M \subseteq ACL(P_{EX_{<i}^M}^M \cup R_{A_1} \cup \dots \cup R_{A_n})$, where R_A are chosen as in (2).

The next lemma corresponds to Lemmas 4.9 and 4.11 in [Ne4].

LEMMA 2.3. *Let* $R_n = ACL(P_E^M \cup \bigcup\{R$ *: for some* A, (R, A) *witnesses* $A \in X_i$ *for some* $i < 5$, *and* (R, AE) *is* n*-determined*$\}$. *Then for some* $n < \omega$, $P_Q^M \subseteq R_n$.

Proof. Suppose not. Then we can choose n_i, $i < \omega$, so that n_i is the minimal k such that for $j = i - 1$, $R_k \neq R_j$. Of course, $P_Q^M \subseteq \bigcup_n R_n$. For $I < \omega$ choose $A_i \in X_j^M$ for some $j < 5$ and R_{A_i} witnessing $A_i \in X_j^M$ so that $(R_{A_i}, A_i E)$ is n_i-determined, and $R_{A_i} \not\subseteq R_{n_{i-1}}$.

For $X \subseteq \omega$ let $R_X = \mathrm{ACL}(P_E^M \cup \bigcup\{R_{A_i} : i \in X\})$. By the omitting types theorem we can find a model M' with $Q = q(M')$ and $P_Q^{M'} = R_X$ and (modulo Claims 1.7 and 2.2) as in the proof of [Ne4, 4.9], we can recover X from (M', E). This contradicts $I(T, \aleph_0) < 2^{\aleph_0}$.

Fix $n^0 < \omega$ with $P_Q^M \subseteq R_{n^0}$. Let $\{r_k : k < \omega\}$ be an enumeration of $\bigcup_{n<\omega} S_n(\mathrm{acl}_{n^0}(\emptyset))$. The proof of the next lemma is similar to that of Lemma 2.3.

LEMMA 2.4. *Let* $R_{n^0}^k = ACL(P_E^M \cup \bigcup\{R$ *: for some* A, (R, A) *witnesses* $A \in X_i^M$ *(for some* $i < 5$*)*, (R, AE) *is* n^0*-determined and corresponds to some* r_i, $i < k\})$. *Then for some* $k < \omega$, $P_Q^M \subseteq_{n^0}^k$.

Fix $k^0 < \omega$ with $P_Q^M \subseteq R_{n^0}^{k^0}$.

THEOREM 2.5. *If* $n^* = 1$, K, L *are fields, and* $[K : L] = 3$ *then* $I(p, \aleph_0)$ *is countable.*

Proof. We can absorb E and $\mathrm{acl}_{n^0}(\emptyset)$ into the signature. We need to find a finite set of invariants determining $p(M)$ up to isomorphism. In order to characterize P_Q^M, we shall decompose it into indecomposables according to the algorithm from §1. We define by induction on $j < k^0$ sets $X_{ij}^M = \{A \in X_i^M :$ for some R witnessing $A \in X_i^M$, (R, AE) is n^0-determined, corresponds to r_j, and $(*)$ $A \perp X_{<i}^M \cup X_{i,<j}^M(E)\}$, where $X_{i,<j}^M = \bigcup_{t<j} X_{it}^M$. By Claim 1.7, $(*)$ in the definition of X_{ij}^M is equivalent to $A \not\subseteq \mathrm{acl}(X_{<i}^M \cup X_{i,<j}^M \cup E)$. Now define by induction sets $A_{ij}^t \in X_{ij}^M$, $t < n_{ij} \leq \omega$, so that $A_{ij}^t \perp X_{<i}^M \cup X_{i<j}^M \cup A_{ij}^0 \cup \cdots \cup A_{ij}^{t-1}(E)$, and $X_{ij}^M \subseteq \mathrm{acl}(X_{<i}^M \cup X_{i,<j}^M \cup \bigcup\{A_{ij} : t < n_{ij}\} \cup E)$.

Let R_{ij}^t witness $A_{ij}^t \in X_{ij}^M$, and let B_{ij}^t be a selector from $\{s(\mathfrak{C}) : s \in R_{ij}^t\}$. Then by Fact 2.1 and Claims 1.7 and 2.2 we have:

(1) $P_Q^M = \mathrm{ACL}(P_E^M \cup \bigcup\{R_{ij}^t : i < 5, j < k^0, t < n_{ij}\})$,
(2) $\{B_{ij}^t A_{ij}^t E : t < n_{ij}\}$ is a Morley sequence in r_j,
(3) $\{B_{ij}^t A_{ij}^t E : i < 5, j < k^0, t < n_{ij}\}$ is independent.

It follows that the isomorphism type of $p(M)$ is determined by n^0, k^0, $n_{ij}(I < 5, j < k^0)$, the isomorphism type of P_E^M, and the dimension of $r(M)$ for any $r \in P_Q^M$. We see that that there are only countably many possibilities.

