§7. The Comparison Process

We prove in this section a comparison lemma for 1-small mice. Our interest is not so much in the lemma itself, but in the method by which it is proved. We shall use that method in a much more important way in the next section.

For bookkeeping purposes we shall use "padded iteration trees". These are just like ordinary iteration trees except that we modify the successor clause in the definition of "iteration tree" so as to allow $\alpha T(\alpha+1)$, $\mathcal{M}_\alpha = \mathcal{M}_{\alpha+1}$, and $i_{\alpha,\alpha+1} =$ identity, and then require that $\alpha T\beta \Rightarrow \beta = \alpha + 1$ or $(\alpha + 1)\ T\beta$. So a padded tree is essentially an ordinary tree with the indexing of the models slowed down by repetition. We shall no doubt often fail to distinguish between iteration trees and their padded counterparts.

Theorem 7.1 (The comparison lemma). *Let $\mathcal{M}$ and $\mathcal{N}$ be n-sound, 1-small, n-iterable premice, where $n \leq \omega$. Then there are n-maximal padded iteration trees $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on $\mathcal{N}$ such that either*

(1) *$\mathcal{T}$ and $\mathcal{U}$ have successor length $\theta + 1$, and either*

(a) *$\mathcal{M}_\theta$ is an initial segment of $\mathcal{N}_\theta$ and $D^{\mathcal{T}} \cap [0,\theta]_T = \varnothing$ and deg$(\alpha+1) = n$ for all $\alpha + 1 \in [0,\theta]_T$, or*

(b) *$\mathcal{N}_\theta$ is an initial segment of $\mathcal{M}_\theta$ and $D^{\mathcal{U}} \cap [0,\theta]_U = \varnothing$ and deg$(\alpha+1) = n$ for all $\alpha + 1 \in [0,\theta]_U$,*

or

(2) *$\mathcal{T}$ and $\mathcal{U}$ have limit length, one of the two is not simple, and in some $V^{\mathrm{Col}(\kappa,\omega)}$ there are wellfounded cofinal branches b of $\mathcal{T}$ and c of $\mathcal{U}$ such that either*

(a) *$\mathcal{M}_b$ is an initial segment of $\mathcal{N}_c$, $D^{\mathcal{T}} \cap b = \varnothing$, and deg$(\alpha+1) = n$ for all $\alpha + 1 \in b$, or*

(b) *$\mathcal{N}_c$ is an initial segment of $\mathcal{M}_b$, $D^{\mathcal{U}} \cap c = \varnothing$, and deg$(\alpha+1) = n$ for all $\alpha + 1 \in c$.*

Proof. We define by induction on γ

$$\mathcal{T} \upharpoonright \gamma = \langle T \cap (\gamma \times \gamma), D^{\mathcal{T}} \cap \gamma, \deg^{\mathcal{T}} \upharpoonright \gamma, \langle E^{\mathcal{T}}_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha + 1 < \gamma\rangle\rangle$$

and

$$\mathcal{U} \upharpoonright \gamma = \langle U \cap (\gamma \times \gamma), D^{\mathcal{U}} \cap \gamma, \deg^{\mathcal{U}} \upharpoonright \gamma, \langle E^{\mathcal{U}}_\alpha, \mathcal{N}^*_{\alpha+1} \mid \alpha + 1 < \gamma\rangle\rangle$$

together with the associated $\mathcal{M}_\alpha$, $\mathcal{N}_\alpha$ for $\alpha < \gamma$, $\rho^{\mathcal{T}}_\alpha$ and $\rho^{\mathcal{U}}_\alpha$ for $\alpha + 1 < \gamma$, and embeddings $i^{\mathcal{T}}_{\alpha\beta}$, $i^{\mathcal{U}}_{\alpha\beta}$ (for (α,β) as appropriate). The method for defining $\mathcal{T}$ and $\mathcal{U}$ is the standard one of "iterating the least disagreement".

We begin with $\gamma = 1$. In this case we need only define $\mathcal{M}_0$ and $\mathcal{N}_0$, which we do by setting

$$\mathcal{M}_0 = \mathcal{M}\,,$$
$$\mathcal{N}_0 = \mathcal{N}\,.$$

Now consider the case γ is a limit ordinal. Then

$$\mathcal{T} \restriction \gamma = \text{“}\bigcup_{\beta<\gamma}\text{”}\, \mathcal{T} \restriction \beta$$
$$\mathcal{U} \restriction \gamma = \text{“}\bigcup_{\beta<\gamma}\text{”}\, \mathcal{U} \restriction \beta$$

where the union is taken along each of the 4 coordinates.

