Replacement $\not\rightarrow$ Collection *

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Summary. Let M be a transitive model for ZF(or [see D] Basic Set Theory + the Collection Scheme). Let $\langle P_i: i \in I\rangle \in$M be a family of notions of forcing and Q be the weak product of ω copies of the family. If H is Q-generic over M and N $= \bigcup\{\mathrm{M[H}|s]: s \subset I\times\omega, s\text{-finite}\}$, then always N$\models$BST + the Replacement Scheme but not necessary N$\models$the Collection Scheme. If M$\models$ZF, then the Power Set Axiom does not hold in N but N$\models \forall\alpha\exists\beta(\beta = \aleph_\alpha)$. If M$\models$WOP (the Well-Ordering Principle), then N$\models$WOP. Thus, even in Set Theory with WOP and as many alephs as ordinals, the principle of collecting sets in a definable manner does not support the principle of collecting sets in a loose manner.

1. Reconstruction.

I have been inspired by Michael Hallett's "Cantorian Set Theory and Limitation of Size," [see H], to look for justifiable Cantorian set theories different than ZF or ZFC but equiconsistent with ZF.

According to [3], p.73, Cantor claimed in his letter to Mittag-Leffler of 14 November 1884 that the continuum could not be of the second power, even more that '...it has no power specifiable by a number,'(Cantor changed his mind next day). In 1904, Jules König from Budapest presented a paper at the Third International Congress of Mathematicians which claimed that the power of Cantor's continuum was not an aleph at all.

Cantor stated a few times that by sets he meant to include only well-orderable collections that could be joined by some rule into a whole ([3], p.245). He clearly believed in existence of $\aleph_\alpha$ for every ordinal α.

If every set is well-orderable and the power of the continuum is not an aleph, then the Power Set Axiom cannot be accepted, so we need another principle that generates alephs. It is the Hartog's functional.

Let $\aleph(x) = \{\alpha: \exists f(f: \alpha \xrightarrow{1-1} x)\}$. Then ZFH! = BST + the Replacement Scheme + $\forall x\exists\beta(\aleph(x) \subseteq \beta)$ and ZFH = ZFH! + the Collection Scheme are theories pretty close to ZF that do not exclude WOP and the continuum being non-well-orderable by any class.

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For logical scrutiny it would be better to use a modified Gödel's Set Theory (see [2]) to express clearly the idea that the continuum is not well-orderable by any class. The modified Gödel's Set Theory, GBH!, has the axioms of groups A, B, and D, the same as in [2] and the axiom 3 of Group C (the Power Set Axiom) is replaced by $\forall x \exists \beta (\aleph(x) \subseteq \beta)$, every set has Hartog's number.

In practice, we may refrain from using the language of classes since in our models $\aleph(\mathit{Continuum}) = \{\alpha : \exists f (f : \alpha \xrightarrow{1-1} \mathit{Continuum})\}$ is an ordinal, not *On*, even if *Continuum* is a proper class. Of course, if X is a proper class and $\aleph(\mathrm{X}) < \mathrm{On}$, then X is non-well-orderable. Otherwise, we would have Burali-Forti's Antinomy.

Kriple-Platek(KP) Set Theory is a subtheory of ZFH. It is not a subtheory of ZFH! but one can analyze [1] or [2] to see that KP! = KP with the Δ_0-Collection Scheme replaced by the Δ_0-Replacement Scheme is strong enough to define the class L of constructible sets and to prove its absoluteness. If we add $\forall x \exists \beta (\aleph(x) \subseteq \beta)$, then we can prove the Condensation Lemma, so ZFH!$\vdash$ [L is an inner model for ZF + V = L]. Thus, ZFC is one of many extensions of ZFH! equiconsistent with ZFH!

In full ZF there is no difference between the Replacement Scheme,

$$\forall x \exists ! y \Phi \Longrightarrow \forall a \exists ! b [\forall x \in a \exists y \in b \Phi \ \&\ \forall y \in b \exists x \in a \Phi],$$

and the Collection Scheme,

$$\forall x \exists y \Phi \Longrightarrow \forall a \exists b [\forall x \in a \exists y \in b \Phi \ \&\ \forall y \in b \exists x \in a \Phi],$$

but from a purely logical point of view they should be different—"attached" sets ought to be more reliable than "loose" sets. The Collection Scheme is a principle of choice for "sets of classes". If $\{\mathrm{X}_j\}_{j \in \mathrm{J}}$ is a "set of classes", then J is a set and $\mathrm{X} = \{\langle j, x \rangle : j \in \mathrm{J} \ \&\ x \in \mathrm{X}_j\}$ is a class that formally represents the family. The Collection Scheme says that for each "set of non-empty classes" there is a family $\{z_j\}_{j \in \mathrm{J}}$ of non-empty sets such that $z_j \subseteq \mathrm{X}_j$ for $j \in \mathrm{J}$. The Replacement Scheme is a more fundamental principle, so I would like to have a model for ZFH! + WOP + $\neg$the Collection Scheme + the continuum is not well-orderable by any class. Actually, a uniform way of constructing many such models with different additional properties.

2. General Description of Models for ZFH! and Consistency Results.

There is a uniform way to construct models for ZFH!. Start with a set I and mutually disjoint collections A_i of subsets of L for $i \in \mathrm{I}$ such that the elements of $\mathrm{A} = \bigcup_{i \in \mathrm{I}} \mathrm{A}_i$ are somehow mutually independent and for each $i \in \mathrm{I}$ the elements of A_i are alike. Suppose that $\Theta(x,y)$ is an absolute formula such that $a \in \mathrm{A}_i \Longrightarrow \Theta(i,a)$; $j \neq i$ & $a \in \mathrm{A}_i \Longrightarrow \neg\Theta(j,a)$ for $i,j \in \mathrm{I}$.

Define $\mathrm{Z(A)} = \bigcup\{\mathrm{L}[s]: s \subset \mathrm{A}, s\text{-finite}\}$. Z(A) is closed on Gödel's operations, so Z(A)$\models$BST (see [1] p.36). Of course, $\mathrm{L}[s] \models$WOP. Hence, Z(A)$\models$BST + WOP.

The mutual independence of elements of A is the key to the Replacement Scheme in Z(A). Its exact meaning will be described in §2 in a more technical fashion. Among many things it means that
$\mathrm{L}[s_1] \cap \mathrm{L}[s_2] = \mathrm{L}[s_1 \cap s_2]$ for $s_1, s_2 \subset \mathrm{A}$, s_1, s_2-finite; if $\Phi(x,\mathbf{d})$ is a formula such that parameters
$\mathbf{d} \in \mathrm{L}[s]$ and $\mathrm{Z(A)} \models \exists! x\Phi(x,\mathbf{d})$, then $(\exists x \in \mathrm{L}[s])\mathrm{Z(A)} \models \Phi(x,\mathbf{d})$.

If $a \in \mathrm{Z(A)}$ and $\mathrm{Z(A)} \models \forall x \in a \exists! y \Psi(x,y,\mathbf{d})$, then there is a finite $s \subset \mathrm{A}$ such that $a, \mathbf{d} \in \mathrm{L}[s]$. Consequently, there is $b \subset \mathrm{L}[s]$ such that $\forall x \in a \exists y \in b \mathrm{Z(A)} \models \Psi(x,y,\mathbf{d})$, $b \subset \mathrm{L}_\alpha[s]$ for some α and $\mathrm{L}_\alpha[s] \in \mathrm{L}[s] \subseteq \mathrm{Z(A)}$. Once we have the Comprehension Scheme in Z(A), we can prove that Z(A) $\models$ the Replacement Scheme.

To kill the Collection Scheme we define $\ell(x) = \{i \in \mathrm{I}\colon \exists u \in \mathrm{L}[x]\Theta(i,u)\}$ for $x \in \mathrm{Z(A)}$.

Of course, $x \in y \Longrightarrow \ell(x) \subseteq \ell(y)$. In many cases
(1) $t = \{\ell(x): x \in \mathrm{Z(A)}\} \in \mathrm{L}$ even if $\langle \ell(x): x \in \mathrm{Z(A)}\rangle \notin \mathrm{L}$,
(2) $\bigcup t = \mathrm{I}$, and
(3) $\mathrm{I} \setminus \ell(x) \neq \emptyset$ for $x \in \mathrm{Z(A)}$.

Now, $\forall u \in t \exists x(\ell(x) = u)$ but there is no $b \in \mathrm{Z(A)}$ such that $\forall u \in t \exists x \in b(\ell(x) = u)$. Otherwise, $\forall u \in t(u \subseteq \ell(b))$, so $\mathrm{I} = \bigcup t \subseteq \ell(b)$ which contradicts (3). Hence, the Collection Scheme fails for a very simple formula $\Phi(x,y) \equiv \ell(y) = x$.

