§8. Solidity and Condensation

In this section we prove the central fine structural result of the theory we are developing, namely that every 1-small mouse is k-solid for all k. We also derive, by the same method, some condensation results we shall need later. Our proofs of these facts trace back to Dodd's proof that the models of [D] satisfy the GCH.

For mice $\mathcal{M}$ up to a strong cardinal (that is, for mice $\mathcal{M}$ such that $\mathcal{J}_\kappa^{\mathcal{M}} \models$ "There are no strong cardinals" whenever $\kappa = \operatorname{crit} E$ for some extender E on the $\mathcal{M}$ sequence), our proof actually shows that $\mathfrak{C}_k(\mathcal{M})$ is an iterate of $\mathfrak{C}_{k+1}(\mathcal{M})$, with the iteration map having critical point $\geq \rho_{k+1}(\mathcal{M})$. That is, every "very small" mouse is an iterate of its core. We suspect that this is not true for arbitrary 1-small mice.

Recall that $u_0(\mathcal{M}) = \varnothing$, and that $u_k(\mathcal{M}) = \langle \rho_k(\mathcal{M}), b_0, \cdots, b_S, \rho_{k-1}^{\mathcal{M}} \rangle$ for $k \geq 1$, where $b_0 \cdots b_S$ are the solidity witnesses for $p_k(\mathcal{M})$ and the last coordinate $\rho_{k-1}^{\mathcal{M}}$ occurs only if it is defined and is smaller than $\mathrm{OR}^{\mathcal{M}}$. Thus $p_{k+1}(\mathcal{M})$ is the appropriate collapse of $\langle r, u_k(\mathcal{M}) \rangle$, where r is the $k+1$st standard parameter of $(\mathfrak{C}_k(\mathcal{M}), u_k(\mathcal{M}))$.

Recall that if $\pi : \mathcal{M} \to \mathcal{N}$ is a k-embedding, then $\pi(u_k(\mathcal{M})) = u_k(\mathcal{N})$.

Theorem 8.1. *Let $\mathcal{M}$ be a k-sound, 1-small, k-iterable premouse, where $k < \omega$. Let r be the $k+1$st standard parameter of $(\mathcal{M}, u_k(\mathcal{M}))$. Then r is $k+1$-solid and $k+1$ universal over $(\mathcal{M}, u_k(\mathcal{M}))$.*

Proof. Let $u = u_k(\mathcal{M})$ and $r = \langle \alpha_0, \cdots, \alpha_S \rangle$, with the ordinals α_s in decreasing order. Let $\alpha_{S+1} = \rho_{k+1}^{\mathcal{M}}$. Let $s \leq S+1$ be least such that

$$\mathrm{Th}_{k+1}^{\mathcal{M}}(\alpha_s \cup \{\alpha_0, \cdots, \alpha_{s-1}, u\}) \notin |\mathcal{M}|.$$

Such an s certainly exists, since $S+1$ will do. Let

$$\mathcal{H} = \mathcal{H}_{k+1}^{\mathcal{M}}(\alpha_s \cup \{\alpha_0, \cdots, \alpha_{s-1}, u\}),$$

let $\pi : \mathcal{H} \to \mathcal{M}$ be the inverse of the collapse (so that π is a k-embedding), and let $\bar{u} = \pi^{-1}(u)$ and $\bar{\alpha}_j = \pi^{-1}(\alpha_j)$ for $j < s$.

Our strategy is to compare $\mathcal{H}$ with $\mathcal{M}$, using k-maximal trees. Suppose that $\mathcal{P}$ is the model produced at the end on the $\mathcal{H}$ side, and $\mathcal{Q}$ the model produced on the $\mathcal{M}$ side. Suppose the branches $\mathcal{H}$ to $\mathcal{P}$ and $\mathcal{M}$ to $\mathcal{Q}$ involve no dropping of any kind, so that we have generalized $r\Sigma_{k+1}$ maps $i \colon \mathcal{H} \to \mathcal{P}$ and $j \colon \mathcal{M} \to \mathcal{Q}$. Suppose $\operatorname{crit} i \geq \alpha_s$ and $\operatorname{crit} j \geq \rho_{k+1}^{\mathcal{M}}$. Then

$$\mathrm{Th}_{k+1}^{\mathcal{H}}(\alpha_s \cup \{\bar{\alpha}_0, \cdots, \bar{\alpha}_{s-1}, \bar{u}\}) = \mathrm{Th}_{k+1}^{\mathcal{P}}(\alpha_s \cup \{i(\bar{\alpha}_0), \cdots, i(\bar{\alpha}_{s-1}), i(\bar{u})\}) \notin \mathcal{Q}$$

and

$$\mathrm{Th}^{\mathcal{M}}_{k+1}(\rho^{\mathcal{M}}_{k+1} \cup \{r, u\}) = \mathrm{Th}^{\mathcal{Q}}_{k+1}(\rho^{\mathcal{M}}_{k+1} \cup \{j(r), j(u)\}) \notin \mathcal{P}$$

so that neither of $\mathcal{P}$ and $\mathcal{Q}$ is a proper initial segment of the other, and hence $\mathcal{P} = \mathcal{Q}$.

Now if $\mathcal{M}$ is not $k+1$-solid then $s < S+1$ and hence $\rho^{\mathcal{M}}_{k+1} < \rho^{\mathcal{H}}_{k+1}$ because we didn't throw α_s as a member into the hull collapsing to $\mathcal{H}$. But we can show $\rho^{\mathcal{H}}_{k+1} \leq \rho^{\mathcal{P}}_{k+1}$, so $\rho^{\mathcal{Q}}_{k+1} \leq \rho^{\mathcal{M}}_{k+1} < \rho^{\mathcal{H}}_{k+1} \leq \rho^{\mathcal{P}}_{k+1}$ contradicting the fact that $\mathcal{P} = \mathcal{Q}$. Thus $\mathcal{M}$ is $k+1$-solid. It follows that $s = S+1$, and $\operatorname{crit} j \geq \rho^{\mathcal{M}}_{k+1}$ so we have $P^{\mathcal{M}}(\rho^{\mathcal{M}}_{k+1}) \subseteq |\mathcal{H}|$. Thus $\mathcal{M}$ is $k+1$-universal.

There are many problems in completing this sketch, but the main one is arranging that $\operatorname{crit} i \geq \alpha_s$. Our strategy will be to modify the comparison. Instead of comparing the models $\mathcal{M}$ and $\mathcal{H}$ by iteration trees $\mathcal{U}$ on $\mathcal{M}$ and $\mathcal{T}$ on $\mathcal{H}$, we will use a iteration tree $\mathcal{U}$ on the model $\mathcal{M}$ and a *pseudo-iteration tree* $\bar{\mathcal{T}}$ on the pair of models $(\mathcal{M}, \mathcal{H})$. The situation can be represented by the following diagram:

$$\begin{tikzcd}
 & \mathcal{M} = \mathcal{P}_0 \arrow[rr, dash, "\mathcal{T}"] & & \mathcal{P}_\theta \\
\mathcal{M} = \bar{\mathcal{P}}_{-1} \arrow[r, dash] \arrow[ur, "\pi_{-1}=\mathrm{id}"] & \mathcal{H} = \bar{\mathcal{P}}_0 \arrow[u, "\pi_0"'] \arrow[rr, dash, "\bar{\mathcal{T}}"] & & \bar{\mathcal{P}}_\theta \arrow[u, "\pi_\theta"'] \\
 & \mathcal{M} = \mathcal{Q}_0 \arrow[rr, dash, "\mathcal{U}"] & & \mathcal{Q}_\theta
\end{tikzcd}$$

The horizontal lines in this diagram indicate that the corresponding models are in the same tree, so that there is an embedding between them just in case they are on the same branch of the tree and there is no dropping on the branch between them. The comparison takes place between $\mathcal{U}$, which is a genuine iteration tree, and the pseudo-iteration tree $\bar{\mathcal{T}}$. The thing which makes $\bar{\mathcal{T}}$ a pseudo- iteration tree, rather than a real one, is that its underlying tree $\bar{T}$ has two separate roots, -1 and 0, corresponding to the models $\bar{\mathcal{P}}_{-1} = \mathcal{M}$ and $\bar{\mathcal{P}}_0 = \mathcal{H}$. We take $\rho_{-1} = \alpha_s$, and then we continue the comparison exactly as if $\bar{\mathcal{T}}$ were a real iteration tree. This means that whenever an extender $\bar{E}_\nu$ appears in the pseudo-tree $\bar{\mathcal{T}}$ such that $\operatorname{crit}(\bar{E}_\nu) < \alpha_s$ then $\bar{T}\text{-Pred}(\nu+1) = -1$, so that the $\nu+1$st model $\bar{\mathcal{P}}_{\nu+1}$ of $\bar{\mathcal{T}}$ is equal to $\mathrm{Ult}(\bar{\mathcal{P}}^*_\nu, E_\nu)$ for some initial segment $\mathcal{P}^*_\nu$ of $\mathcal{M}$.