Now suppose $\gamma = \lambda + 1$, where λ is a limit ordinal. If $\mathcal{T} \restriction \lambda$ or $\mathcal{U} \restriction \lambda$ is not simple, we stop our induction. Suppose $\mathcal{T} \restriction \lambda$ and $\mathcal{U} \restriction \lambda$ are simple. As $\mathcal{M}$ and $\mathcal{N}$ are n-iterable we have wellfounded branches b of $\mathcal{T} \restriction \lambda$ and c of $\mathcal{U} \restriction \lambda$ which are cofinal in λ. Set

$$\mathcal{M}_\lambda = \mathcal{M}_b\,,$$
$$\mathcal{N}_\lambda = \mathcal{N}_c\,,$$

$$T \cap (\gamma \times \gamma) = (T \cap (\lambda \times \lambda)) \cup \{(\alpha, \lambda) \mid \alpha \in b\}$$
$$U \cap (\gamma \times \gamma) = (U \cap (\lambda \times \lambda)) \cup \{(\alpha, \lambda) \mid \alpha \in c\}\,,$$

$$i^{\mathcal{T}}_{\alpha\lambda} = i^{\mathcal{T}}_{\alpha b} \quad \text{for } \alpha \in b - \sup(D^{\mathcal{T}} \cap b)\,,$$
$$i^{\mathcal{U}}_{\alpha\lambda} = i^{\mathcal{U}}_{\alpha c} \quad \text{for } \alpha \in c - \sup(D^{\mathcal{N}} \cap c)\,.$$

The rest of $\mathcal{T} \restriction \gamma$ and $\mathcal{U} \restriction \gamma$ is determined by this.

Finally, we have the case $\gamma = \eta{+}2$. Here we must define $E^{\mathcal{T}}_\eta$, $\mathcal{M}^*_{\eta+1}$, $D^{\mathcal{T}} \cap \{\eta{+}1\}$, $\deg^{\mathcal{T}}(\eta{+}1)$, T-pred$(\eta{+}1)$, and similarly for the $\mathcal{U}$ side. We are given the models $\mathcal{M}_\eta$ and $\mathcal{N}_\eta$.

If $\mathcal{M}_\eta$ is an initial segment of $\mathcal{N}_\eta$, or vice-versa, then we stop our inductive definition. Otherwise we have a least ordinal γ such that $\mathcal{J}^{\mathcal{M}_\eta}_\gamma \neq \mathcal{J}^{\mathcal{N}_\eta}_\gamma$. Set

$$E^{\mathcal{T}}_\eta = \begin{cases} \varnothing & \text{if } \mathcal{J}^{\mathcal{M}_\eta}_\gamma \text{ is passive}\,, \\ \dot{F}^{\mathcal{J}^{\mathcal{N}_\eta}_\gamma} & \text{if } \mathcal{J}^{\mathcal{N}_\eta}_\gamma \text{ is active}\,, \end{cases}$$

$$E^{\mathcal{U}}_\eta = \begin{cases} \varnothing & \text{if } \mathcal{J}^{\mathcal{N}_\eta}_\gamma \text{ is passive}\,, \\ \dot{F}^{\mathcal{J}^{\mathcal{N}_\eta}_\gamma} & \text{if } \mathcal{J}^{\mathcal{N}_\eta}_\gamma \text{ is active}\,. \end{cases}$$

The rest is determined by the rules for forming non-overlapping, n-maximal, padded iteration trees. So, on the $\mathcal{T}$ side:

If $E_\eta^{\mathcal{T}} = \varnothing$, then $\mathcal{T}$-pred$(\eta+1) = \eta$, $\mathcal{M}^*_{\eta+1} = \mathcal{M}_{\eta+1} = \mathcal{M}_\eta$, $i^{\mathcal{T}}_{\eta,\eta+1} =$ identity. In this case, we also set $\rho_\eta^{\mathcal{T}} = 0$.