In some cases, especially when I is finite, Z(A) $\models$ the Collection Scheme. Prior to this paper I constructed Z(A) models (see [4]) and their modifications for the following theories

(i) ZFH + AC + DC + $\neg$WOP,
(ii) ZFH + WOP + DC + Continuum is non-well-orderable + every set of reals has cardinality $\leq \aleph_1$.

Actually, DC had not been discussed in [4], but the same proof that shows the Collection Scheme also works for DC$\equiv$the Principle of Dependent Choices as a scheme. Also, in both models alephs are exactly constructible alephs.

In this paper, by a suitable choice of A_i's, a model Z(A) is constructed such that $\langle \mathrm{Z(A)}, \in, \mathrm{A}\rangle \models$ZFH! + WOP + $\aleph_1 = \aleph_\omega^{\mathrm{L}}$ + every set of reals is countable. In the above model the Δ_0-Collection Scheme fails, so ZFH! + WOP $\not\vdash$ KP. Therefore, KP has some flavor of non-constructiveness which is a rather peculiar fact for the Foundations of Abstract Computability.

The last model has also startling consequences for Foundations of Real Analysis since in the model the Cauchy's definition of continuity is not equivalent to the Heine's definition even at the presence of WOP. Apparently, DC fails in the model. Notice that at the same time we have WOP & $cf(\aleph_1) = \aleph_0$.

The new consistency results proven in this paper:
if ZF is consistent, then the following theories are consistent

(iii) ZFH! + WOP + ¬the Collection Scheme + ¬DC,
(iv) ZFH! + WOP + ¬the Collection Scheme + DC.

The models Z(A) are not designed to show consistency of ZFH! + WOP + the Collection Scheme + ¬DC, i.e. ZFH + WOP + ¬DC. The last theory is equivalent to ZFH + WOP + ¬the Reflection Principle.

The Reflection Principle is a scheme. For each formula $\Phi(x_1, x_2, \ldots, x_n)$ the corresponding axiom is the following statement:
$\forall a \exists t\big(a \in t \ \&\ t \text{ is transitive } \&\ \forall x_1, x_2, \ldots, x_n \in t\big[\Phi(x_1, x_2, \ldots, x_n) \longleftrightarrow \Phi^t(x_1, x_2, \ldots, x_n)\big]\big)$.

I believe that ZFH + WOP + ¬the Reflection Principle is consistent with no clue how to prove it.

3. Mutual Independence.

Let $\wp_i = \langle P_i, \leq_i \rangle$ be a notion of forcing for $i \in \mathrm{I}$. Let $\langle \mathrm{M}, \in \rangle$ be a transitive structure such that (i) if y is a finite subset of M, then $y \in$M; (ii) I$\subset$M, and (iii) $P_i, \leq_i \subseteq$ M for every$i \in$I.
Let $\wp = \big\{\langle i, u, v\rangle : u \leq_i v \ \&\ i \in \mathrm{I}\big\}$. Then $\wp \subseteq$ M.

Definition 3.1. *$X \subseteq M$ is specifiable in $\langle \mathrm{M}, \in, \wp\rangle$ if*
$X = D_{\Phi,\mathbf{m}} = \big\{x \in \mathrm{M}: \langle \mathrm{M}, \in, \wp\rangle \models \Phi[x, m_1, m_2, \ldots, m_k]\big\}$ *for some* $\mathbf{m} = \langle m_1, m_2, \ldots, m_k\rangle \in M$ *and some formula Φ of the language of the structure* $Sp\big(\langle M, \in, \wp\rangle\big)$ *= the collection of all specifiable subsets of M.*

Definition 3.2 (partial, finite, choice functions). *Let $\mathcal{B} = \langle B_i : i \in \mathrm{I}\rangle$ be a family of some collections. Then*
$pcf(g, \mathcal{B}) \iff Func(g) \ \&\ g \text{ is finite } \&\ dom(g) \subset \mathrm{I} \times \omega \ \&\ \forall \langle i, n\rangle \in dom(g)\big(g(i, n) \in B_i\big)$,
$pci(g, \mathcal{B}) \iff pcf(g, \mathcal{B}) \ \&\ g \text{ is one-to-one}$.

Definition 3.3. $FinInj(\mathcal{B}) = \{g : pci(g, \mathcal{B})\}$ *and it is ordered by reverse inclusion.*

Definition 3.4. *Let 1_i be the weakest element of $\wp_i$ and $B_i = P_i \setminus \{1_i\}$ for $i \in \mathrm{I}$. Then* $Q = Q(\langle \wp_i : i \in \mathrm{I}\rangle) = \{g : pcf(g, \mathcal{B})\}$.

Proposition 3.1. *Q is specifiable in $\langle \mathrm{M}, \in, \wp\rangle$.*

Definition 3.5. $f \leq_{\mathrm{Q}} g \iff f, g \in \mathrm{Q} \ \&\ dom(f) \supseteq dom(g) \ \&\ \forall \langle i, n\rangle \in dom(g)\big(f(i, n) \leq_i g(i, n)\big)$.

Proposition 3.2. *$\leq_{\mathrm{Q}}$ is specifiable in $\langle \mathrm{M}, \in, \wp\rangle$.*

Proposition 3.3 (the weak product of ω copies of $\langle \wp_i : i \in \mathrm{I}\rangle$). *$\langle \mathrm{Q}, \leq_{\mathrm{Q}}\rangle$ is a notion of forcing.*

Definition 3.6 (mutual independence). *$\mathcal{A} = \langle \mathrm{A}_i : i \in \mathrm{I}\rangle$ is a family of mutually independent generics (MIG) if*

(i) $a \in \mathrm{A}_i \Longrightarrow a$ is $\wp_i$-generic over $Sp(\langle M, \in, \wp\rangle)$,
(ii) A_i's are mutually disjoint,
(iii) if $a_{i_\ell, n_\ell} \in \mathrm{A}_{i_l}$ for $\ell = 1, 2, \ldots, k$ and $a_{i_1,n_1}, a_{i_2,n_2}, \ldots, a_{i_k,n_k}$ are distinct, then $a_{i_1,n_1} \times a_{i_2,n_2} \times \ldots \times a_{i_k,n_k}$ is $\wp_{i_1} \times \wp_{i_2} \times \ldots \times \wp_{i_k}$-generic over $Sp(\langle M, \in, \wp\rangle)$,
(iv) for every $p \in P_i$ there are infinitely many $a \in \mathrm{A}_i$ such that $p \in a$;

Definition 3.7. *If $f \in Q$ and $F \in FinInj(\mathcal{A})$, then*
$pc(f, F) \iff dom(f) \subseteq dom(F)$ & $\forall \langle i, n\rangle \in dom(f)\big(f(i,n) \in F(i,n)\big)$.
If $D \in Sp(\langle M, \in, \wp\rangle)$ and $\$D \subseteq Q$, then $D(\mathcal{A}) = \{F \in FinInj(\mathcal{A}) : \exists f \in D\; pc(f, F)\}$.

Lemma 3.1. *If D is dense in Q, then $D(\mathcal{A})$ is dense in $FinInj(\mathcal{A})$.*

Proof. Let $F \in FinInj(\mathcal{A})$ and $t = dom(F)$. t is a finite subset of $\mathrm{I} \times \omega$, so $t \in \mathrm{M}$. $D|t = \{g|t : g \in D\} \in Sp(\langle \mathrm{M}, \in, \wp\rangle)$ and $D|t$ is dense in $\mathrm{Q}|t = \{f \in \mathrm{Q} : dom(f) \subseteq t\}$.

By the assumption, $\prod_{\langle i,n\rangle \in t} F(i,n)$ is $\mathrm{Q}|t$-generic over $Sp(\langle \mathrm{M}, \in, \wp\rangle)$, so $D|t \cap \prod_{\langle i,n\rangle \in t} F(i,n) \neq \emptyset$. Therefore, there is $f \in D$ such that $f|t \in \prod_{\langle i,n\rangle \in t} F(i,n)$, i.e. $f|t$ is a partial choice function for F.
Let $\{\langle i_1, n_1\rangle, \langle i_2, n_2\rangle, \ldots, \langle i_k, n_k\rangle\} = dom(f) \setminus t$. By the assumption(iv), there are <u>distinct</u> $a_{i_1,n_1}, a_{i_2,n_2}, \ldots, a_{i_k,n_k}$ such that $a_{i_l,n_l} \in \mathrm{A}_{i_l}$ for $l = 1, 2, \ldots, k$, $f(i_l, n_l) \in a_{i_l,n_l}$ for $l = 1, 2, \ldots, k$, and $\{a_{i_1,n_1}, a_{i_2,n_2}, \ldots, a_{i_k,n_k}\} \cap \{F(i,n) : \langle i,n\rangle \in t\} = \emptyset$.
Let $G = F \cup \big\{\big\langle \langle i_l, n_l\rangle, a_{i_l,n_l}\big\rangle : l = 1, 2, \ldots, k\big\}$. Then $G \supseteq F$, $G \in FinInj(\mathcal{A})$, and f is a partial choice function for G, so $G \in D(\mathcal{A})$.