Since $\bar{\mathcal{T}}$ is not a genuine iteration tree, we don't know directly that it has well founded branches. For this we use the iteration tree $\mathcal{T}$ and embeddings π_α, which are defined by setting $\pi_{-1} = \mathrm{id}$, letting π_0 be the inverse of the collapse map, and then using the shift lemma to copy $\bar{\mathcal{T}}$. Since $\mathcal{T}$ is a genuine iteration tree, theorem 6.2 implies that it is simple. Thus it has well founded branches at every stage, and the embeddings $\pi_\nu : \bar{\mathcal{P}}_\nu \to \mathcal{P}_\nu$ ensure that the corresponding branches of $\bar{\mathcal{T}}$ are also well founded.

We will show that $0\bar{T}\theta$ and that there is no dropping along the main branch of either tree. Thus the maps $\bar{\imath}_{0,\theta} : \mathcal{H} \to \bar{\mathcal{P}}^\theta$ and $i^{\mathcal{U}}_{0,\theta} : \mathcal{M} \to \mathcal{Q}_\theta$ are defined. In addition we show that $\bar{\mathcal{P}}_\theta = \mathcal{Q}_\theta$ and finally that $\bar{\imath}_{0,\theta} = i^{\mathcal{U}}_{0,\theta} \circ \pi_0$.

We now begin the actual proof of theorem 8.1. Notice first that if $\rho^{\mathcal{M}}_{k+1} > \operatorname{lh} E$ for all extenders E from the $\mathcal{M}$-sequence, then $\mathcal{H}$ is already an initial segment of $\mathcal{M}$ (since $\alpha_s \geq \rho^{\mathcal{M}}_{k+1}$). In this case, no iteration is necessary, and we have that $\mathcal{H} = \mathcal{M}$, which easily gives the desired results. Thus we may and do assume that $\rho^{\mathcal{M}}_{k+1} \leq \operatorname{lh} E$ for some extender E on the $\mathcal{M}$-sequence. According to the strong uniqueness theorem then, every k-maximal iteration tree on $\mathcal{M}$ is simple. This fact will make the Dodd-Jensen lemma applicable in what follows.

We now define by induction on length: (1) a "psuedo iteration tree" $\bar{\mathcal{T}}$ on the pair $(\mathcal{H}, \mathcal{M})$, (2) a tree $\mathcal{T}$ on $\mathcal{M}$ "enlarging" $\bar{\mathcal{T}}$, and (3) a tree $\mathcal{U}$ on $\mathcal{M}$. We use $\bar{\mathcal{P}}_\alpha$, $\mathcal{P}_\alpha$, and $\mathcal{Q}_\alpha$ for the αth models of $\bar{\mathcal{T}}$, $\mathcal{T}$, and $\mathcal{U}$ respectively. We use T for the tree ordering of $\mathcal{T}, \bar{T}$ for that of $\bar{\mathcal{T}}$, and U for the tree ordering of $\mathcal{U}$. The rest we indicate with superscripts; e.g., $\rho^{\mathcal{T}}_\alpha$, $\bar{\rho}_\alpha$, and $\rho^{\mathcal{U}}_\alpha$ or $i^{\mathcal{T}}_{\alpha\beta}$, $\bar{\imath}_{\alpha\beta}$, and $i^{\mathcal{U}}_{\alpha\beta}$.

The systems $\mathcal{T}$ and $\mathcal{U}$ will literally be a padded iteration trees on $\mathcal{M}$; they will be k-maximal and non-overlapping. $\bar{T}$ will not literally be a tree ordering in our sense, as it will have two roots, but will agree with T on $\mathrm{OR} - \{0, -1\}$.

Simultaneously with $\mathcal{T}$, $\bar{\mathcal{T}}$, and $\mathcal{U}$ we define $\pi_\alpha : \bar{\mathcal{P}}_\alpha \to \mathcal{P}_\alpha$ such that the map π_α is a weak $\overline{\deg}(\alpha)$-embedding.

We begin by setting

$$\bar{\mathcal{P}}_{-1} = \mathcal{M},\ \bar{\mathcal{P}}_0 = \mathcal{H},\ \mathcal{P}_0 = \mathcal{M},\ \mathcal{Q}_0 = \mathcal{M}$$

and

$$\pi_{-1} = \text{identity},\ \pi_0 = \text{inverse of collapse}.$$

Notice that π_{-1} and π_0 are k-embeddings.

Now suppose that we have defined $\mathcal{T} \restriction \theta$, $\bar{\mathcal{T}} \restriction \theta$, and $\mathcal{U} \restriction \theta$. (This means we have defined the models $\bar{\mathcal{P}}_\alpha$, $\mathcal{P}_\alpha$, and $\mathcal{Q}_\alpha$ for $\alpha < \theta$, together with the extenders $\bar{E}_\alpha$, $E^{\mathcal{T}}_\alpha$, and $E^{\mathcal{U}}_\alpha$, for $\alpha + 1 < \theta$, etc.) Suppose we have also defined $\pi_\alpha : \bar{\mathcal{P}}_\alpha \to \mathcal{P}_\alpha$ for $\alpha < \theta$ with the following commutativity and agreement properties.

(i) If $\alpha \bar{T} \beta$ and $\bar{D} \cap (\alpha, \beta]_{\bar{T}} = \varnothing$ then $i^{\mathcal{T}}_{\alpha\beta} \circ \pi_\alpha = \pi_\beta \circ \bar{\imath}_{\alpha\beta}$.

(ii) If $0 \leq \alpha < \beta < \theta$, then $\bar{\mathcal{P}}_\alpha$ agrees with $\bar{\mathcal{P}}_\beta$ below $\operatorname{lh} \bar{E}_\alpha$; moreover letting $\gamma = \operatorname{lh} \bar{E}_\alpha$ and $N = J^{\bar{\mathcal{P}}_\alpha}_\gamma = J^{\bar{\mathcal{P}}_\beta}_\gamma$, we have $\pi_\alpha \restriction N = \pi_\beta \restriction N$.

Remark. Some simple observations about $\mathcal{H}$.

(1) We may assume $\alpha_s \in |\mathcal{H}|$. For otherwise $\mathcal{H}$ is an initial segment of $\mathcal{M}$ (if $\mathcal{M}$ and hence $\mathcal{H}$ is active, then the initial segment condition on good extender sequences implies $\dot{F}^{\mathcal{H}} = \dot{F}^{\mathcal{M}} \restriction \mathrm{OR}^{\mathcal{H}}$ is on the $\mathcal{M}$-sequence) but then $\mathcal{H} = \mathcal{M}$, and we are done.

(2) $\mathcal{H} \models \alpha_s$ is a cardinal, since $\alpha_s = \operatorname{crit} \pi_0$ if $s < S + 1$ and, $\alpha_s = \pi(\alpha_s) = \rho^{\mathcal{M}}_{k+1}$ if $s = S + 1$.

(3) For $\beta \geq 0$ and $\kappa < \alpha_s$, $P(\kappa) \cap |\bar{\mathcal{P}}_\beta| = P(\kappa) \cap |\mathcal{H}| = P(\kappa) \cap |\mathcal{J}^{\mathcal{M}}_{\alpha_s}|$. However, it seems possible at this point that $P(\kappa) \cap |\mathcal{M}|$ might be larger than $P(\kappa) \cap |\mathcal{H}|$.

We now define $\mathcal{T} \restriction \theta + 1$, $\bar{\mathcal{T}} \restriction \theta + 1$ and $\mathcal{U} \restriction \theta + 1$.

Case 1. θ is a limit ordinal.

In this case, we have only to pick cofinal wellfounded branches in each of our trees.

As $\mathcal{T} \restriction \theta$ is k-maximal and $\rho^{\mathcal{M}}_{k+1} \leq \mathrm{lh}\, E$ for some extender E from the $\mathcal{M}$ sequence, $\mathcal{T} \restriction \theta$ is simple. As $\mathcal{M}$ is k-iterable, there is a cofinal wellfounded branch b of $\mathcal{T}$. Similarly, there is a cofinal wellfounded branch c of $\mathcal{U}$. Finally, let $\bar{b} = b$ or $\bar{b} = (b - \{0\}) \cup \{-1\}$, whichever is a branch of $\bar{\mathcal{T}}$. Set

$$\begin{aligned} \mathcal{P}_\theta &= \text{direct limit of } \mathcal{P}_\alpha\,,\ \alpha \in b - \sup D^{\mathcal{T}} \\ \bar{\mathcal{P}}_\theta &= \text{direct limit of } \bar{\mathcal{P}}_\alpha\,,\ \alpha \in \bar{b} - \sup \bar{D} \\ Q_\theta &= \text{direct limit of } Q_\alpha\,,\ \alpha \in c - \sup D^{\mathcal{U}} \end{aligned}$$

and extend T, $\bar{T}$ and U to $\theta+1$ correspondingly. For $\alpha \in \bar{b} - \sup \bar{D}$ and $x \in |\bar{\mathcal{P}}_\alpha|$ we can set

$$\pi_\theta(\bar{\imath}_{\alpha,\theta}(x)) = i^{\mathcal{T}}_{\alpha,\theta}(\pi_\alpha(x))$$

(where of course $\bar{\imath}_{\alpha,\beta} = \bar{\imath}_{\alpha,\bar{b}}$, etc.), and by induction hypotheses (i) and (ii) this gives a well-defined $\pi_\theta : \bar{\mathcal{P}}_\theta \to \mathcal{P}_\theta$. Clearly π_θ is a $\overline{\deg}\,(\theta)$-embedding and (i) and (ii) continue to hold.