Now suppose $E_\eta^{\mathcal{T}} \neq \varnothing$. Let $\kappa =$ crit $E_\eta^{\mathcal{T}}$, and let $\beta \leq \eta$ be least such that $\kappa < \rho_\beta^{\mathcal{T}}$. Then we set $\mathcal{T}$-pred$(\eta+1) = \beta$. Let

$$\begin{aligned}\mathcal{M}^*_{\eta+1} &= \text{longest initial segment } \mathcal{P} \text{ of } \mathcal{M}_\beta \text{ such that}\\ &\quad P(\kappa) \cap |\mathcal{P}| = P(\kappa) \cap |\mathcal{J}_\gamma^{\mathcal{M}_\eta}|\\ &= \text{longest initial segment } \mathcal{P} \text{ of } \mathcal{M}_\beta \text{ such that}\\ &\quad P(\kappa) \cap |\mathcal{P}| = P(\kappa) \cap |\mathcal{J}^{\mathcal{M}_\beta}_{\mathrm{lh}\, E_\beta}|.\end{aligned}$$

We let

$$\eta + 1 \in D^{\mathcal{T}} \Leftrightarrow \mathcal{M}^*_{\eta+1} \neq \mathcal{M}_\beta .$$

Let

$$\begin{aligned}k &= \text{largest } m \leq \omega \text{ such that } \kappa < \rho_m^{\mathcal{M}^*_{\eta+1}} \text{ and}\\ &\quad D^{\mathcal{T}} \cap [0, \eta+1]_T = \varnothing \Rightarrow m \leq n .\end{aligned}$$

We let $\deg^{\mathcal{T}}(\eta+1) = k$, and

$$\mathcal{M}_{\eta+1} = \mathrm{Ult}_k(\mathcal{M}^*_{\eta+1}, E_\eta^{\mathcal{T}}),$$

and let $i^*_{\eta+1} : \mathcal{M}^*_{\eta+1} \to \mathcal{M}_{\eta+1}$ be the canonical embedding, and for $\alpha T(\eta+1)$ such that $D^{\mathcal{T}} \cap (\alpha, \eta+1]_T = \varnothing$, let $i^{\mathcal{T}}_{\alpha,\eta+1} = i^*_{\eta+1} \circ i^{\mathcal{T}}_{\alpha,\beta}$.

This completes the definition of $\mathcal{T} \restriction \eta+2$. We obtain $\mathcal{U} \restriction \eta+2$ from $E_\eta^{\mathcal{U}}$ in a similar fashion.

This completes the definitions of $\mathcal{T}$ and $\mathcal{U}$. It is easy to see they are iteration trees.

Claim. If $\alpha < \beta$, then $\max(\rho_\alpha^{\mathcal{T}}, \rho_\alpha^{\mathcal{U}}) < \min(\rho_\beta^{\mathcal{T}}, \rho_\beta^{\mathcal{U}})$.

Proof. $\gamma = \mathrm{lh}\, E_\alpha^{\mathcal{T}}$ is a cardinal of $\mathcal{M}_\beta$, and hence a cardinal of $J^{\mathcal{N}_\beta}_{\mathrm{lh}\, E_\beta^{\mathcal{N}}}$. As γ is a cardinal of $J^{\mathcal{M}_\beta}_{\mathrm{lh}\, E_\beta^{\mathcal{T}}}$, $\gamma \leq \rho_\beta^{\mathcal{T}}$. As γ is a cardinal of $J^{\mathcal{N}_\beta}_{\mathrm{lh}\, E_\beta^{\mathcal{T}}}$, $\gamma \leq \rho_\beta^{\mathcal{U}}$. So $\rho_\alpha^{\mathcal{T}} < \gamma \leq \min(\rho_\beta^{\mathcal{T}}, \rho_\beta^{\mathcal{U}})$. Symmetrically, $\rho_\alpha^{\mathcal{U}} < \min(\rho_\beta^{\mathcal{T}}, \rho_\beta^{\mathcal{U}})$.

Lemma 7.2. *Let $\alpha+1, \beta+1 < \mathrm{lh}\,\mathcal{T}$. Suppose $E_\alpha^{\mathcal{T}} \neq \varnothing$ and $E_\beta^{\mathcal{U}} \neq \varnothing$. Suppose crit $E_\alpha^{\mathcal{T}} =$ crit $E_\beta^{\mathcal{U}} = \kappa$. Then there is a parameter $a \in [\rho_\alpha^{\mathcal{T}} \cap \rho_\beta^{\mathcal{U}}]^{<\omega}$, and a set $A \subseteq [\kappa]^{\mathrm{card}\, a}$ such that*

$$A \in (E_\alpha^{\mathcal{T}})_a \quad \textit{and} \quad A \notin (E_\beta^{\mathcal{U}})_a .$$