Definition 3.8. *Let $\overline{\mathcal{A}} = \big\{\langle \Phi, \mathrm{m}, F\rangle : F \in D_{\Phi,\mathrm{m}}(\mathcal{A})\big\}$. Let H be $FinInj(\mathcal{A})$-generic over $Sp(\langle M, \in, \wp, \overline{\mathcal{A}}\rangle)$. Then $\overline{H} = \big\{f \in \mathrm{Q} : pc\big(f, (\bigcup H) \mid dom(f)\big)\big\}$, $\overline{H}_{i,n} = \{f(i,n) : f \in \overline{H}\}$, and $\overline{H}_i = \{\overline{H}_{i,n} : n < \omega\}$.*

Theorem 3.1. *Let $\mathcal{A}$ be a MIG family and H a $FinInj(\mathcal{A})$-generic over $Sp(\langle M, \in, \wp, \overline{\mathcal{A}}\rangle)$. Then $\overline{H}$ is Q-generic over $Sp(\langle M, \in, \wp\rangle)$ and $\overline{H}_i = \mathrm{A}_i$ for each $i \in \mathrm{I}$.*

Thus every MIG family can be reduced to the axes of an appropriate Q-generic. Whatever we can prove about axes of a Q-generic we may apply it to MIG families. One can skip the last theorem to obtain all other results of this paper. It only shows that the mutual independence can be reduced to the properties of $\langle \mathrm{Q}, \leq_{\mathrm{Q}}\rangle$.

4. Basics and Homogeneity Arguments.

We assume that M is a transitive model for ZF (for a proof of general properties ZF can be replaced by BST + the Collection Scheme; the Collection Scheme is essential even if we want to show only the Replacement Scheme in M[G]). $\langle \wp_i : i \in \mathrm{I}\rangle \in$M is a family of notions of forcing and Q is the weak product of ω copies of the family.

If H is Q-generic over M, and $\mathrm{H}_{i,n} = \{f(i,n): f \in \mathrm{H}\}$, then $\mathrm{H}_{i,n}$ is $\wp_i$-generic over M (if $\langle i,n\rangle$ is not in $dom(f)$ and $f(i,n)$ is used, then it stands for 1_i). In general, if $s \in$M and $s \subseteq \mathrm{I}\times\omega$, and $\mathrm{H}|s = \{f|s: f \in \mathrm{H}\}$, then H$|s$ is Q$|s$-generic over M. Of course, $\mathrm{Q}|s = \{f|s: f \in \mathrm{Q}\} = \{f \in \mathrm{Q}: dom(f) \subseteq s\}$.

By the natural isomorphism, if $\langle i_l, n_l\rangle \in \mathrm{I}\times\omega$, $l = 1,2,\ldots,k$, are distinct pairs, then $\mathrm{H}_{i_1,n_1}\times\mathrm{H}_{i_2,n_2}\times\ldots\times\mathrm{H}_{i_k,n_k}$ is $\wp_{i_1}\times\wp_{i_2}\times\ldots\times\wp_{i_k}$-generic over M and $\mathrm{M}[\mathrm{H}_{i_1,n_1}\times\mathrm{H}_{i_2,n_2}\times\ldots\times\mathrm{H}_{i_k,n_k}] = \mathrm{M}[\mathrm{H}|s]$ for $s = \{\langle i_l,n_l\rangle: l = 1,2,\ldots,k\}$.

Let $P \in$M be any notion of forcing. M^P is the class of all P-names, so if $\underline{x} \in \mathrm{M}^P$ and $v \in \underline{x}$, then there are $\underline{y} \in \mathrm{M}^P$ and $p \in P$ such that $v = \langle \underline{y}, p\rangle$. $\check{a} = \{\langle \check{b}, 1\rangle: b \in a\}$ is the canonical name for a.
The ground model M is represented by the predicate S, where
$p\Vdash^* S(\underline{x}) \equiv \exists a(p\Vdash^* \underline{x} = \check{a})$, $p\Vdash S(\underline{x}) \equiv p\Vdash^* \neg\neg S(\underline{x})$.
If σ is an automorphism of P and $\sigma \in$M, then it induces the automorphism of M^P, denoted also by σ, such that $\sigma(\underline{x}) = \{\langle\sigma(\underline{y}),\sigma(p)\rangle: \langle \underline{y},p\rangle \in \underline{x}\}$. If G is P-generic over M, then $val_G(a) = \{val_G(b): \exists p \in \mathrm{G}(\langle b,p\rangle \in a)\}$ and $\mathrm{M}[\mathrm{G}] = \{val_G(a): a \in \mathrm{M}^P\}$.

Lemma 4.1 (Permutation Lemma). *If $\Phi(v_1,v_2,\ldots,v_n)$ is a formula of the extended language(S may be used) and $a_1,a_2,\ldots,a_n \in \mathrm{M}^P$, then for any automorphism σ of P, $\sigma \in M$, and any $p \in P$*
$p\Vdash\Phi(a_1,a_2,\ldots,a_n) \longleftrightarrow \sigma(p)\Vdash\Phi(\sigma(a_1),\sigma(a_2),\ldots,\sigma(a_n))$.

Lemma 4.2 (Fundamental Lemma). *If G is P-generic over M, then M[G] is the smallest transitive model for ZF (BST + the Collection Scheme, respectively) such that M[G]$\supseteq$M and G$\in$M[G]. Moreover, $\langle$M[G]$, \in, \mathrm{M}\rangle \models$the Collection Scheme and M is almost universal in M[G], i.e. if $x \in$M[G] and $x \subset$M, then there is $y \in$M such that $x \subseteq y$. If M$\models$WOP, then M[G]$\models$WOP.*

Lemma 4.3 (Product Lemma). *If $P_1, P_2 \in M$ are notions of forcing and G is $P_1\times P_2$-generic over M, then $\mathrm{G}_1 = \{p_1 \in P_1: \exists p_2(\langle p_1,p_2\rangle \in \mathrm{G})\}$ is P_1-generic over M and $\mathrm{G}_2 = \{p_2: \exists p_1(\langle p_1,p_2\rangle \in \mathrm{G})\}$ is P_2-generic over $\langle \mathrm{M}[\mathrm{G}_1], \in, \mathrm{M}\rangle$. Also, $G = \mathrm{G}_1\times\mathrm{G}_2$, $M[\mathrm{G}_1][\mathrm{G}_2] = \mathrm{M}[\mathrm{G}_1\times\mathrm{G}_2] = \mathrm{M}[\mathrm{G}_2\times\mathrm{G}_1] = \mathrm{M}[\mathrm{G}_2][\mathrm{G}_1]$ and $\mathrm{M}[\mathrm{G}_1]\cap\mathrm{M}[\mathrm{G}_2] = \mathrm{M}$.*

Let $\langle\sigma_i: i \in \mathrm{I}\rangle \in$M be a family of permutations of ω. If $f \in$Q, then $\sigma(f)$ is a finite function such that $dom(\sigma(f)) = \{\langle i,\sigma_i(n)\rangle: \langle i,n\rangle \in dom(f)\}$ and $\sigma(f)(i,\sigma_i(n)) = f(i,n)$. If $s \subseteq \mathrm{I}\times\omega$, then $\sigma[s] \stackrel{\mathrm{df}}{=} \{\langle i,\sigma_i(n)\rangle: \langle i,n\rangle \in s\}$.

Proposition 4.1. *If $\underline{x} \in \mathrm{M}^{\mathrm{Q}|s}$, then $\sigma(\underline{x}) \in \mathrm{M}^{\mathrm{Q}|\sigma[s]}$.*

Proposition 4.2. *Let $f, g \in Q$. If s is a finite subset of $I\times\omega$ and $f|s$ is compatible with $g|s$, then there is σ such that $\sigma|\mathrm{M}^{\mathrm{Q}|s}$ = identity and f is compatible with $\sigma(g)$.*

Canonical names:

$$\underline{\mathrm{H}}_{i,n} = \{\langle \check{p}, f|\{\langle i,n\rangle\}\rangle : p = f(i,n) \;\&\; f \in \mathrm{Q}\},$$
$$\underline{\mathrm{H}}_s = \{\langle \check{f|s}, f|s\rangle : f \in \mathrm{Q}\},$$
$$\underline{h} = \{\langle \underline{\mathrm{H}}_{i,n}, \emptyset\rangle : i \in \mathrm{I} \;\&\; n < \omega\}.$$

$\underline{\mathrm{H}}_{i,n}$ is a name for $\mathrm{H}_{i,n}$; $\underline{\mathrm{H}}_s$ is a name for $\mathrm{H}|s$, and $\underline{h}$ is a name for $\mathrm{A} = \{\mathrm{H}_{i,n} : i \in \mathrm{I}, n < \omega\}$ = the collection of all axes of H.
$\sigma(\underline{\mathrm{H}}_{i,n}) = \underline{\mathrm{H}}_{i,\sigma_i(n)}$, $\sigma(\underline{\mathrm{H}}_s) = \underline{\mathrm{H}}_{\sigma[s]}$, and $\sigma(\underline{h}) = \underline{h}$.