At the beginning we assumed that p_a is non-isolated. Now we will show how to omit this assumption, at least in some cases. So we assume now that

p is a stationary, non-isolated complete type over $\emptyset$ of U-rank 2 and p is not almost orthogonal to some $q \in S(\emptyset)$ of U-rank 1. For b satisfying p choose a realizing q with $a \in \mathrm{acl}(b)$, and let $p_a = \mathrm{tp}(b/a)$. Now we admit that p_a is possibly isolated. Again, we can dispose easily with some trivial cases, and after some manipulation we can assume that q is stationary, and p_a is non-trivial, properly weakly minimal. Again we may restrict ourselves to the case when $Q = q(M)$ has dimension $\aleph_0$, and now we focus our attention on P^* (defined in the introduction). If $p_a \perp p_b$ for $a, b \in Q$ with $a \downarrow b$, then a standard argument (using ideas from [Bu2]) shows that either $I(T,\aleph_0) = 2^{\aleph_0}$ or $I(p,\aleph_0) \leq \aleph_0$. Indeed, for $a \in Q$ consider the set $S_a(M) = \{p_b(M) : b \in \mathrm{acl}(a) \cap Q\}$. This set is a countable union of w.m. sets and we can apply [Bu2] to show that there are (in T^{eq}) properly w.m. types $q_0, \ldots, q_{n-1}$ over a, of finite multiplicity, such that for any M with $Q = q(M)$, $S_a(M)$ is prime over $\{a\} \cup I_0 \cup \cdots \cup I_{n_1}$, where I_j is a Morley sequence in q_j. As for $a \downarrow b$, $p_a \perp p_b$, the structures of $S_a(M)$ and $S_b(M)$ may be chosen independently, hence unless $n = 0$ and every $S_a(M)$ is prime just over a, we get $I(T,\aleph_0) = 2^{\aleph_0}$.

But in case when $S_a(M)$ is prime over a, $S_a(M)$ is unique up to isomorphism, hence we get $I(p,\aleph_0) \leq \aleph_0$.

So we can assume that $p_a \not\perp p_b$ for $a, b \in Q$ with $a \downarrow b$. As for c realizing p_b, $b \in \mathrm{acl}(c)$ and $\mathrm{tp}(c/\emptyset)$ is stationary, we get easily that for all $a, b \in Q$, all stationarizations of p_a, p_b are non-orthogonal. Hence P^* is a family on non-orthogonal types.

FACT 2.6. *If $n(r) = 0$ for every $r \in P^*$ then for some finite $C \subseteq acl(\emptyset)$ there are stationary $q_0, \ldots, q_n \in S(C)$ such that for every M with $Q = q(M)$, $P^*_Q(M) \subseteq acl(I_0 \cup \cdots \cup I_n \cup C) \subseteq M$ for some Morley sequences $I_0, \ldots, I_n$ in $q_0, \ldots, q_n$ respectively.*

Proof. ACL-dimension of $P^*_\emptyset$ is finite (see [Ne4, 0.3] or [Bu2, 5.2(a)]).

Later we shall see how to conclude the computation of $I(p,\aleph_0)$ in case when $n(r) = 0$ for any $r \in P^*$. Now assume $n(r) > 0$ for some $r \in P^*$. Then by the proofs in [Bu4, §2], Q is locally modular. Of course we can assume that Q is non-trivial, hence again we can assume that q is modular and generic of a weakly minimal type-definable over $\emptyset$ group $G = (G, +)$. Let K be the division ring corresponding to q, and L be the division ring corresponding to any stationarization of p_a. We shall prove the following theorem.

THEOREM 2.7. *If K, L are finite and $DIM(P^*_a) > 1$ then $I(p,\aleph_0) \leq \aleph_0$.*

Proof. In [Ne4] we worked with P_Q, translating the ACL-dependence on P_Q into a linear dependence. But the same proofs work for P^*_Q as well, hence we get an $n^* = \max\{n(r) : r \in P^*\}$, and an embedding of L into the ring of matrices $M_{n^* \times n^*}(K)$ so that the ACL-dependence on P^* translates into L-dependence on $(Q \cup \{0\})^{n^*}$. Now by [DR], unless $n^* = 1$ and $[K : L] \leq 3$, $I(T,\aleph_0) = 2^{\aleph_0}$ (because $\mathcal{K}(K, L)$ is of infinite representation type then). If $n^* = 1$ and $[K : L] \leq 3$ then [Ne4, 4.13] and the proof of Theorem 2.5 show that there is a finite set $C \subseteq \mathrm{acl}(\emptyset)$, and finitely many stationary types $q_0, \ldots, q_n$ such that for every M with $Q = q(M)$ there are Morley sequences $I_0, \ldots I_n \subseteq M$ in $q_0, \ldots, q_n$ respectively such that $P^*(M) \cup Q \subseteq \mathrm{acl}(I_0 \cup \cdots \cup I_n)$. Notice that this is also the conclusion of Fact 2.6. As in [Bu2, §5] we prove the following claim which concludes the proof of the theorem and shows that when the assumptions of Claim 2.6 hold then $I(p, \aleph_0) \leq \aleph_0$ as well.