Case 2. $\theta = \eta + 1$. In this case we "iterate the least disagreement" between $\bar{\mathcal{P}}_\eta$ and Q_η, as in the proof of the comparison lemma.

Let γ be least $\leq \mathrm{OR}^{\bar{\mathcal{P}}_\eta} \wedge \mathrm{OR}^{Q_\eta}$ such that

$$\mathcal{J}^{\bar{\mathcal{P}}_\eta}_\gamma \neq \mathcal{J}^{Q_\eta}_\gamma\,;$$

if no such γ exists then we stop the construction of $\mathcal{T}$, $\bar{\mathcal{T}}$, and $\mathcal{U}$. Set

$$\bar{E}_\eta = \begin{cases} \dot{F}^{\mathcal{J}^{\bar{\mathcal{P}}_\eta}_\gamma}\,, & \text{if } \mathcal{J}^{\bar{\mathcal{P}}_\eta}_\gamma \text{ is active} \\ \varnothing & \text{otherwise} \end{cases}$$

$$E^{\mathcal{U}}_\eta = \begin{cases} \dot{F}^{\mathcal{J}^{Q_\eta}_\gamma}\,, & \text{if } \mathcal{J}^{Q_\eta}_\gamma \text{ is active} \\ \varnothing & \text{otherwise}\,. \end{cases}$$

On the $\mathcal{U}$ side the rest is determined by the demands of a k-maximal iteration tree. So $\mathcal{U}$-pred$(\eta + 1) = \xi$, where

$$\xi = \text{least } \alpha \text{ such that crit } E^{\mathcal{U}}_\eta < \rho^{\mathcal{U}}_\alpha\,.$$

(Assuming now $E_\eta^{\mathcal{U}} \neq \varnothing$; if $E_\eta^{\mathcal{U}} = \varnothing$ we just pad for one step.) Let $\kappa = \text{crit } E_\eta^{\mathcal{U}}$, let $Q_{\eta+1}^*$ be the longest initial segment $\mathcal{N}$ of Q_ξ such that $P(\kappa) \cap |\mathcal{N}| = P(\kappa) \cap |Q_\eta|$ and let

$$Q_{\eta+1} = \text{Ult}_n(Q_{\eta+1}^*, E_\eta^{\mathcal{U}})$$

where

$$n = \text{largest } s \text{ such that } \kappa < \rho_s^{Q_{\eta+1}^*} \text{ and } s \leq k \text{ if } D^{\mathcal{U}} \cap [0, \eta+1]_U = \varnothing\,.$$

On the $\bar{\mathcal{T}}$ side we proceed similarly. We assume $\bar{E}_\eta \neq \varnothing$; otherwise we pad for one step. Set $\kappa = \text{crit}(\bar{E}_\eta)$ and let let $\bar{\mathcal{T}}$-pred$(\eta+1) = \xi$, where

$$\xi = \text{least } \alpha \text{ such that } \kappa < \bar{\rho}_\xi\,,$$

so that in particular $\xi = -1$ if $\kappa < \alpha_i = \bar{\rho}_{-1}$. Now set $\bar{\mathcal{P}}_{\eta+1}^*$ equal to the longest initial segment $\mathcal{N}$ of $\bar{\mathcal{P}}_\xi$ such that $P(\kappa) \cap |\mathcal{N}| = P(\kappa) \cap |\bar{P}_\eta|$ and

$$\bar{\mathcal{P}}_{\eta+1} = \text{Ult}_n(\bar{\mathcal{P}}_{\eta+1}^*, \bar{E}_\eta)$$

where n is the largest integer s such that $\kappa < \rho_s^{Q_{\eta+1}^*}$ and such that $s \leq k$ if $\bar{D} \cap \{\alpha \mid \alpha \bar{T}(\eta+1) \vee \alpha = \eta+1\} = \varnothing$.

That $\bar{\mathcal{P}}_{\eta+1}$ agrees with $\bar{\mathcal{P}}_\eta$ below $\text{lh}\,\bar{E}_\eta$ is proved as usual. Notice that if $\xi = -1$, then as $P(\kappa) \cap |\mathcal{J}_{\alpha_i}^{\mathcal{M}}| = P(\kappa) \cap |\bar{\rho}_\eta|$, there is an $\mathcal{N}$ as called for in the definition of $\bar{\mathcal{P}}_{\eta+1}^*$.

Finally, we extend $\mathcal{T}$ by "copying" what we just did with $\bar{\mathcal{T}}$. Let γ be least such that $\mathcal{J}_\gamma^{\bar{\mathcal{P}}_\eta} \neq \mathcal{J}_\gamma^{Q_\eta}$. Assume that $\mathcal{J}_\gamma^{\bar{\mathcal{P}}_\eta}$ is active; otherwise we just pad $\mathcal{T}$ for one step. Let

$$E_\eta^{\mathcal{T}} = \dot{F}^{\mathcal{N}}, \quad \text{where} \quad \mathcal{N} = \mathcal{J}_{\pi_\eta(\gamma)}^{\mathcal{P}_\eta}\,,$$

where as usual we let $\pi_\eta(\text{OR}^{\bar{\mathcal{P}}_\eta}) = \text{OR}^{\mathcal{P}_\eta}$.

Subcase A. $\bar{\mathcal{T}}$-pred$(\eta+1) = -1$.

Let $\mathcal{T}$-pred$(\eta+1) = 0$,

$$\begin{aligned}
&\mathcal{P}_{\eta+1}^* = \bar{\mathcal{P}}_{\eta+1}^* \\
&\mathcal{P}_{\eta+1} = \text{Ult}_n(\mathcal{P}_{\eta+1}^*, E_\eta^{\mathcal{T}})\,, \quad \text{where } n = \overline{\deg}\ (\eta+1)\,.
\end{aligned}$$

We get $\pi_{\eta+1} : \bar{\mathcal{P}}_{\eta+1} \to \mathcal{P}_{\eta+1}$ by the shift lemma, lemma 5.2 which implies that $\pi_{\eta+1}$ is a $\overline{\deg}\,(\eta+1)$-embedding with the required commutativity and agreement properties (i) and (ii).

Subcase B. $\bar{\mathcal{T}}$-pred$(\eta+1) = \xi \geq 0$. Let $\mathcal{T}$-pred$(\eta+1) = \xi$. Let $\bar{\mathcal{P}}_{\eta+1}^* = \mathcal{J}_\nu^{\bar{\mathcal{P}}_\xi}$; then

$$\mathcal{P}_{\eta+1}^* = \mathcal{J}_{\pi_\xi(\nu)}^{\mathcal{P}_\xi}$$

where $\pi_\xi(\mathrm{OR}^{\bar{\mathcal{P}}_\xi}) = \mathrm{OR}^{\mathcal{P}_\xi}$. Let $n = \overline{\deg}(\eta + 1)$, then

$$\mathcal{P}_{\eta+1} = \mathrm{Ult}_n(\mathcal{P}^*_{\eta+1}, E^{\mathcal{T}}_\eta).$$

Finally, we get the desired $\pi_{\eta+1}$ by the shift lemma.

This completes the construction of $\mathcal{T}$, $\bar{\mathcal{T}}$, and $\mathcal{U}$. We leave it to the reader to check the many details we ought to have verified in the course of the construction. (In particular, that $\mathcal{T}$ is a k-maximal iteration tree, and that the π_η's have the required commutativity and agreement properties.)

Now because $\mathcal{T}$ and $\mathcal{U}$ are simple we must reach an ordinal θ such that $\bar{\mathcal{P}}_\theta$ is an initial segment of Q_θ or vice-versa. The proof is exactly the same as the proof in §7 that the comparison process stops.

We shall say that a branch b of $\mathcal{U}$ *drops* if either $D^{\mathcal{U}} \cap b \neq \varnothing$ or $\exists \alpha \in b\ (\overline{\deg}^{\mathcal{U}}(\alpha) \neq k)$, and similarly for branches of $\bar{\mathcal{T}}$ or $\mathcal{T}$.

We need to verify that, just as with the comparison in section 7, at most one side of the comparison drops, and that the side which drops is the longer. That is, if the main branch $\{\,\beta : \beta \bar{T} \theta\,\}$ of $\bar{\mathcal{T}}$ drops then the main branch $[0, \theta]_U$ of $\mathcal{U}$ does not drop and $\mathcal{P}_\theta$ is not a proper initial segment of Q_θ, while if the main branch of $\mathcal{U}$ drops then the main branch of $\mathcal{T}$ does not drop and Q_θ is not a proper initial segment of $\bar{\mathcal{P}}_\theta$.