Lemma 4.4 (Restriction Lemma). *For each formula Φ of the extended language and any $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t \in \mathrm{M}^{\mathrm{Q}|s}$, where s is a finite subset of $\mathrm{I}\times\omega$*
$f \Vdash \Phi(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t, \underline{h}) \longleftrightarrow f|s \Vdash \Phi(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t, \underline{h})$.

Proof. Proposition 4.2 and the Permutation Lemma.

Definition 4.1. *A finite function, φ, from a subset of $\mathrm{I}\times\omega$ to $\mathrm{I}\times\omega$ is <u>acceptable</u> if φ is one-to-one and for each argument, $\langle i,n\rangle$, there is $n' < \omega$ such that $\varphi(i,n) = \langle i,n'\rangle$.*

Lemma 4.5. *Let s and s' be disjoint, finite subsets of $\mathrm{I}\times\omega$. Let $q \in \mathrm{Q}$ be a condition such that $dom(q) = s'$. Let $\mathrm{D}_{q,s,s'} = \{f \in \mathrm{Q} : \exists\varphi(\varphi$ is acceptable & $dom(q) = s'$ & $\varphi[s'] \cap (s \cup s') = \emptyset$ & $dom(f) \supseteq \varphi[s']$ & $\forall\langle i,n\rangle \in s'\; f(\varphi(i,n)) = q(i,n)\}$. Then $\mathrm{D}_{q,s,s'}$ is a dense section of Q.*

Proof. Let $f_0 \in \mathrm{Q}$ and $\bar{s} = dom(f_0) \cup s \cup s'$. $\bar{s}$ is a finite subset of $\mathrm{I}\times\omega$, so for every $\langle i,n\rangle \in s'$ there are infinitely many $n' < \omega$ such that $\langle i,n'\rangle \notin \bar{s}$. Therefore, there is an acceptable φ such that $dom(\varphi) = s'$ and $\varphi(i,n) \notin \bar{s}$ for $\langle i,n\rangle \in dom(\varphi)$. Hence, $f_0 \cup \{\langle\varphi(i,n), q(i,n)\rangle : \langle i,n\rangle \in s'\} \in \mathrm{Q}$ and it is an extension of f_0 that belongs to $\mathrm{D}_{q,s,s'}$.

Corollary 4.1. *If $f_0, f \in \mathrm{H}$, $dom(f_0) = s$, $dom(f) = s \cup s'$ and s, s' are disjoint, then there is $\sigma \in \mathrm{M}$ such that $\sigma|\mathrm{M}^{\mathrm{Q}|s}$ = identity, $\sigma[s'] \cap (s \cup s') = \emptyset$, $dom(\sigma(f)) = s \cup \sigma[s']$, and $\sigma(f) \in \mathrm{H}$.*

Proof. $\mathrm{H} \cap \mathrm{D}_{q,s,s'} \neq \emptyset$, where $q = f|s'$. σ is an extension of an appropriate, acceptable φ.

Theorem 4.1 (Theorem H; H≡Homogeneity). *If $x \in \mathrm{M[H]}$, $x \subseteq \mathrm{M[H}|s_0]$ for some finite $s_0 \subseteq \mathrm{I}\times\omega$, and x is definable in $\langle \mathrm{M[H]}, \in, \mathrm{M}\rangle$ with the parameter $A = \{\mathrm{H}_{i,n} : i \in \mathrm{I}, n < \omega\}$ and parameters from $\mathrm{M[H}|s]$ for some finite $s \subset \mathrm{I}\times\omega$, then $x \in \mathrm{M[H}|s]$.*

Proof. Let $-s_0 = (\mathrm{I}\times\omega)\setminus s_0$. By the Product Lemma $\mathrm{M[H]} = \mathrm{M}[\mathrm{H}|s_0][\mathrm{H}|-s_0]$, and by the Fundamental Lemma there is $y \in \mathrm{M}[\mathrm{H}|s_0]$ such that $x \subseteq y$. y has a name $\underline{y} \in \mathrm{M}^{\mathrm{Q}|s_0}$. Let $\Phi(v, x_1, x_2, \ldots, x_t, \mathrm{A})$ be a definition of x, where $x_1, x_2, \ldots, x_t \in \mathrm{M}[\mathrm{H}|s]$, so $\langle \mathrm{M[H]}, \in, \mathrm{M}\rangle \models x = v \longleftrightarrow \Phi(v, x_1, x_2, \ldots, x_t, \mathrm{A})$. Hence, $\langle \mathrm{M[H]}, \in, \mathrm{M}\rangle \models x = \{u \in y : \exists v[u \in v \;\&\; \Phi(v, x_1, x_2, \ldots, x_t, \mathrm{A})]\}$. Let $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t \in \mathrm{MQ}|s$ be names for $x_1, x_2, \ldots, x_t$, respectively. Let $\Psi(u) \equiv u \in \underline{y} \;\&\; \exists v(u \in v \;\&\; \Phi(v, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t, \underline{h}))$. If $\langle \underline{u}, g\rangle \in \underline{y}$, then $\underline{u} \in \mathrm{M}^{\mathrm{Q}|s_0}$ and, by the Restriction Lemma, for every $f \in \mathrm{Q}$ $\;f \Vdash \Psi(\underline{u}) \longleftrightarrow (f|s \cup s_0) \Vdash \Psi(\underline{u})$. Thus, $\underline{x} = \{\langle \underline{u}, f\rangle : f \in \mathrm{Q}|s \cup s_0 \;\&\; \exists g(\langle \underline{u}, g\rangle \in \underline{y}) \;\&\; f \Vdash \Psi(\underline{u})\} \in \mathrm{M}^{\mathrm{Q}|s \cup s_0}$ is a name for x. Therefore, $x \in \mathrm{M}[\mathrm{H}|s \cup s_0]$. We want to show that $x \in \mathrm{M}[\mathrm{H}|s]$. Let s' be disjoint with s and $s \cup s' = s \cup s_0$. By the Restriction Lemma, there is $f_0 \in \mathrm{H}|s$ such that $f_0 \Vdash \forall v_1, v_2 \big(\Phi(v_1, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t, \underline{h}) \;\&\; \Phi(v_2, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t, \underline{h}) \Longrightarrow v_1 = v_2\big)$.

By the Truth Lemma, there is $f \in \mathrm{H}$ that extends f_0 and $f \Vdash \Phi(\underline{x}, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t, \underline{h}) \;\&\; \underline{x} \in S[\underline{\mathrm{H}}_s \times \underline{\mathrm{H}}_{s'}]$.

By the Restriction Lemma we may assume that $f \in \mathrm{H}|s \cup s'$.

By Corollary 4.1, there is $\sigma \in \mathrm{M}$ such that $\sigma|\mathrm{M}^{\mathrm{Q}|s} =$ identity, $\sigma[s'] \cap (s \cup s') = \emptyset$, and $\sigma(f) \in \mathrm{H}$.

Let $g = f \cup \sigma(f)$. Then $g \in \mathrm{Q}$ and $g \in \mathrm{H}$. Of course, g extends f_0, f, and $\sigma(f)$.

By the Permutation Lemma, $\sigma(f) \Vdash \Phi(\sigma(\underline{x}), \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t, \underline{h}) \;\&\; \sigma(\underline{x}) \in S[\underline{\mathrm{H}}_s \times \underline{\mathrm{H}}_{\sigma[s']}]$.

By the Extension Lemma,

$g \Vdash \forall v_1, v_2 \big[\Phi(v_1, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t, \underline{h}) \;\&\; \Phi(v_2, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t, \underline{h}) \longrightarrow v_1 = v_2\big]$,
$g \Vdash \Phi(\underline{x}, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t, \underline{h}) \;\&\; \underline{x} \in S[\underline{\mathrm{H}}_s \times \underline{\mathrm{H}}_{s'}]$, and
$g \Vdash \Phi(\sigma(\underline{x}), \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_t, \underline{h}) \;\&\; \sigma(\underline{x}) \in S[\underline{\mathrm{H}}_s \times \underline{\mathrm{H}}_{\sigma[s']}]$.