CLAIM 2.8. *Assume $Q = q(M)$, $I_0, \ldots, I_m \subseteq M$ are Morley sequences in $q_0, \ldots, q_n$ respectively, and $P^*(M) \cup Q \subseteq acl(I_0 \cup \cdots \cup I_n)$. Then $Q \cup \bigcup_{a \in Q} p_a(M)$ is prime (i.e., atomic) over $I_0 \cup \cdots \cup I_n$.*

Proof. Let $N \subseteq M$ be a prime model over $I_0 \cup \cdots \cup I_n$ such that $\bigcup_{a \in Q} p_a(M)$ is maximal. We have to show $Q \cup \bigcup_{a \in Q} p_a(M) \subseteq N$. Suppose not. Take $b \in p_a(M) \backslash N$. As in the proof of [Bu2, 5.1], there is a finite $B \subseteq \bigcup_{a \in Q} p_a(N) \cup Q$ such that $\mathrm{tp}(b/B)$ is non-isolated. We can assume also that $a \in B$, and for every $c \in B \backslash Q$ there is $d \in B$ such that c realizes p_d. Let $D = B \cap Q$. Applying [Bu2, 4.1] to $T(D)$, we get an $r \in P^*_D$ such that $r \upharpoonright B \not\perp^a \mathrm{stp}(b/B)$. So there is c realizing r with $c \not\perp b(B)$. Thus $c \in M \backslash N$. On the other hand, $c \in P^*(M) \subseteq \mathrm{acl}(I_0 \cup \cdots \cup I_n) \subseteq N$, a contradiction.

If T is superstable then in general there are no prime models over infinite sets. However, if in addition T is small then there are prime models over indiscernible sets. The first author conjectures that in case of a model M of a superstable T of finite U-rank, if $A \subseteq M$ is a skeleton of M, then, although M may not be prime over A, there are finitely many Morley sequences $I_0, \ldots, I_n \subseteq M$ with $A \subseteq \mathrm{acl}(I_0 \cup \cdots \cup I_n)$ such that M is prime over $I_0 \cup \cdots \cup I_n$. This seems to work in case of classification of types of U-rank 2.

REFERENCES

[Ba] J. T. BALDWIN, ***Fundamentals of stability theory***, Springer 1987.

[BK] J. T. BALDWIN, R. N. MCKENZIE, *Counting models in universal Horn classes*, ***Algebra universalis***, vol. 15 (1982), pp. 359–384.

[Bu1] S. BUECHLER, *The geometry of weakly minimal types*, **The journal of symbolic logic**, vol. 50 (1985), pp. 1044–1054.

[Bu2] S. BUECHLER, *The classification of small weakly minimal sets I*, ***Classification theory, Chicago 1985***, ed. J. Baldwin, Springer 1987, pp. 32–71.

[Bu3] S. BUECHLER, *Classification of small weakly minimal sets III $\frac{1}{2}$*, manuscript.

[Bu4] S. BUECHLER, *Rank 2 types vis-a-vis Vaught's conjecture*, preprint 1989.

[DR] V. DLAB, C. M. RINGEL, *On algebras of finite representation type*, ***Journal of algebra***, vol. 33 (1975), pp. 306–394.

[H] E. HRUSHOVSKI, *Locally modular regular types*, ***Classification theory, Chicago 1985***, ed. J. Baldwin, Springer 1987, pp. 132–164.

[Ne1] L. NEWELSKI, *A proof of Saffe's conjecture*, ***Fundamenta Mathematicae***, to appear.

[Ne2] L. NEWELSKI, *Weakly minimal unidimensional formulas: a global approach*, ***Annals of pure and applied logic***, vol. 46 (1990), pp. 65–94.

[Ne3] L. NEWELSKI, *Classifying U-rank 2 types*, manuscript, February 1989.

[Ne4] L. NEWELSKI, *On U-rank 2 types*, ***Transactions of the American Mathematical Society***, submitted.

[P] A. PILLAY, ***Geometrical stability theory***, Oxford University Press, in preparation.

[PS] A. PILLAY, C. STEINHORN, *A note on non-multidimensional superstable theories*, ***The journal of symbolic logic***, vol. 50 (1985), pp. 1020–1024.

[PR] M. PREST, *Rings of finite representation type and modules of finite Morley rank*, ***Journal of algebra***, vol. 88 (1984), pp. 502–533.

[Sa] J. SAFFE, *On Vaught's conjecture for superstable theories*, preprint 1982.

[Sh] S. SHELAH, ***Classification theory***, North Holland 1978.

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