PROOF. We may as well assume $\alpha \leq \beta$. Notice then $\rho^{\mathcal{T}}_\alpha \leq \rho^{\mathcal{U}}_\beta$, and

$$P(\kappa) \cap J^{\mathcal{M}_\alpha}_{\mathrm{lh}\, E^{\mathcal{T}}_\alpha} = P(\kappa) \cap J^{\mathcal{M}_\beta}_{\mathrm{lh}\, E^{\mathcal{U}}_\beta} = P(\kappa) \cap J^{\mathcal{N}_\beta}_{\mathrm{lh}\, E^{\mathcal{U}}_\beta}$$

by 5.1 and the fact that $\mathcal{M}_\beta$ and $\mathcal{N}_\beta$ agree below $\mathrm{lh}\, E^{\mathcal{U}}_\beta$. So $E^{\mathcal{T}}_\alpha$ and $E^{\mathcal{U}}_\beta$ are defined on the same subsets of κ, and it will suffice to show

$$E^{\mathcal{T}}_\alpha \restriction \rho^{\mathcal{T}}_\alpha \neq E^{\mathcal{U}}_\beta \restriction \rho^{\mathcal{T}}_\alpha .$$

Suppose $E^{\mathcal{T}}_\alpha \restriction \rho^{\mathcal{T}}_\alpha = E^{\mathcal{U}}_\beta \restriction \rho^{\mathcal{T}}_\alpha$. Now $E^{\mathcal{T}}_\alpha$ is the trivial completion of $E^{\mathcal{T}}_\alpha \restriction \rho^{\mathcal{T}}_\alpha$, and so the initial segment condition on $\mathcal{N}_\beta$ gives us two possibilities. We may have $E^{\mathcal{T}}_\alpha$ on the sequence of $\mathcal{N}_\beta$ at or before the position of $E^{\mathcal{U}}_\beta$. But $E^{\mathcal{T}}_\alpha$ is not on the $\mathcal{N}_\alpha$ sequence because it is part of the least disagreement at α, and by coherence then $E^{\mathcal{T}}_\alpha$ is not on the $\mathcal{N}_\beta$ sequence for all $\beta \geq \alpha$. Thus the second possibility from the initial segment condition is realized: $\rho^{\mathcal{T}}_\alpha \in \mathrm{dom}\, \dot{E}^{\mathcal{N}_\beta}$ and $E^{\mathcal{T}}_\alpha$ is on the sequence of $\mathrm{Ult}_0(\mathcal{P}, F)$, where $F = (\dot{E}^{\mathcal{N}_\beta})_{\rho^{\mathcal{T}}_\alpha}$ and $\mathcal{P} = J^{\mathcal{N}_\beta}_{\rho^{\mathcal{T}}_\alpha}$. In this case, F is on the sequence of $\mathcal{N}_\beta$, and hence of $\mathcal{M}_\beta$ as $\rho^{\mathcal{T}}_\alpha < \mathrm{lh}\, E^{\mathcal{U}}_\beta$. Thus F is on the sequence of $\mathcal{M}_\alpha$ as $\rho^{\mathcal{T}}_\alpha < \mathrm{lh}\, E^{\mathcal{T}}_\alpha$. Also $\mathcal{P} = J^{\mathcal{M}_\alpha}_{\rho^{\mathcal{T}}_\alpha}$. By coherence, F is on the sequence of $\mathrm{Ult}_0(\mathcal{P}, E^{\mathcal{T}}_\alpha)$ and F is not on the sequence of $\mathrm{Ult}_0\,(\mathrm{Ult}_0(\mathcal{P}, F), E^{\mathcal{T}}_\alpha)$. This is a contradiction, as these ultrapowers agree past $\mathrm{lh}\, E^{\mathcal{T}}_\alpha$. □

CLAIM. The inductive definitions of $\mathcal{T}$ and $\mathcal{U}$ halt at some ordinal γ such that $\gamma \leq \max(\mathrm{card}\ \mathcal{M},\ \mathrm{card}\ \mathcal{N})^+$.

PROOF. Let $\theta = \max(\mathrm{card}\ \mathcal{M},\ \mathrm{card}\ \mathcal{N})^+$. If the claim is false, then $\mathcal{M}_\theta$ and $\mathcal{N}_\theta$ are defined. Let $b = [0, \theta)_T$ and $c = [0, \theta)_U$. So b and c are club in θ. By the standard closure argument we can find a club $d \subseteq b \cup c$ such that

(i) $D^{\mathcal{T}} \cap d = D^{\mathcal{N}} \cap d = \varnothing$.

(ii) $\alpha \in d \Rightarrow \alpha = \mathrm{crit}\ i^{\mathcal{T}}_{\alpha b} = \mathrm{crit}\ i^{\mathcal{U}}_{\alpha c}$.

(iii) $\alpha, \beta \in d \wedge \alpha < \beta \Rightarrow (i^{\mathcal{T}}_{\alpha\beta}(\alpha) = \beta \wedge i^{\mathcal{U}}_{\alpha\beta}(\alpha) = \beta)$.