By the Truth Lemma, $x = val_{\mathrm{H}}(\underline{x}) = val_{\mathrm{H}}(\sigma(\underline{x}))$ and $x \in \mathrm{M}[\mathrm{H}|s \times \mathrm{H}|s'] \cap \mathrm{M}[\mathrm{H}|s \times \mathrm{H}|\sigma[s']]$.

By the Product Lemma (s, s', and $\sigma[s']$ are mutually disjoint), $\mathrm{M}[\mathrm{H}|s \times \mathrm{H}|s'] = \mathrm{M}[\mathrm{H}|s][\mathrm{H}|s']$, $\mathrm{M}[\mathrm{H}|s \times \mathrm{H}|\sigma[s']] = \mathrm{M}[\mathrm{H}|s][\mathrm{H}|\sigma[s']]$, $\mathrm{H}|s \times (\mathrm{H}|s' \times \mathrm{H}|\sigma[s'])$ is $\mathrm{Q}|s \cup (s' \cup \sigma[s'])$-generic over M, so $\mathrm{M}[\mathrm{H}|s][\mathrm{H}|s'] \cap \mathrm{M}[\mathrm{H}|s][\mathrm{H}|\sigma[s']] = \mathrm{M}[\mathrm{H}|s]$. Therefore, $x \in \mathrm{M}[\mathrm{H}|s]$.

Using Theorem H we can easily show a strong form of the Replacement Scheme in N, $\mathrm{N} = \bigcup\{\mathrm{M}[\mathrm{H}|s] : s \subset \mathrm{I} \times \omega, s\text{-finite}\}$. Let $\mathrm{M}^{F\mathrm{Q}} = \bigcup\{\mathrm{M}^{\mathrm{Q}|s} : s \subset \mathrm{I} \times \omega, s\text{-finite}\}$. Then $\mathrm{N} = \{val_{\mathrm{H}}(\underline{x}) : \underline{x} \in \mathrm{M}^{F\mathrm{Q}}\}$. N is definable in $\langle \mathrm{M[H]}, \in, \mathrm{M}\rangle$ with the parameter A. Indeed, $x \in \mathrm{N} \iff \exists g(g$ is a finite subset of A & $x \in \mathrm{M}[g])$. $\mathrm{A} \notin \mathrm{N}$, so we add an additional predicate to the language and denote it also by A.

We want to show that $\langle \mathrm{N}, \in, \mathrm{M}, \mathrm{A}\rangle \models$the Replacement Scheme. Actually, we can show even more. Let $\langle \mathrm{M[H]}, \in, \mathrm{M}, \mathrm{A}\rangle \models \forall x, y_1, y_2 \big[\Phi(x, y_1, \mathbf{d}) \;\&\; \Phi(x, y_2, \mathbf{d}) \longrightarrow y_1 = y_2\big]$, where Φ is a formula of the extended language and parameters, $\mathbf{d} \in \mathrm{N}$. Let $a \in \mathrm{N}$ and $b = \{y \in \mathrm{N} : \exists x \in a\, \Psi(x, y, \mathbf{d})\}$. There is a finite $s \subset \mathrm{I} \times \omega$ such that $a, \mathbf{d} \in \mathrm{M}[\mathrm{H}|s]$. If $x \in a$ and $\Phi(x, y, \mathbf{d}) \;\&\; y \in \mathrm{N}$, then for some finite $s_0 \subset \mathrm{I}\times\omega$ $\;y \subset \mathrm{M}[\mathrm{H}|s_0]$ and y is definable in $\langle \mathrm{M[H]}, \in, \mathrm{M}\rangle$ with the parameter A and parameters $x, \mathbf{d} \in \mathrm{M[H}|s]$. By Theorem H, $y \in \mathrm{M[H}|s]$. Thus, $b \subset \mathrm{M[H}|s]$ and its definition is shown. Hence, $b \in \mathrm{M[H}|s]$. This proves

Lemma 4.6 (Lemma R; R≡Replacement). *Let* $\langle \mathrm{M[H]}, \in, \mathrm{M}, \mathrm{A}\rangle \models \forall x \in \mathrm{N}\ \exists! y \in \mathrm{N}\ \Phi[x, y, \mathbf{d}]$, *where* $\mathbf{d} \in \mathrm{N}$. *If* $a \in \mathrm{N}$ *and, in* $\langle \mathrm{M[H]}, \in, \mathrm{M}, \mathrm{A}\rangle$, $b = \{y \in \mathrm{N}\colon \exists x \in a \Phi(x, y, \mathbf{d})\}$, *then* $b \in \mathrm{N}$.

Theorem 4.2 (Theorem R). *Let M be a transitive model at least for BST + the Collection Scheme. Let* $\langle \wp_i\colon i \in \mathrm{I}\rangle \in \mathrm{M}$ *be a family of notions of forcing and Q be the weak product of* ω *copies of the family. Let A be the collection of all axes of H. Then*

(i) $\langle \mathrm{N}, \in, \mathrm{M}, \mathrm{A}\rangle \models$*BST + the Replacement Scheme;*
(ii) if M$\models$WOP, then N$\models$WOP;
(iii) if M$\models$ZFH, then N$\models$ZFH!;
(iv) M is almost universal in N but N is not almost universal in M[H];
(v) if $x \in \mathrm{N}$, $u_1, u_2, \ldots, u_k \in \mathrm{M[H}|s]$ *and* $y \in x \longleftrightarrow \langle \mathrm{N}, \in, \mathrm{M}, \mathrm{A}\rangle \models \Phi[y, u_1, u_2, \ldots, u_k]$, *then* $x \in \mathrm{M[H}|s]$.
(vi) if $\mathrm{M[H]} \models \kappa$ *is a cardinal, then* $\mathrm{N} \models \kappa$ *is a cardinal;*
(vii) it may happen that κ *is a cardinal in N but not in M[H].*

For somebody who is not interested in technical details, it is important to have the following

Corollary 4.2. *If* $\mathcal{A}$ *is a MIG family over L, then* $\langle \mathrm{Z(A)}, \in, \mathrm{A}\rangle \models$*ZFH! + WOP.*

5. The A-functional and its imitation.

Let $Fin(\mathrm{A})$ be the collection of all finite subsets of A. By the Product Lemma, if $g, g' \in Fin(\mathrm{A})$ and $a \in \mathrm{A}$, then $a \in \mathrm{M}[g] \iff a \in g$ and $\mathrm{M}[g] \cap \mathrm{M}[g'] = \mathrm{M}[g \cap g']$. If $x \in \mathrm{N}$, then $x \in \mathrm{M}[g]$ for some $g \in Fin(\mathrm{A})$ and $\mathrm{A} \cap \mathrm{L}[x] \subseteq g$, so $\mathrm{A} \cap \mathrm{L}[x]$ is a finite subset of A.

Proposition 5.1. *If* $x \in \mathrm{N}$ *and* $x \subset \mathrm{A}$, *then* x *is finite.*

Definition 5.1. $F_{\mathrm{A}}(x) = |\mathrm{A} \cap \mathrm{L}[x]|$ *for* $x \in \mathrm{N}$.

Proposition 5.2. *If* $g \in Fin(\mathrm{A})$, *then* $F_{\mathrm{A}}(g) = |g|$. *If* $y \in x$, *then* $F_{\mathrm{A}}(y) \leq F_{\mathrm{A}}(x)$.

Proposition 5.3. $\forall n < \omega \exists x \subset \mathrm{A}(|x| = n)$ *but there is no set b in N such that* $\forall n < \omega$ $\exists x \in b(F_{\mathrm{A}}(x) = n)$.

Definition 5.2 (Heine-Cauchy Functional). *If* x *is a real, then* $H\text{-}C_{\mathrm{A}}(x) = x \cdot F_{\mathrm{A}}(x)$.