It is immediate that if either branch drops then its final model is not ω-sound, and hence cannot be a proper initial segment of the final model of the other branch. If follows that if both branches dropped then we would have $\bar{\mathcal{P}}_\theta = Q_\theta$. This implies that if the last drop on $\{\,\beta : \beta \bar{T} \theta\,\}$ occurs at $\alpha + 1$ and the last drop on $[0, \theta]_u$ at $\beta + 1$, then

$$\begin{aligned} \deg^{\mathcal{U}}(\beta + 1) &= \text{the least } n \text{ such that } Q_\theta \text{ is not } n + 1 \text{ sound} \\ &= \text{the least } n \text{ such that } \bar{\mathcal{P}}_\theta \text{ is not } n + 1 \text{ sound} \\ &= \overline{\deg}(\alpha + 1). \end{aligned}$$

Moreover, if $n = \deg^{\mathcal{U}}(\beta + 1)$,

$$\begin{aligned} Q^*_{\beta+1} &= \mathfrak{C}_{n+1}(Q^*_{\beta+1}) = \mathfrak{C}_{n+1}(Q_\theta) \\ &= \mathfrak{C}_{n+1}(\bar{\mathcal{P}}_\theta) = \mathfrak{C}_{n+1}(\bar{\mathcal{P}}^*_{\alpha+1}) = \bar{\mathcal{P}}^*_{\alpha+1}. \end{aligned}$$

Also

$$\mathrm{crit}\ i^{\mathcal{U}}_{\beta+1,\theta} \circ i^{*\mathcal{U}}_{\beta+1} \geq \rho^{Q^*_{\beta+1}}_{n+1} = \rho^{Q_\theta}_{n+1} = \rho^{\bar{\mathcal{P}}_\theta}_{n+1},$$

and

$$\mathrm{crit}\ \bar{\imath}_{\alpha+1,\theta} \circ \bar{\imath}^*_{\alpha+1} \geq \rho^{\bar{\mathcal{P}}^*_{\alpha+1}}_{n+1} = \rho^{\bar{\mathcal{P}}_\theta}_{n+1}.$$

Also

$$\begin{aligned}\text{crit } E^{\mathcal{U}}_{\beta} &= \text{least } \kappa \text{ such that } \kappa \neq \tau^{Q_\theta}[\bar{\alpha}, p_{n+1}(Q_\theta)] \text{ for any}\\ &\quad \tau \in \mathrm{Sk}_{n+1} \text{ and } \bar{\alpha} \in (\rho_{n+1}^{Q_\theta})^{<\omega}\\ &= \text{least } \kappa \text{ such that } \kappa \neq \tau^{\bar{\mathcal{P}}_\theta}[\bar{\alpha}, p_{n+1}(\bar{\mathcal{P}}_\theta)] \text{ for any}\\ &\quad \tau \in \mathrm{Sk}_{n+1} \text{ and } \bar{\alpha} \in (\rho_{n+1}^{Q_\theta})^{<\omega}\\ &= \text{crit } \bar{E}_\alpha .\end{aligned}$$

Finally, if $A \in Q^*_{\beta+1}$ and $A \subseteq [\text{crit } E^{\mathcal{N}}_\beta]^n$, then letting

$$A = \tau^{Q^*_{\beta+1}}[\bar{\alpha}, p_{n+1}(Q^*_{\beta+1})]$$

where $\bar{\alpha} \in (\rho_{n+1}^{Q^*_{\beta+1}})^{<\omega}$,

$$\begin{aligned} i^{\mathcal{U}}_{\beta+1,\theta} \circ i^{*\mathcal{U}}_{\beta+1}(A) &= \tau^{Q_\theta}[\bar{\alpha}, p_{n+1}(Q_\theta)]\\ &= \tau^{\bar{\mathcal{P}}_\theta}[\bar{\alpha}, p_{n+1}(\bar{\mathcal{P}}_\theta)]\\ &= \bar{\imath}_{\alpha+1,\theta} \circ \bar{\imath}^*_{\alpha+1}(A) .\end{aligned}$$

It follows that one of $E^{\mathcal{U}}_\beta$ and $\bar{E}_\alpha$ is an initial segment of the other, and this is a contradiction as in the proof of the comparison lemma. Thus at most one of the trees $\bar{\mathcal{T}}$ and $\mathcal{U}$ can have a drop along its main branch.

CLAIM 1. $0\bar{T}\theta$, that is, $\bar{\mathcal{P}}_\theta$ lies above $\mathcal{P}_0 = \mathcal{H}$ in the $\bar{\mathcal{T}}$ system.

PROOF. Assume not; that is, assume that $-1\bar{T}\theta$, so that $\bar{\mathcal{P}}_\theta$ lies above $\bar{\mathcal{P}}_{-1} = \mathcal{M}$. We know that at least one of the branches $[-1,\theta]_{\bar{\mathcal{T}}}$ and $[0,\theta]_{\mathcal{U}}$ does not drop.

CASE 1. $[-1,\theta]_{\bar{\mathcal{T}}}$ drops.

Then $\bar{\mathcal{P}}_\theta$ is not ω-sound, so is not a proper initial segment of Q_θ. Suppose first Q_θ is a proper initial segment of $\bar{\mathcal{P}}_\theta$; say $Q_\theta = \mathcal{J}^{\bar{\mathcal{P}}_\theta}_\gamma$. Let $\sigma = \pi_\theta \restriction \mathcal{J}^{\bar{\mathcal{P}}_\theta}_\gamma$, so that $\sigma : Q_\theta \to \mathcal{J}^{\mathcal{P}_\theta}_{\pi_\theta(\gamma)}$ is fully elementary. Then the map $\sigma \circ i^{\mathcal{U}}_{0,\theta}$ is a weak k-embedding from $\mathcal{M}$ to a proper initial segment of $\mathcal{P}_\theta$. As $\mathcal{T}$ is k-bounded and simple, this contradicts the Dodd-Jensen lemma.

Suppose next that $Q_\theta = \bar{\mathcal{P}}_\theta$. Then as Q_θ is k-sound, $\overline{\deg}(\theta) \geq k$, so that π_θ is a weak k-embedding. Thus $\pi_\theta \circ i^{\mathcal{U}}_{0,\theta}$ is a weak k-embedding from $\mathcal{M}$ to $\mathcal{P}_\theta$. But by case hypothesis, $[0,\theta]_T$ drops. This contradicts the Dodd-Jensen lemma. (As $\deg^{\mathcal{T}}(\theta) \geq k$, we must have $D^{\mathcal{T}} \cap [0,\theta]_T \neq \varnothing$.)

CASE 2. $[0,\theta]_U$ drops.

In this case, $[-1,\theta]_{\bar{\mathcal{T}}}$ doesn't drop and $\bar{\mathcal{P}}_\theta$ is an initial segment of Q_θ. If proper, then $\bar{\imath}_{-1,\theta}$ is a k-embedding of $\mathcal{M}$ to a proper initial segment of Q_θ, which lies on a k-bounded, simple iteration tree with base model $\mathcal{M}$. This contradicts Dodd-Jensen. If $\bar{\mathcal{P}}_\theta = \mathcal{Q}_\theta$, then as $\bar{\mathcal{P}}_\theta$ is k-sound, $\deg^{\mathcal{U}}(\theta) \geq k$. But then the fact that $[0,\theta]_U$ drops contradicts Dodd-Jensen.

CASE 3. Neither $[-1,\theta]_{\mathcal{T}}$ nor $[0,\theta]_U$ drops.

If $\bar{\mathcal{P}}_\theta$ is a proper initial segment of $\mathcal{Q}_\theta$, then $\bar{\imath}_{-1,\theta}$ contradicts Dodd-Jensen. If $\mathcal{Q}_\theta$ is proper initial segment of $\bar{\mathcal{P}}_\theta$, then $\pi_\theta \circ i^{\mathcal{U}}_{0,\theta}$ contradicts Dodd-Jensen. So we must have $\bar{\mathcal{P}}_\theta = \mathcal{Q}_\theta$. We now use the minimality of the iteration maps $i^{\mathcal{U}}_{0,\theta}$, $i^{\mathcal{T}}_{0,\theta}$ given by the Dodd-Jensen lemma. Let $\leq_{\mathrm{L}}$ be the order of construction in premice.