(iv) $(\alpha \in d \wedge A \subseteq [\alpha]^n \wedge A \in |\mathcal{M}_\alpha| \cap |\mathcal{N}_\alpha|) \Rightarrow i^{\mathcal{T}}_{\alpha b}(A) = i^{\mathcal{U}}_{\alpha c}(A)$.

Now let d satisfy (i)-(iv) and take $\alpha \in d$. Let $\beta + 1$ and $\gamma + 1$ be the successor of α in b and c respectively, so that T-$\mathrm{Pred}(\beta + 1) = U$-$\mathrm{Pred}(\gamma + 1) = \alpha$. Since $\mathcal{T}$ and $\mathcal{U}$ are non-overlapping, $\mathrm{crit}\ i^{\mathcal{T}}_{\beta+1,b} \geq \rho^{\mathcal{T}}_\beta$ and $\mathrm{crit}\ i^{\mathcal{U}}_{\gamma+1,c} \geq \rho^{\mathcal{U}}_\gamma$. By (iv) we see that for all $A \subseteq [\alpha]^n$, $A \in |\mathcal{M}_\alpha| \cap |\mathcal{N}_\alpha|$,

$$i^{\mathcal{T}}_{\alpha,\beta+1}(A) \cap [\rho]^n = i^{\mathcal{U}}_{\alpha,\gamma+1}(A) \cap [\rho]^n$$

where $\rho = \rho^{\mathcal{T}}_\beta \cap \rho^{\mathcal{U}}_\gamma$. It follows that $E^{\mathcal{T}}_\beta \restriction \rho = E^{\mathcal{U}}_\gamma \restriction \rho$, contradicting the lemma. This proves our claim. □

There are two ways the construction of $\mathcal{T}$ and $\mathcal{U}$ can halt. Suppose first we reach $\theta + 1$ such that $\mathcal{M}_\theta$ is an initial segment of $\mathcal{N}_\theta$ or vice-versa. If $\mathcal{M}_\theta$ is a proper initial segment of $\mathcal{N}_\theta$, then there's no dropping along $[0,\theta]_T$ because $\mathcal{N}_\theta$ is a premouse so that $\mathcal{M}_\theta$ is ω-sound. So we have (1) (a) of our desired conclusion. If $\mathcal{N}_\theta$ is a proper initial segment of $\mathcal{M}_\theta$ we have (1) (b) of our desired conclusion. Finally, if $\mathcal{M}_\theta = \mathcal{N}_\theta$ then on one of $[0,\theta]_T$ and $[0,\theta]_U$ there's no dropping; the proof is just like that of Claim 4 in the proof of 6.2, so we omit it.

Suppose next the construction halts because we reach a limit ordinal θ such that one of $\mathcal{T} \upharpoonright \theta$ and $\mathcal{U} \upharpoonright \theta$ is not simple. Say $\mathcal{T} \upharpoonright \theta$ is not simple, so there are distinct wellfounded cofinal branches of $\mathcal{T} = \mathcal{T} \upharpoonright \theta$. Just as in the proof of the first 4 claims of 6.2, we can find a cofinal wellfounded branch b of $\mathcal{T}$ such that $D^{\mathcal{T}} \cap b = \varnothing$ and $\deg(\alpha+1) = n$ for all $\alpha+1 \in b$, and $\mathcal{M}_b$ has no extenders with length $\geq \delta = \delta(\mathcal{T})$. Let c be any cofinal, wellfounded branch of $\mathcal{U}$. If $\mathcal{M}_b$ is an initial segment of $\mathcal{N}_c$, we are done. If $\mathcal{N}_c$ is a proper initial segment of $\mathcal{M}_b$ then $\mathcal{N}_c$ is ω-sound, so there's no dropping along c and we're done. The remaining possibility (since $\mathcal{M}_b$ and $\mathcal{N}_c$ agree below δ) is that $\mathcal{N}_c$ has an extender F such that $\delta \leq \operatorname{lh} F \leq \mathrm{OR}^{\mathcal{M}_b}$. But this contradicts the 1-smallness of $\mathcal{N}_c$. □

Remark. We haven't ruled out the possibility that (1) of our Theorem 7.1 holds, that $\mathcal{M}_\theta = \mathcal{N}_\theta$, and that one (but not both!) of $[0,\theta]_T$ and $[0,\theta]_U$ has a drop. Nor have we ruled out the analogous situation in case (2) of 7.1. One can show that this cannot happen in the case $n = \omega$, but for $n < \omega$ we don't know.