Proposition 5.4. *If x, r are reals, $r \in$ M and $r \neq 0$, then $F_{\mathrm{A}}(x \pm r) = F_{\mathrm{A}}(x) = F_{\mathrm{A}}(r \cdot x) = F_{\mathrm{A}}(\frac{x}{r})$. If $x \neq 0$, then $F_{\mathrm{A}}(\frac{1}{x}) = F_{\mathrm{A}}(x)$.*

If $\langle x_n : n < \omega \rangle \in$ N, then there is $m \in \omega$ such that $m = \mathrm{Max}\{F_{\mathrm{A}}(x_n): n < \omega\}$. Thus,

Proposition 5.5. *If $\lim\limits_{n \to \infty} x_n = 0$, then $\lim\limits_{n \to \infty} H\text{-}C_{\mathrm{A}}(x_n) = 0 = H\text{-}C_{\mathrm{A}}(0)$.*

Suppose that $\{F_{\mathrm{A}}(x): x \text{ is a real}\}$ is unbounded (as a subset of ω). Let $m > 0$. There is a real, x, such that $F_{\mathrm{A}}(x) > 2m^2$. Let $\mathrm{E}(x) =$ the integer part of x. If $\overline{x} = \frac{x - \mathrm{E}(x)}{2m} + \frac{1}{2m}$, then $\frac{1}{2m} < \overline{x} < \frac{1}{m}$ and $\mathrm{H\text{-}C_A}(\overline{x}) = \overline{x} \cdot F_{\mathrm{A}}(\overline{x}) > \frac{1}{2m} \cdot 2m^2 > m$, so $\forall m > 0 \exists x (0 < x < \frac{1}{m}$ & $\mathrm{H\text{-}C_A}(x) > m)$. Notice that $F_{\mathrm{A}}(\overline{x}) = F_{\mathrm{A}}(x)$ by Proposition 5.4

To imitate the A-functional F_{A}, without using the class A, we need some assumptions about $\langle \wp_i : i \in \mathrm{I} \rangle$.

Definition 5.3. *$\langle \wp_i : i \in \mathrm{I} \rangle$ are almost independent if I is infinite and for every finite $s \subset \mathrm{I}$ there is a finite $s' \supseteq s$ such that for every finite $t \subset \omega$ $\forall j \in \mathrm{I} \setminus s' (\Vdash^*_{\mathrm{Q}|s \times t}$ "there is no $\check{\wp}_j$-generic over S").*

Definition 5.4. *If $\langle \wp_i : i \in \mathrm{I} \rangle$ are almost independent, then $l(x) = \big\{ j \in \mathrm{I} : \exists G \in \mathrm{L}[x](G$ is $\wp_j$-generic over $\mathrm{M}) \big\}$ for $x \in$ N.*

If $x \in$ N, then $x \in \mathrm{M}[\mathrm{H}|s \times t]$ for some finite s and t. Thus $l(x) \subseteq s'$ is a finite subset of I.

Definition 5.5. $F(x) = |l(x)|$

It is clear that

(i) $\{F(x): x \in \mathrm{N}\}$ is unbounded(as a subset of ω),
(ii) $y \in x \Longrightarrow F(y) \leq F(x)$,
(iii) if $\langle x_n : n < \omega \rangle \in$ N, then there is $m < \omega$ such that $m = \mathrm{Max}\{F(x_n): n < \omega\}$,
(iv) if x and r are reals, $r \in$ M and $r \neq 0$, then $F(x \pm r) = F(x) = F(x \cdot r) = F(\frac{x}{r})$.

Also, $\forall n < \omega \exists x (F(x) \geq n)$ but there is no b such that $\forall n < \omega \exists x \in b (F(x) \geq n)$.

Theorem 5.1 (Theorem ¬Coll; ¬DC). *Let $M \models WOP + ZFH$.*
If $\langle \wp_i : i \in \mathrm{I} \rangle$ are almost independent, then $N \models WOP + ZFH! + \neg$the Collection Scheme + ¬DC.

Proof. The Collection Scheme fails for $\Phi(n, x) \equiv F(x) \geq n$ and DC fails for $\Psi(x, y) \equiv \exists j \in \mathrm{I}\big[(y$ is $\wp_j$-generic over M) & $\forall z \in \mathrm{L}[x](z$ is not $\wp_j$-generic over M$)\big]$.

Theorem 5.2 (Theorem H-C). *If $M \models WOP + ZFH$ and $\langle \wp_i : i \in \mathrm{I} \rangle$ are almost independent and, additionally, $\{F(x): x$ is a real $\}$ is unbounded, then — in the model N—$H\text{-}C(x) = x \cdot F(x)$ is continuous at $x = 0$ according to Heine's definition of continuity and it is unbounded in any neighborhood of 0.*

6. Weakly Independent Forcings and Applications.

To illustrate how the machinery described in §3 works, we will use very simple forcings. We are working in M, a transitive model for ZFC.

Definition 6.1. $P(\mathrm{X}, \mathrm{Y}; \mu) = \{f : Func(f) \; \& \; dom(f) \subseteq \mathrm{Y} \; \& \; |f| < \mu\}$, *where $\mu \geq \omega$. $P(\mathrm{X}, \mathrm{Y}; \mu)$ is ordered by the reverse inclusion.*

Note: If $\mu = \omega$ and X,Y are sets, then $P(\mathrm{X}, \mathrm{Y}; \mu)$ is a set in BST + the Δ_0-Collection Scheme. If $\mu > \omega$, then we need ZF to prove that $P(\mathrm{X}, \mathrm{Y}; \mu)$ is a set.

Proposition 6.1. *If $\mathrm{X} = \mathrm{X}_0 \cup \mathrm{X}_1$ and $\mathrm{X}_0 \cap \mathrm{X}_1 = \emptyset$, then $P(\mathrm{X}, \mathrm{Y}; \mu) \cong P(\mathrm{X}_0, \mathrm{Y}; \mu) \times P(\mathrm{X}_1, \mathrm{Y}; \mu)$.*

Proposition 6.2. *Let $P = P(\kappa, \lambda; \mu)$, where $\omega \leq \mu \leq \kappa$ are regular cardinals and $\lambda \geq 2$. Let $m \in \mathrm{M}$ and $\bigcup m = \kappa$. If G is P-generic over M and $g = \bigcup G$, then $S_0 = \{\alpha < \kappa : g(\alpha) = 0\} \notin \mathrm{M}$ and $m \not\subseteq S_0$.*

Proof. If $m \subseteq S_0$, then $m \subset On$, so $|m| \geq \kappa \geq \mu$, since κ is regular. $D_m = \{f \in P : \exists \alpha \in m \, (f(\alpha) \neq 0)\}$ is dense in P.

Suppose that $\lambda > \mu$ and g is P-generic over M. By Proposition 6.1 $G = G_1 \times G_2$ where G_1 is $P(\mu, \lambda; \mu)$-generic over M and G_2 is $P(\kappa \backslash \mu, \lambda; \mu)$-generic. Let $g = \bigcup G_1$. By transfinite recursion we define $\alpha_\xi = \min_\alpha [\forall \eta < \xi (g(\alpha_\eta) < g(\alpha))]$ for $\xi < \mu$ and $S_0 = \{g(\alpha_\xi) : \xi < \mu\}$. Let $m \in \mathrm{M}$ and $\bigcup m = \mu$, $m \subset On$.

Proposition 6.3. $D_m = \{f \in P(\mu, \lambda; \mu) : dom(f) \in \mu \; \& \; \exists \alpha + 1 \in dom(f) \, [\forall \beta < \alpha \; (f(\beta) < f(\alpha)) \; \& \; f(\alpha) < f(\alpha+1) \; \& \; m \cap [f(\alpha+1) \setminus f(\alpha)] \neq \emptyset]\}$ *is dense in $P(\mu, \lambda; \mu)$.*

Lemma 6.1. *$S_0 \notin \mathrm{M}$ and $m \not\subseteq S_0$.*

Definition 6.2. *A set $b \subset On$ is generic γ-cofinal over M if γ is a limit ordinal, $\bigcup b = \gamma$ and $\forall x \in \mathrm{M}(x \subset On \; \& \; \bigcup x = \gamma \Longrightarrow x \not\subseteq b)$.*

Lemma 6.2. *Let $\gamma = \max(\kappa, \lambda)$. Then $P(\kappa, \lambda; \mu)$ adds a set that is generic γ-cofinal over M.*

Lemma 6.3. *Let $P \in \mathrm{M}$ be any notion of forcing such that $|P| = \kappa$. Let θ be a regular cardinal $> \kappa$. Suppose that $p_0 \Vdash$ "$\underline{b}$ is a subset of $\check{\theta}$ cofinal with $\check{\theta}$". Let $q \leq p_0$ and $\mathrm{A}_\alpha = \{r \leq q : r \Vdash \check{\alpha} \in \underline{b}\}$ for $\alpha < \theta$. Then there is $r \leq q$ such that $|\{\alpha : r \in \mathrm{A}_\alpha\}| = \theta$.*

Proof. Let $\mathrm{A} = \{\alpha \in \theta : \mathrm{A}_\alpha \neq \emptyset\}$. $|\mathrm{A}| = \theta$ since $p_0 \Vdash$ "$\underline{b}$ is cofinal with $\check{\theta}$". Let f be a 1–1 function from κ onto P and $r_\alpha = f(\beta(\alpha))$ for $\alpha \in \mathrm{A}$, where $\beta(\alpha) = \min\limits_{\beta}[f(\beta) \leq q \ \&\ f(\beta) \Vdash \check{\alpha} \in \underline{b}]$. There is r such that $|\{\alpha \in \mathrm{A} : r_\alpha = r\}| = \theta$, since $|P| = \kappa < \theta$.