Fix any $x \in |\mathcal{M}|$. By Dodd-Jensen,

$$i^{\mathcal{U}}_{0,\theta}(x) \leq_{\mathrm{L}} \bar{\imath}_{-1,\theta}(x)$$

since $i^{\mathcal{U}}_{0,\theta}$ is a "k-bounded" iteration map, and $\bar{\imath}_{-1,\theta}$ a k-embedding. But also

$$i^{\mathcal{T}}_{0,\theta}(x) \leq_{\mathrm{L}} \pi_\theta(i^{\mathcal{U}}_{0,\theta}(x))$$

since $i^{\mathcal{T}}_{0,\theta}$ is a k-bounded iteration map and $i^{\mathcal{U}}_{0,\theta} \circ \pi_\theta$ is a weak k-embedding. Then

$$\pi_\theta(\bar{\imath}_{-1,\theta}(x)) = i^{\mathcal{T}}_{0,\theta}(x) \leq_{\mathrm{L}} \pi_\theta(i^{\mathcal{U}}_{0,\theta}(x))$$

so

$$\bar{\imath}_{-1,\theta}(x) \leq_{\mathrm{L}} i^{\mathcal{U}}_{0,\theta}(x)$$

so

$$\bar{\imath}_{-1,\theta}(x) = i^{\mathcal{U}}_{0,\theta}(x).$$

But if $\bar{\imath}_{-1,\theta} = i^{\mathcal{U}}_{0,\theta}$, then the first extender used on $[-1,\theta]_{\mathcal{T}}$ is compatible with the first extender used on $[0,\theta]_U$, which is impossible. □

This proves Claim 1, and it follows that $[0,\theta]_{\mathcal{T}}$ is the main branch of $\bar{\mathcal{T}}$. Again, we know that at most one of the branches $[0,\theta]_{\mathcal{T}}$ and $[0,\theta]_U$ drops.

CLAIM 2. $[0,\theta]_{\mathcal{T}}$ doesn't drop.

PROOF. Suppose it did drop. Then $[0,\theta]_U$ does not drop and $\bar{\mathcal{P}}_\theta$ is not a proper initial segment of $\mathcal{Q}_\theta$. Suppose $\mathcal{Q}_\theta$ is a proper initial segment of $\bar{\mathcal{P}}_\theta$. Then $\pi_\theta \circ i^{\mathcal{U}}_{0,\theta}$ is a weak k-embedding from $\mathcal{M}$ to a proper initial segment of $\mathcal{P}_\theta$, contrary to Dodd-Jensen. Suppose $\mathcal{Q}_\theta = \bar{\mathcal{P}}_\theta$. Then as $\mathcal{Q}_\theta$ is k-sound, $\overline{\deg}(\theta) \geq k$, so that $\pi_\theta \circ i^{\mathcal{U}}_{0,\theta}$ is a weak k-embedding from $\mathcal{M}$ to $\mathcal{P}_\theta$. As $[0,\theta]_T$ drops and $\overline{\deg}^{\mathcal{T}}(\theta) \geq k$, we have $D^{\mathcal{T}} \cap [0,\theta]_T \neq \varnothing$. This contradicts Dodd-Jensen. □

CLAIM 3. $\mathcal{Q}_\theta$ is an initial segment of $\bar{\mathcal{P}}_\theta$.

PROOF. Claims 1 and 2 together imply that

$$\mathrm{Th}^{\mathcal{H}}_{k+1}(\alpha_s \cup \{\langle \bar{\alpha}_0, \cdots, \bar{\alpha}_{s-1}, \bar{u}\rangle\}) = \\ \mathrm{Th}^{\bar{\mathcal{P}}_\theta}_{k+1}(\alpha_s \cup \{\bar{\imath}_{0,\theta}(\langle \bar{\alpha}_0, \cdots, \bar{\alpha}_{s-1}\bar{u}\rangle)\}) \notin |\bar{\mathcal{P}}_\theta|.$$

Moreover, as $\mathrm{Th}_{k+1}^{\mathcal{H}}(\alpha_s \cup \{\langle \bar{\alpha}_0, \cdots, \bar{\alpha}_{s-1}, \bar{u}\rangle\})$ is essentially a subset of α_s, and is not in $\mathcal{M}$, it is not in Q_θ. (Note here that $P(\alpha_s)^{Q_1} \subseteq |Q_0|$, and that if $\xi \geq 1$ and $E_\xi^{\mathcal{U}} \neq \varnothing$, then $\mathrm{lh}\, E_\xi^{\mathcal{U}} > \alpha_s$, so that $P(\alpha_s) \cap |Q_{\xi+1}| \subseteq P(\alpha_s) \cap |Q_\xi|$, with equality holding after the least such ξ.) It follows that $\bar{\mathcal{P}}_\theta$, over which the subset of α_s in question is definable, is not a proper initial segment of Q_θ. □

CLAIM 4. $[0, \theta]_U$ doesn't drop.

Suppose otherwise. Then Q_θ is not ω-sound, so $Q_\theta = \bar{\mathcal{P}}_\theta$. But then Q_θ is k-sound, so that $\deg^{\mathcal{U}}(\theta) \geq k$.

Let $\gamma + 1$ be the largest member of $D^{\mathcal{U}} \cap [0, \theta]_{\mathcal{U}}$. Thus $\deg^{\mathcal{U}}(\xi) \geq k$ for all $\xi \geq \gamma + 1$ such that $\xi \in [0, \theta]_U$.

For any $X \subseteq |\mathcal{N}|$, any j, let

$$\overline{\mathrm{Th}}_j^{\mathcal{N}}(X) = \{(\varphi, \bar{a}) \in \mathrm{Th}_j^{\mathcal{N}}(X) \mid \varphi \quad \text{is pure } r\Sigma_j\}.$$

Then set

$$A = \overline{\mathrm{Th}}_{k+1}^{\bar{\mathcal{P}}_\theta}(\alpha_s \cup \{\bar{\imath}_{0,\theta}(\langle \bar{\alpha}_0, \cdots, \bar{\alpha}_{s-1}, \bar{u}\rangle)\}).$$

Thus A is $r\Sigma_{k+1}^{\bar{\mathcal{P}}_\theta}$, and by Lemma 2.10, $A \notin |\bar{\mathcal{P}}_\theta|$.

CASE 1. $\mathrm{crit}(i_{\gamma+1,\theta}^{\mathcal{U}} \circ i_{\gamma+1}^{*\mathcal{U}}) \geq \alpha_s$.

As in the proof of Lemma 4.5, we can show by induction on $\beta \in [\gamma + 1, \theta]_U$ that any set $X \subseteq \alpha_s$ which is $r\Sigma_{k+1}^{Q_\beta}$ is in fact $r\Sigma_{k+1}^{Q_{\gamma+1}^*}$. Thus A is $r\Sigma_{k+1}^{Q_{\gamma+1}^*}$. Thus $A \in Q_\xi$, where $\xi = U\text{-pred}(\gamma + 1)$. But then $A \in |\mathcal{M}| = |Q_0|$, since $A \subseteq \alpha_s$. But then the proof of 2.10 shows that

$$\mathrm{Th}_{k+1}^{\bar{\mathcal{P}}_\theta}(\alpha_s \cup \{\bar{\imath}_{0,\theta}(\langle \bar{\alpha}_0, \cdots, \bar{\alpha}_{s-1}, \bar{u}\rangle)\}) \in |\mathcal{M}|,$$

a contradiction.

CASE 2. Otherwise. Let

$$\kappa = \mathrm{crit}(i_{\gamma+1,\theta}^{\mathcal{U}} \circ i_{\gamma+1}^{*\mathcal{U}}) < \alpha_s.$$

Since $\kappa = \mathrm{crit}(i_{\gamma+1}^{*\mathcal{U}}) = \mathrm{crit}\ E_\gamma^{\mathcal{U}}$, and $\gamma + 1 \in D^{\mathcal{U}}$, we have

$$P(\kappa) \cap |Q_{\gamma+1}^*| \not\subseteq P(\kappa) \cap |\mathcal{M}|.$$

Let ξ be least such that $E_\xi^{\mathcal{U}} \neq \varnothing$; thus $\xi \leq \gamma$ and $Q_\xi = Q_0 = \mathcal{M}$. Now $\mathcal{M}$ agrees with $Q_{\gamma+1}^*$ below $\mathrm{lh}\, E_\xi^{\mathcal{U}}$, and $\mathrm{lh}\, E_\xi^{\mathcal{U}} \geq \alpha_s$; thus there must be a subset of κ in $\mathcal{M}$ but not in $\mathcal{J}_{\alpha_s}^{\mathcal{M}}$. So

$$\mathcal{M} \models \mathrm{card}(\alpha_s) \leq \kappa.$$

Thus $\alpha_s \neq \rho^{\mathcal{M}}_{k+1}$ and $s < S+1$ and $\alpha_s =$ crit π_0 where $\pi_0 : \mathcal{H} \to \mathcal{M}$ is the inverse of the collapse. But then

$$\mathcal{H} \models \alpha_s = \kappa^+ .$$

Thus $P(\kappa) \cap |\mathcal{H}| = P(\kappa) \cap J^{\mathcal{H}}_{\alpha_s} = P(\kappa) \cap |\bar{\mathcal{P}}_\eta|$, all $\eta \geq 0$. Now since Q_ξ agrees with $\bar{\mathcal{P}}_\xi$ below $\mathrm{lh}\, E^{\mathcal{U}}_\xi$, and $\mathrm{lh}\, E^{\mathcal{U}}_\xi$ is a cardinal of Q_η for $\eta > \xi$, we have

$$P(\kappa) \cap |Q_\eta| = P(\kappa) \cap J^{Q_\eta}_{\alpha_s} \qquad (\text{all } \eta > \xi),$$

and

$$Q_\eta \models \alpha_s = \kappa^+ \qquad (\text{all } \eta > \xi).$$

It follows, since $\gamma + 1 \in D^{\mathcal{U}}$, that U-pred$(\gamma+1) = \xi$. Also,

$$\mathrm{crit}\,(i^{\mathcal{U}}_{\gamma+1,\theta}) \geq \rho^{\mathcal{U}}_\gamma \geq (\kappa^+)^{J^{Q_\gamma}_{\mathrm{lh}\, E^{\mathcal{U}}_\gamma}} \geq \alpha_s .$$

We can then show by an induction using the proof of 4.5 that

$$A \in r\Sigma^{Q_{\gamma+1}}_{k+1} .$$

Say A is $r\Sigma^{Q_{\gamma+1}}_{k+1}$ in the parameter p, where $p = [a, f]^{Q^*_{\gamma+1}}_{E^{\mathcal{U}}_\gamma}$. It will be enough to show that $E^{\mathcal{U}}_\gamma \restriction \alpha_s \cup a$ is a member of $\mathcal{M}$, for then, since $Q^*_{\gamma+1} \in Q_\xi = \mathcal{M}$, we get that $A \in |\mathcal{M}|$, a contradiction.

Suppose first $\gamma = \xi$. Since $\gamma + 1 \in D^{\mathcal{U}}$, $E^{\mathcal{U}}_\gamma \neq \dot{F}^{Q_\xi}$, and thus $E^{\mathcal{U}}_\gamma \in Q_\xi$, as desired.

Now let $\gamma > \xi$. Since $\xi U \eta$ for all $\eta > \xi$, $\xi U \gamma$. If $E^{\mathcal{U}}_\gamma \neq \dot{F}^{Q_\gamma}$, then $E^{\mathcal{U}}_\gamma \in |Q_\gamma|$, and since $E^{\mathcal{U}}_\gamma \restriction \alpha_s \cup a$ is a subset of α_s, $E^{\mathcal{U}}_\gamma \restriction \alpha_s \cup a \in |Q_\xi|$, as desired. So we may assume that $E^{\mathcal{U}}_\gamma = \dot{F}^{Q_\gamma}$.

Now $D^{\mathcal{U}} \cap [\xi, \gamma]_{\mathcal{U}} \neq \emptyset$, as otherwise since crit $\dot{F}^{Q_\gamma} = \kappa$, crit $i^{\mathcal{U}}_{\xi,\gamma} > \kappa$ and $P(\kappa) \cap |Q_\gamma| = P(\kappa) \cap |Q_\xi|$.

So let $\eta + 1$ be largest in $D^{\mathcal{U}} \cap [\xi, \gamma]_{\mathcal{U}}$. So $\dot{F}^{Q^*_{\eta+1}}$ has critical point κ, and $i^{\mathcal{U}}_{\eta+1,\gamma} \circ i^*_{\eta+1}$ has critical point $> \kappa$, hence $\geq \alpha_s$. But now $\dot{F}^{Q_\gamma} \restriction (\alpha_s \cup a)$ is an $r\Sigma^{Q_\gamma}_1$ subset of α_s, and hence (as in the proof of 4.5) $\dot{F}^{Q_\gamma} \restriction \alpha_s \cup a$ is $r\Sigma^{Q^*_{\eta+1}}_1$. Since $\eta + 1 \in D^{\mathcal{U}}$, we get $\dot{F}^{Q_\gamma} \restriction (\alpha_s \cup a) \in Q_\xi$, as desired. □

CLAIM 5. $\bar{\mathcal{P}}_\theta = Q_\theta$.

PROOF. Otherwise Q_θ is a proper initial segment of $\bar{\mathcal{P}}_\theta$. But then $\pi_\theta \circ i^{\mathcal{U}}_{0,\theta}$ is a weak k-embedding from $\mathcal{M}$ to a proper initial segment of $\mathcal{P}_\theta$, which is on a k-bounded simple iteration tree based on $\mathcal{M}$. This contradicts Dodd-Jensen. □

CLAIM 6. $\bar{\imath}_{0,\theta}(\bar{u}) = i_{0,\theta}(u)$, and for $j \leq s-1$, $\bar{\imath}_{0,\theta}(\bar{\alpha}_j) = i^{\mathcal{U}}_{0,\theta}(\alpha_j)$.

PROOF. $\bar{\imath}_{0,\theta}(\bar{u}) = u_k(\bar{\mathcal{P}}_\theta) = u_k(Q_\theta) = i^{\mathcal{U}}_{0,\theta}(u)$, since $\bar{\imath}_{0,\theta}$ and $i_{0,\theta}(u)$ are k-embeddings.

We show the second assertion by induction on j. Assume it for $p < j$. As $i^{\mathcal{U}}_{0,\theta}$ is a k-embedding, the proofs of 4.6 and 4.7 show that

$$\mathrm{Th}^{Q_\theta}_{k+1}(i^{\mathcal{U}}_{0,\theta}(\alpha_j) \cup \{i^{\mathcal{U}}_{0,\theta}(\langle\alpha_0, \cdots, \alpha_{j-1}, u\rangle)\}) \in |Q_\theta|\,.$$

On the other hand

$$\mathrm{Th}^{\bar{\mathcal{P}}_\theta}_{k+1}(\bar{\imath}_{0,\theta}(\bar{\alpha}_j + 1) \cup \{\bar{\imath}_{0,\theta}(\langle\bar{\alpha}_0, \cdots, \bar{\alpha}_{j-1}, \bar{u}\rangle)\} \notin \bar{\mathcal{P}}_\theta\,.$$

So our induction hypothesis implies that $i^{\mathcal{U}}_{0,\theta}(\alpha_j) \leq \bar{\imath}_{0,\theta}(\bar{\alpha}_j)$. On the other hand, since the iteration map $i^{\mathcal{T}}_{0,\theta}$ is minimal and $\pi_\theta \circ i^{\mathcal{U}}_{0,\theta}$ is a k-embedding of $\mathcal{M}$ into $\mathcal{P}_\theta$, we have

$$i^{\mathcal{T}}_{0,\theta}(\alpha_j) \leq \pi_\theta(i^{\mathcal{U}}_{0,\theta}(\alpha_j))$$

or

$$\pi_\theta(\bar{\imath}_{0,\theta}(\bar{\alpha}_j)) \leq \pi_\theta(i^{\mathcal{U}}_{0,\theta}(\alpha_j))$$

so that $\bar{\imath}_{0,\theta}(\bar{\alpha}_j) \leq i^{\mathcal{U}}_{0,\theta}(\alpha_j)$, and thus $\bar{\imath}_{0,\theta}(\bar{\alpha}_j) = i^{\mathcal{U}}_{0,\theta}(\alpha_j)$, as desired. □

CLAIM 7. crit $i^{\mathcal{U}}_{0,\theta} \geq \rho^{\mathcal{M}}_{k+1}$.

PROOF. Assume not, and let $\kappa = \text{crit } i^{\mathcal{U}}_{0,\theta} = \text{crit } E^{\mathcal{U}}_\beta$, where $\beta + 1 \in [0,\theta]_{\mathcal{U}}$ is such that $U\text{-pred}(\beta+1) = 0$. Then

$$\mathrm{Th}^{\mathcal{M}}_{k+1}(\kappa \cup \{\langle\alpha_0, \cdots, \alpha_{s-1}, u\rangle\}) \in |\mathcal{M}|\,.$$

It follows as in the proof of 4.6 that

$$\mathrm{Th}^{Q_{\beta+1}}_{k+1}(i_{0,\beta+1}(\kappa) \cup \{i_{0,\beta+1}(\langle\alpha_0, \cdots, \alpha_{s-1}, u\rangle)\}) \in |Q_{\beta+1}|\,.$$

But now $\alpha_s \leq \mathrm{lh}\, E^{\mathcal{U}}_\beta < i_{0,\beta+1}(\kappa)$, so

$$\mathrm{Th}^{Q_{\beta+1}}_{k+1}(\alpha_s \cup \{i^{\mathcal{U}}_{0,\beta+1}(\langle\alpha_0, \cdots, \alpha_{s-1}, u\rangle)\}) \in |Q_{\beta+1}|\,.$$

So, again using the proof of 4.6 if crit $i^{\mathcal{U}}_{\beta+1,\theta} < \alpha_s$ (which seems possible; we may have $\rho^{\mathcal{U}}_\beta < \alpha_s$),

$$\mathrm{Th}^{Q_\theta}_{k+1}(\alpha_s \cup \{i^{\mathcal{U}}_{0,\theta}(\langle\alpha_0, \cdots, \alpha_{s-1}, u\rangle)\}) \in |Q_\theta|\,.$$

This contradicts the conjunction of our previous claims. □

CLAIM 8. $s = S+1$; that is, $\langle\alpha_0, \cdots, \alpha_S\rangle$ is $k+1$-solid over $(\mathcal{M}, u)$.