Conclusion: $\{p \in P : |\{\alpha \in \theta : p \Vdash \check{\alpha} \in \underline{b}\}| = \theta\}$ is dense below p_0.

Corollary 6.1. *If $M \models ZFC + GCH$, $P = P(\kappa, \lambda; \mu)$ and $\gamma = \max(\kappa, \lambda)$, then P doesn't add any set that is generic θ-cofinal over M for any regular cardinal $\theta \geq \gamma^+$.*

Definition 6.3. *$\langle \wp_i : i \in \mathrm{I}\rangle$ are weakly independent if*
$\forall s \in Fin(\mathrm{I}) \exists j \in \mathrm{I}\, \forall t \in Fin(\omega) \Vdash_{\mathrm{Q}|s \times t}$ *"there is no $\check{\wp}_j$-generic over S".*

If $\langle \wp_i : i \in \mathrm{I}\rangle$ are weakly independent, then $\mathrm{N} \models \forall i \in \mathrm{I} \exists G (G$ is $\wp_i$-generic over M) but there is no $b \in \mathrm{N}$ such that $\mathrm{N} \models \forall i \in \mathrm{I} \exists G \in b (G$ is $\wp_i$-generic over M).
If $\mathrm{I} = \omega$, then $G(i, u) \equiv u$ is $\wp_i$-generic over M, $GSeq(x) \equiv x$ is a finite sequence such that $\forall i \in dom(x)\ G(i, u)$, and $\Phi(x, y) \equiv GSeq(x) \Longrightarrow [GSeq(y) \ \&\ y \supset x]$. Let x_0 be $\wp_0$-generic over M. Then $\forall x \exists y \Phi(x, y)$ and there is no f such that $f(0) = x_0$ and $\forall n < \omega \Phi(f(n), f(n+1))$. So, $\mathrm{N} \models \neg$DC.

Theorem 6.1 (Theorem $\neg$Coll). *Let $M \models WOP + ZFH$. If $\langle \wp_i : i \in \mathrm{I}\rangle$ are weakly independent, then $N \models WOP + ZFH! + \neg$the Collection Scheme.*

Theorem 6.2 (Theorem $\neg$DC). *If $\mathrm{I} = \omega$ and M, $\langle \wp_i : i \in \mathrm{I}\rangle$ are as above, then $N \models \neg DC$.*

Theorem 6.3. *Let $M \models ZFC + GCH$ and $\mathrm{I} \in \mathrm{M}$. Let $P_i = P(\kappa_i, \lambda_i; \mu_i)$ and $\gamma_i = \max(\kappa_i, \lambda_i)$. If $\{\gamma_i : i \in \mathrm{I}\}$ doesn't have a maximal element and H is Q-generic over M, thenN$\models WOP + ZFH! + \neg$the Collection Scheme. If $\mathrm{I} = \omega$, then also $N \models \neg DC$.*

Proof. $\langle P_i : i \in \mathrm{I}\rangle$ are weakly independent.

Now, the machinery is ready to produce models with unusual combinations of choice principles.

Notation: CUC $\equiv$ For any countable family of countable sets the union is countable;

$\mathrm{AC}_\alpha \equiv$ For any family of α non-empty sets there is a choice function;
ADC $\equiv \forall a \forall r \subseteq a \times a \big[\forall x \in a \exists y \in a(\langle x, y\rangle \in r) \Longrightarrow \forall x_0 \in a \exists f \big(Func(f) \ \&\ dom(f) = \omega \ \&\ f(0) = x_0 \ \&\ \forall n < \omega(\langle f(n), f(n+1)\rangle \in r)\big)\big]$;

DC_α stands for the Principle of α Dependent Choices:
for each formula $\Phi(x,y)$ $\forall x \exists y \Phi(x,y) \Longrightarrow \exists f(Func(f)$ & $dom(f) = \alpha$ & $\forall \xi < \alpha\ \Phi(f|\xi, f(\xi)))$;
$\mathrm{DC}_{<On}$ stands for $\forall x\ \exists y\ \Phi(x,y) \Longrightarrow \forall \alpha\ \exists f\ (Func(f)$ & $dom(f) = \alpha$ & $\forall \xi < \alpha\ \Phi(f|\xi, f(\xi)))$;
DC_ω is equivalent to DC.

It is possible to prove (not shown in this paper) that ZFH + $\forall \alpha \mathrm{AC}_\alpha \vdash$DC.

Lemma 6.4. *(i) BST + WOP$\vdash$AC and AC$\vdash \forall \alpha AC_\alpha$;*
(ii) BST + the Δ_0-Replacement Scheme$\vdash$ $\forall a \exists! b(b = SFin(a))$, where $SFin(a)$ = the collection of all finite sequences with terms in a;
(iii) BST + the Δ_0-Replacement Scheme + AC$\vdash$ADC;
(iv) BST + AC_ω + Δ_0-Collection$\vdash$CUC;
(v) BST + ADC + Δ_0-Collection$\vdash$CUC;
(vi) BST + DC$\vdash$CUC;
(vii) BST + $\alpha = \aleph(x)$ + $DC_\alpha \vdash x$ is well-orderable;
(viii) ZFH! + $DC_{<On} \vdash$WOP + the Collection Scheme + the Principle of Reflection.

Notice that ZFH + AC + the Principle of Reflection$\vdash$DC but in [4] a model for ZFH + WOP + DC + the Principle of Reflection + $\neg\mathrm{DC}_{<On}$ is constructed.

Now, I want to show some applications of **Theorem 6.3**.

Theorem 6.4. *Let $M \models$ WOP + ZFH, $\mathrm{I} = \omega$, and $P_n = P(\aleph_0, \aleph_n; \aleph_0)$. Then $N \models$ WOP + ZFH! + $\aleph_1 = \aleph_\omega^M$ + $\neg$CUC, so $N \models \neg$DC + $\neg\Delta_0$-Collection. If $F(x) = \max_n(\mathrm{L}[x] \models \aleph_n^{\mathrm{M}}$ is countable) and $C_k(x) = x^k \cdot F(x) (k \geq 1, k$ fixed) for every real x, then — in N — $C_k(x)$ is unbounded in every neighborhood of 0 and for every sequence $\langle x_n : n < \omega \rangle$ of reals $\lim_{n\to\infty} x_n = 0 \Longrightarrow \lim_{n\to\infty} C_k(x_n) = 0$.*

Notice that N$\not\models$KP.

Theorem 6.5. *Let $\mathrm{M} = \mathrm{M}_\kappa$, where κ is On or a regular cardinal in L and M_κ is the model obtained by adding κ Cohen's reals to L.*

(i) If $\kappa < \aleph_\omega^{\mathrm{L}}$, then $N \models \aleph_1 = \aleph_\omega^{\mathrm{L}}$ + every set of reals is countable.
(ii) If $\aleph_\omega^{\mathrm{L}} < \kappa < On$, then $N \models \aleph_1 = \aleph_\omega^{\mathrm{L}}$ + κ is a cardinal + there is a set of reals of cardinality κ + every set of reals has cardinality $\leq \kappa$.
(iii) If $\kappa = On$, then $N \models$ WOP + ZFH! + for every cardinal μ there is a set of reals of cardinality μ.

Theorem 6.6. *Let $M \models$ZFC. Let $P_n = P(\kappa_n, 2; \kappa_n)$, where $\langle \kappa_n : n < \omega \rangle$ is a sequence of regular cardinals such that $|P_n| < \kappa_{n+1}$ for $n < \omega$. Then $N \models$ WOP + ZFH! + $\neg$DC + $\neg$the Collection Scheme and $\kappa \in \{\kappa_n : n < \omega\}$ $\longleftrightarrow$ N $\models \kappa$ is a regular cardinal & there is a subset of κ that is generic*

κ-cofinal over M.
If M$\models$GCH, then $cf^{\mathrm{N}} = cf^{\mathrm{M}}$. If $\mathrm{M} = \mathrm{L}[a]$, where a is Cohen's generic over L, and $c_n =$ the n^{th} element of a, then we define $\kappa_n = \aleph_{1+c_n}$ for $n < \omega$.
In this case, $\mathrm{N} \models m \in a \longleftrightarrow$ there is a subset of $\aleph_{1+m}$ that is generic $\aleph_{1+m}$-cofinal over L.

It is not necessary to destroy CUC or collapse cardinals to get a model with a Heine-Cauchy functional.

Theorem 6.7. *Let M$\models$ZFC. Let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of regular cardinals such that $\kappa_0 \geq 2^{\aleph_0}$. If $P_n = P(\kappa_n \times \omega, 2; \aleph_0)$ for $n < \omega$, then N$\models$WOP + ZFH! + $cf = cf^{\mathrm{M}}$ + CUC + $\neg$DC + $\neg$Collection.*
If $F(x) = \min_n(\mathrm{L}[x] \models 2^{\aleph_0} \leq \kappa_n)$ and $C_k(x) = x^k \cdot F(x)$ for every real x, then C_k is a Heine-Cauchy functional.

Proof. Everything is obvious except CUC. Suppose that $\langle a_n : n < \omega \rangle \in \mathrm{N}$ and $\mathrm{N} \models \forall n < \omega \exists f (f : \omega \xrightarrow{\text{onto}} a_n)$. There are finite $s, t \subset \omega$ such that $\langle a_n : n < \omega \rangle \in \mathrm{M}[\mathrm{H}|s \times t]$. $\mathrm{M}[\mathrm{H}] = \mathrm{M}[\mathrm{H}|s \times t][\mathrm{H}| - (s \times t)]$ and $\mathrm{M}[\mathrm{H}] \models$ "every a_n is countable". By c.c.c., $M[\mathrm{H}|s \times t] \models$ "every a_n is countable". But $M[\mathrm{H}|s \times t] \models$ZFC, so $M[\mathrm{H}|s \times t] \models \bigcup_{n<\omega} a_n$ is countable. Therefore, $\mathrm{N} \models \bigcup_{n<\omega} a_n$ is countable.

Above theorems show that ZFH! + WOP + $cf = cf^{\mathrm{L}}$ + $\neg$DC + $\neg$Collection is consistent. As already stated, ZFH! + the Collection Scheme + AC$\vdash$ADC but nothing is known about consistency of ZFH + WOP + $\neg$DC. So, we may think only about consistency of ZFH! + WOP + $cf = cf^{\mathrm{L}}$ + DC + $\neg$Collection. The easiest way to obtain a model for the above theory is by some modification of the construction of N.

Sketch of the modification: M is a transitive model for ZFC, $\langle P_i : i \in \mathrm{I} \rangle \in$ M is a family of notions of forcing. $\mathrm{Q}(\omega_1) =$ the set of all functions f such that $dom(f) \subset \mathrm{I} \times \omega_1, |dom(f)| < \omega_1$, and $f(i,n) \in P_i \setminus \{1_i\}$ for every $\langle i, \alpha \rangle \in dom(f)$.
If H is $\mathrm{Q}(\omega_1)$-generic over M, then $\mathrm{N}(\omega_1) = \bigcup \{\mathrm{M}[\mathrm{H}|s] : s \subset \mathrm{I} \times \omega_1, |s| < \omega_1\}$.

One can modify the proofs to see that $\mathrm{N}(\omega_1) \models$ZFH! + WOP.
Suppose that every P_i is ω_1-closed in M. Then $\mathrm{Q}(\omega_1)$ is ω_1-closed. If $t \in \mathrm{M}$ is a subset of $\mathrm{I} \times \omega_1$, then $\mathrm{Q}(\omega_1)|t = \{f \in \mathrm{Q}(\omega_1) : dom(f) \subseteq t\}$ and $\mathrm{Q}(\omega_1)| - t = \{f \in \mathrm{Q}(\omega_1) : dom(f) \cap t = \emptyset\}$. Of course, $\mathrm{Q}(\omega_1) \cong \mathrm{Q}(\omega_1)|t \times \mathrm{Q}(\omega_1)| - t$, $\mathrm{M}[\mathrm{H}] = \mathrm{M}[\mathrm{H}|t][\mathrm{H}| - t]$ and $\mathrm{Q}(\omega_1)| - t$ is ω_1-closed in $\mathrm{M}[\mathrm{H}|t]$.

Let $\forall x \in \mathrm{N}(\omega_1) \exists y \in \mathrm{N}(\omega_1) \mathrm{M}[\mathrm{H}] \models \Phi[x, y, a_1, a_2, \ldots, a_l]$. Then there is $F \in \mathrm{M}[\mathrm{H}]$ such that $\forall n < \omega F(n) \in \mathrm{N}(\omega_1)$ and $\mathrm{M}[\mathrm{H}] \models \forall n < \omega \Phi[F(n), F(n+1), a_1, a_2, \ldots, a_l]$. By the definition of $\mathrm{N}(\omega_1)$, for every n there is $s \in \mathrm{M}$ such that $F(n) \in \mathrm{M}[\mathrm{H}|s]$ and $|s| < \omega_1$. M[H]$\models$WOP + the Collection Scheme, so there is $G \in \mathrm{M}[\mathrm{H}]$ such that $\forall n < \omega \big(G(n) \in \mathrm{M}$ & $|G(n)| < \omega_1$ & $F(n) \in \mathrm{M}[\mathrm{H}|G(n)]\big)$. $\mathrm{Q}(\omega_1)$ is ω_1-closed, so $G \in \mathrm{M}$ and $t = \bigcup_{n<\omega} G(n)$ also belongs to

M. In full ZFC, t must be countable and $\forall n(F(n) \in \mathrm{M}[\mathrm{H}|t])$. $\mathrm{Q}(\omega_1)|-t$ is ω_1-closed in $\mathrm{M}[\mathrm{H}|t]$, so $F \in \mathrm{M}[\mathrm{H}|t]$ and $\mathrm{M}[\mathrm{H}|t] \subset \mathrm{N}(\omega_1)$. Therefore, $F \in \mathrm{N}(\omega_1)$ which shows a strong version of DC in $\mathrm{N}(\omega_1)$.

Up to now, $\mathrm{N}(\omega_1) \models$ZFH! + WOP + DC.

To destroy the Collection Scheme we need ω_1 independent notions of forcing. So, $\mathrm{I} = \omega_1$, $\langle \kappa_\alpha : \alpha < \omega_1 \rangle$ is a sequence of regular cardinals such that $\kappa_0 < \omega$ and $|\prod_{\beta<\alpha} P_\beta| < \kappa_\alpha$, where $P_\alpha = P(\kappa_\alpha, 2; \kappa_\alpha)$ for $\alpha < \omega_1$. It is clear that $\mathrm{N}(\omega_1) \models$ZFH! + WOP + DC + $\neg$the Collection Scheme. If M$\models$GCH, then $cf^{\mathrm{N}(\omega_1)} = cf^{\mathrm{M}}$.

Theorem 6.8. *ZFH! + WOP + $cf = cf^{\mathrm{L}}$ + $\neg$Collection $\pm$ DC are consistent theories.*

7. Final Remarks.

More can be done with the machinery and all combinatorial properties of forcings like Sacks, Laver, Mathias, Grigorieff, Jensen-Solovay (almost disjoint), etc. forcings. At least at the first stage of using them. If we want to use them in N, then many problems arise. For example, Jensen-Solovay Forcing is not necessary c.c.c. or even set-c.c. There is no problem with forcing in N as long as the notion of forcing is a set in N. Indeed, if $\langle \mathrm{C}, \leq \rangle \in \mathrm{N}$ and G is C-generic over N, then $\langle \mathrm{C}, \leq \rangle \in \mathrm{M}[\mathrm{H}|s_0] \subset \mathrm{N}$ for some $s_0 \in \mathrm{M}$ and $\mathrm{N}[G] = \bigcup\{\mathrm{M}[\mathrm{H}|s_0][\mathrm{H}|t][G] : t \cap s_0 = \emptyset, t \subset \mathrm{I} \times \omega, t\text{-finite}\} = \bigcup\{\mathrm{M}[\mathrm{H}|s_0][G][\mathrm{H}|t] : t$ as above $\}$, but $\mathrm{M}[\mathrm{H}|s_0][G] = \mathrm{M}[\mathrm{H}|s_0 \times G] \models$ZFH, so $\mathrm{N}[G] \models$ZFH!.

Another meaningful discussion could refer to <u>local</u> properties of large cardinals. Let M$\models$ZFC + κ is a large cardinal. We may construct two types of N.

Type 1. Let $P_{2\xi+1} = P(\aleph_0, \aleph_{\xi+1}; \aleph_0)$ for $\xi < \kappa$ and $P_{2\xi}$ is whatever is needed ($\xi < \kappa$), assuming that $|P_{2\xi}| < \kappa$.

Then N$\models$WOP + ZFH! + $\aleph_1 = \kappa$ + $\neg$the Collection Scheme.

Type 2. Let $P = \{f : Func(f) \;\&\; |f| < \omega \;\&\; dom(f) \subset \kappa \times \omega \;\&\; \forall \langle \alpha, n \rangle \in dom(f)(f(\alpha, n) < \alpha)\}$ and $|\mathrm{I}| = 1$. Then N$\models$WOP + ZFH(the Collection Scheme is included) + $\aleph_1 = \kappa$. P may be combined with other useful forcings.

It is interesting to know what original properties of κ are inherited by $\aleph_1$ in N. But that is another story.

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