PROOF. Let $A \subseteq \rho_{k+1}^{\mathcal{M}}$ be $r\Sigma_{k+1}^{\mathcal{M}}$ but not a member of $|\mathcal{M}|$. Then A is $r\Sigma_{k+1}^{Q_\theta}$, hence $r\Sigma_{k+1}^{\bar{\mathcal{P}}_\theta}$, hence $r\Sigma_{k+1}^{\mathcal{H}}$. But if $s < S+1$, this means A is $r\Sigma_{k+1}^{\mathcal{M}}$ in a parameter from $(\alpha_s \cup \{\alpha_0, \cdots, \alpha_{s-1}, u\})^{<\omega}$, hence in u and a parameter $<_{\text{lex}}$ $\langle \alpha_0, \cdots, \alpha_S \rangle$. This contradicts the minimality of $\langle \alpha_0, \cdots, \alpha_S \rangle$. □

CLAIM 9. $\mathcal{P}(\rho_{k+1}^{\mathcal{M}})^{\mathcal{M}} = \mathcal{P}(\rho_{k+1}^{\mathcal{M}})^{\mathcal{H}}$; that is, r is $k+1$-universal over $(\mathcal{M}, u)$.

PROOF. This follows easily from the facts that $\bar{\mathcal{P}}_\theta = Q_\theta$ and crit $\bar{\imath}_{0,\theta} \geq \rho_{k+1}^{\mathcal{M}}$, crit $i_{0,\theta}^{\mathcal{U}} \geq \rho_{k+1}$. □

This completes the proof of Theorem 8.1. □

The method by which 8.1 was proved gives some condensation results for 1-small coremice. One which will be of use to us is the following.

Theorem 8.2. *Let $\mathcal{H}$ and $\mathcal{M}$ be 1-small coremice, and suppose there is a non-trivial fully elementary $\pi : \mathcal{H} \to \mathcal{M}$ such that crit $(\pi) = \rho_\omega^{\mathcal{H}}$. Then either*

(a) *$\mathcal{H}$ is a proper initial segment of $\mathcal{M}$*

or

(b) *There is an extender E on the $\mathcal{M}$ sequence such that* $\operatorname{lh} E = \rho_\omega^{\mathcal{H}}$ *and $\mathcal{H}$ is a proper initial segment of* $\operatorname{Ult}_0(\mathcal{M}, E)$.

Remark. In case (b), H is not an initial segment of $\mathcal{M}$. The following example shows that case (b) can occur. Suppose $\mathcal{P}$ is an active 1-small coremouse, $\kappa =$ crit $\dot{F}^{\mathcal{P}}$, and $\dot{F}^{\mathcal{P}} \restriction \alpha$ is on the $\mathcal{P}$ sequence for some $\alpha > (\kappa^+)\mathcal{P}$. (We shall later construct such a $\mathcal{P}$.) Let

$$\sigma : \operatorname{Ult}_0(\mathcal{P}, \dot{F}^{\mathcal{P}} \restriction \alpha) \to \operatorname{Ult}_0(\mathcal{P}, \dot{F}^{\mathcal{P}})$$

be the natural embedding. It is easy to see $\alpha =$ crit (σ). Let

$$\mathcal{H} = (J_{\alpha+1}^{\dot{E}^{\mathcal{P}} \restriction \alpha}, \in, \dot{E}^{\mathcal{P}} \restriction \alpha)$$

and

$$\mathcal{M} = \sigma(\mathcal{H}), \quad \pi = \sigma \restriction \mathcal{H}.$$

Clearly $\alpha = \operatorname{crit}(\pi) = \rho_\omega^{\mathcal{H}}$, π is fully elementary, and $\mathcal{H}$ is not an initial segment of $\mathcal{M}$.

PROOF OF 8.2. Suppose first that $\operatorname{lh} E < \rho_\omega^{\mathcal{H}}$ for all extenders E from the $\mathcal{H}$ sequence. Then either $\mathcal{H}$ is an initial segment of $\mathcal{M}$, so that (a) holds, or we have a first E from the $\mathcal{M}$ sequence such that $\rho_\omega^{\mathcal{H}} \leq \operatorname{lh} E \leq \mathrm{OR}^{\mathcal{H}}$. As $\mathcal{M}$ is internally iterable, $\operatorname{lh} E$ is a cardinal of $L[\dot{E}^{\mathcal{M}} \restriction \rho_\omega^{\mathcal{H}}]$. But $\operatorname{card}(\mathrm{OR}^{\mathcal{H}}) \leq \rho_\omega^{\mathcal{H}}$ in $L[\dot{E}^{\mathcal{M}} \restriction \rho_\omega^{\mathcal{H}}]$, so $\operatorname{lh} E = \rho_\omega^{\mathcal{H}}$. Moreover, $\mathcal{H}$ is an initial segment of $\operatorname{Ult}_0(\mathcal{M}, E)$ as otherwise again we have a cardinal of $L[\dot{E}^{\mathcal{M}} \restriction \rho_\omega^{\mathcal{H}}]$ strictly between $\rho_\omega^{\mathcal{H}}$ and $\mathrm{OR}^{\mathcal{H}}$. So we have alternative (b).

So we may assume $\rho_\omega^{\mathcal{H}} \leq \operatorname{lh} E$ for some E from the $\mathcal{H}$ sequence, and hence $\rho_\omega^{\mathcal{M}} \leq \operatorname{lh} E$ for some E from the $\mathcal{M}$ sequence.

The next section of the the proof will be almost the same as the start of the proof of theorem 8.1. We will compare $\mathcal{H}$ with $\mathcal{M}$ as in in theorem 8.1, with $\rho_\omega^{\mathcal{H}}$ in the place of α_s and ω in the place of k, noticing that the proof of the strong uniqueness theorem gives easily that every ω-maximal iteration tree on $\mathcal{H}$ or $\mathcal{M}$ is simple. Everything will go through almost exactly as before until the point where we used the fact that there was a subset A of α_s which is definable in $\mathcal{H}$ and not in $\mathcal{M}$. Thus we will conclude that $0\bar{T}\theta$, that there is no dropping along $[0,\theta]_{\bar{\mathcal{T}}}$, and that $\bar{\mathcal{P}}_\theta \leq \mathcal{Q}_\theta$. It will follow immediately that $\bar{\imath}_{0,\theta}$ is the identity, since the use of any extender with critical point greater than or equal to $\rho_\omega^{\mathcal{H}}$ would cause a drop.

We now continue with the detailed proof. As before, we define three ω-maximal trees by induction on length:

(1) a"psuedo iteration tree" $\bar{\mathcal{T}}$ on the pair $(\mathcal{H},\mathcal{M})$, with models $\bar{\mathcal{P}}_\alpha$; (2) an iteration tree $\mathcal{T}$ on $\mathcal{M}$ enlarging $\mathcal{T}$, with models $\mathcal{P}_\alpha$, and (3) an iteration tree $\mathcal{U}$ on $\mathcal{M}$ with models $\mathcal{Q}_\alpha$. We also have embeddings

$$\pi_\alpha : \bar{\mathcal{P}}_\alpha \to \mathcal{P}_\alpha$$

such that π_α is a $\overline{\deg}\,(\alpha)$ embedding. The π_α's have the natural commutativity and agreement properties they had in 8.1.

Set

$$\bar{\mathcal{P}}_0 = \mathcal{H},\ \bar{\mathcal{P}}_{-1} = \mathcal{M},\ \mathcal{P}_0 = \mathcal{Q}_0 = \mathcal{M}$$

and

$$\pi_0 = \pi,\ \pi_{-1} = \text{identity}\,.$$

The remainder of $\mathcal{T}$, $\bar{\mathcal{T}}$, and $\mathcal{U}$ is defined by induction just as in 8.1: we get $\bar{\mathcal{P}}_{\alpha+1}$ and $\mathcal{Q}_{\alpha+1}$ by "iterating the least disagreement" between $\bar{\mathcal{P}}_\alpha$ and $\mathcal{Q}_\alpha$, as in the comparison process. We get $\pi_{\alpha+1}$ and $\mathcal{P}_{\alpha+1}$ by copying. The role of α_s in the proof of 8.1 is played here by $\rho_\omega^{\mathcal{H}}$; that is, if crit $\bar{E}_\alpha < \rho_\omega^{\mathcal{H}}$, then $-1\bar{T}(\alpha+1)$.

As before, we get θ such that $\bar{\mathcal{P}}_\theta$ is an initial segment of $\mathcal{Q}_\theta$ or vice-versa.

We say a branch b of $\mathcal{U}$ drops if either $D^{\mathcal{U}} \cap b \neq \varnothing$ or $\deg^{\mathcal{U}}(\alpha) < \omega$ for some $\alpha \in b$. Similarly for branches of $\mathcal{T}$ and $\bar{\mathcal{T}}$. Since we are dealing with ω-maximal trees on fully sound mice, we have that

(a) if $\{\beta \mid \beta\bar{T}\theta\}$ drops, then $\mathcal{Q}_\theta$ is a proper initial segment of $\bar{\mathcal{P}}_\theta$ and $[0,\theta]_{\mathcal{U}}$ doesn't drop and
(b) if $[0,\theta]_{\mathcal{U}}$ drops, then $\bar{\mathcal{P}}_\theta$ is a proper initial segment of $\mathcal{Q}_\theta$ and $\{\beta \mid \beta\bar{T}\theta\}$ doesn't drop.

CLAIM 1. $\{\beta \mid \beta\bar{T}\theta\}$ doesn't drop.

PROOF. By (a) above, if $\{\beta \mid \beta \bar{T} \theta\}$ drops then $\pi_\theta \circ i^{\mathcal{U}}_{0,\theta}$ is a fully elementary embedding from $\mathcal{M}$ to a proper initial segment of $\mathcal{P}_\theta$, which lies on a simple iteration tree based on $\mathcal{M}$. This contradicts the Dodd-Jensen lemma.

CLAIM 2. $0\bar{T}\theta$.

PROOF. Suppose $-1\bar{T}\theta$.

CASE 1. $[0,\theta]_U$ drops. Then $\bar{\imath}_{-1,\theta}$ is a fully elementary embedding from $\mathcal{M}$ to a proper initial segment of Q_θ. This contradicts the Dodd-Jensen lemma.

CASE 2. $[0,\theta]_U$ doesn't drop. If one of $\bar{\mathcal{P}}_\theta$ and Q_θ is a proper initial segment of the other, then we have a contradiction to the Dodd-Jensen lemma. So suppose $\bar{\mathcal{P}}_\theta = Q_\theta$. Then as in Case 3 of the proof of Claim 1 of 8.1, $\bar{\imath}_{-1,\theta} = i^{\mathcal{U}}_{0,\theta}$. This means that the first extender used along $[-1,\theta]_{\mathcal{T}}$ is compatible with the first extender used along $[0,\theta]_{\mathcal{U}}$, which is impossible. □

CLAIM 3. $\bar{\imath}_{0,\theta} =$ identity.

PROOF. Otherwise, since $\rho^{\bar{\mathcal{P}}_0}_\omega \le \operatorname{crit} \bar{\imath}_{0,\theta}$, $[0,\theta]_{\mathcal{T}}$ drops. This contradicts Claim 1. □

CLAIM 4. $\bar{\mathcal{P}}_\theta = \mathcal{H}$ is a proper initial segment of Q_θ.

PROOF. If $[0,\theta]_{\mathcal{U}}$ drops, then in fact $\bar{\mathcal{P}}_\theta$ must be a proper initial segment of Q_θ, as $\bar{\mathcal{P}}_\theta$ is ω-sound. If $[0,\theta]_{\mathcal{U}}$ doesn't drop, then $\bar{\mathcal{P}}_\theta$ is an initial segment of Q_θ as otherwise $\pi_\theta \circ i^{\mathcal{U}}_{0,\theta}$ contradicts the Dodd-Jensen lemma. But $\rho^{\mathcal{H}}_\omega < \rho^{\mathcal{M}}_\omega \le i^{\mathcal{U}}_{0,\theta}(\rho^{\mathcal{M}}_\omega) = \rho^{Q_\theta}_\omega$, so $\bar{\mathcal{P}}_\theta = \mathcal{H} = Q_\theta$ is impossible. □

Our proof now deviates from that of theorem 8.1. In order to show that $\mathcal{U}$ is the desired tree we must verify that either (a) or (b) of the statement of 8.2 holds. Suppose (a) fails, that is, $\mathcal{U}$ is nontrivial. So $E^{\mathcal{U}}_0 \ne \varnothing$. Now $\rho^{\mathcal{H}}_\omega \le \operatorname{lh} E^{\mathcal{U}}_0$ since $\operatorname{crit}(\pi) = \rho^{\mathcal{H}}_\omega$, and $\operatorname{lh} E^{\mathcal{U}}_0 \le \mathrm{OR}^{\mathcal{H}}$, since otherwise $\mathcal{H}$ would be an initial segment of $\mathcal{M}$. But now $\operatorname{lh} E^{\mathcal{U}}_0$ is a cardinal of Q_θ, and $\mathcal{H}$ is a proper initial segment of Q_θ, so that $\operatorname{card}(\mathrm{OR}^{\mathcal{H}}) \le \rho^{\mathcal{H}}_\omega$ in Q_θ. So we must have $\operatorname{lh} E^{\mathcal{U}}_0 = \rho^{\mathcal{H}}_\omega$. Similarly, if $E^{\mathcal{U}}_1$ exists, then $\mathrm{OR}^{\mathcal{H}} < \operatorname{lh} E^{\mathcal{U}}_1$. So in fact $E^{\mathcal{U}}_1$ doesn't exist, that is, $\theta = 1$ and $\mathcal{H}$ is a proper initial segment of $Q_1 = \mathrm{Ult}_k(\mathcal{M}, E^{\mathcal{U}}_0)$, where $k = \deg^{\mathcal{U}}(1)$. We can take $k = 0$ because $\mathrm{Ult}_0(\mathcal{M}, E^{\mathcal{U}}_0)$ and $\mathrm{Ult}_k(\mathcal{M}, E^{\mathcal{U}}_0)$ agree to their common value for $(\rho^{\mathcal{H}}_\omega)^+$ and beyond. □

Remark. The hypothesis that $\operatorname{crit}(\pi) = \rho^{\mathcal{H}}_\omega$ is necessary in 8.2. For notice that $\operatorname{crit}(\pi) > \rho^{\mathcal{H}}_\omega$ is impossible since π is fully elementary. (That is, this case is vacuous.) On the other hand, $\operatorname{crit}(\pi) < \rho^{\mathcal{H}}_\omega$ can occur while conclusions (a) and (b) of 8.2 fail: e.g., let $\mathcal{M} = \mathrm{Ult}_\omega(\mathcal{H}, E)$ where E is on the $\mathcal{H}$ sequence and $\operatorname{crit}(E) < \rho^{\mathcal{H}}_\omega$, and let π be the canonical embedding.

One can also derive a version of 8.2 with $\rho^{\mathcal{H}}_{n+1}$ replacing $\rho^{\mathcal{H}}_\omega$. Namely, suppose $\mathcal{H}$ and $\mathcal{M}$ are 1-small, $n+1$ sound mice, and $\pi : \mathcal{H} \to \mathcal{M}$ is $r\Sigma_{n+1}$ elementary with $\operatorname{crit}(\pi) \ge \rho^{\mathcal{H}}_{n+1}$. Then either

(a) $\mathcal{H}$ is a proper initial segment of $\mathcal{M}$, or
(b) $\rho_{n+1}^{\mathcal{H}} = \mathrm{lh}\, E$ for some E from the $\mathcal{M}$ sequence, and $\mathcal{H}$ is a proper initial segment of $\mathrm{Ult}_0(\mathcal{M}, E)$.

The example following the statement of 8.2 shows alternative (b) is necessary.

The proof of this version is almost the same as that of 8.2. We use n-maximal trees in the comparison and modify the uses of Dodd-Jensen slightly to accommodate this change. Note that in this case we don't know that if e.g. $[0,\theta]_{\mathcal{U}}$ drops then $\bar{\mathcal{P}}_\theta$ is a *proper* initial segment of $\mathcal{Q}_\theta$. Also notice that we can assume that there is an extender E from $\mathcal{M}$ sequence with $\mathrm{lh}(\mathcal{M}) \geq \rho^{\mathcal{M}}$, since the result is trivial otherwise.

Notice that alternative (b) of 8.2 (or its "$n+1$ version") cannot arise when $\rho_\omega^{\mathcal{H}}$ (respectively $\rho_{n+1}^{\mathcal{H}}$) is a cardinal of $\mathcal{M}$, simply because $\mathrm{lh}\, E$ is never a cardinal of $\mathcal{M}$ when E is on the $\mathcal{M}$ sequence and $\mathrm{lh}\, E < \mathrm{OR}^{\mathcal{M}}$.

As a sample application of the $n+1$-version: let $\mathcal{M}$ be a 1-small, 1-sound mouse, and let $\alpha < \rho_1^{\mathcal{M}}$, α a cardinal of $\mathcal{M}$. Let $p = p_1(\mathcal{M})$, and $\mathcal{H} = \mathcal{H}_1^{\mathcal{M}}(\alpha \cup \{p\})$. Let $\pi : \mathcal{H} \to \mathcal{M}$ be the inverse of the collapse. Clearly $\alpha = \rho_1^{\mathcal{H}} \leq \mathrm{crit}(\pi)$, and π is $r\Sigma_1$ elementary. Suppose α is large enough that the solidity witnesses for p are all of the form $\tau^{\mathcal{M}}[\bar{\beta}, p]$ for some $\bar{\beta} \in \alpha^{<\omega}$ and $\tau \in \mathrm{Sk}_1$. This guarantees that $\pi^{-1}(p)$ is the first standard parameter of $\mathcal{H}$, and that $\mathcal{H}$ is 1-sound. We can then conclude that $\mathcal{H}$ is a proper initial segment of $\mathcal{M}$.

We don't know whether the assumption that $\mathcal{H}$ is $n+1$ sound can be reduced to n soundness. If this can be done, then in the application just mentioned we needn't assume $p = p_1(\mathcal{M})$ or make the largeness assumption about α.