How to characterize provably total functions by the Buchholz operator method *

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Summary. Inspired by Buchholz' technique of operator controlled derivations (which were introduced for simplifying Pohlers' method of local predicativity) a straightforward, perspicuous and conceptually simple method for characterizing the provably recursive functions of Peano arithmetic in terms of Kreisel's ordinal recursive functions is given. Since only amazingly little proof and hierarchy theory is used, the paper is intended to make the field of ordinally informative proof theory accessible even to non-prooftheorists whose knowledge in mathematical logic does not exceed a first graduate level course.

1. Introduction and Motivation

A fascinating result of ordinally informative proof theory due to Kreisel (1952) is as follows:

Theorem: (*)
The provably recursive functions of Peano arithmetic are exactly the ordinal recursive functions.

Folklore (proof-theoretic) proofs for (*) [cf., for example, Schwichtenberg (1977), Takeuti (1987), Buchholz (1991) or Friedman and Sheard (1995) for such proofs] rely on non trivial metamathematical evaluations of the Gentzen- or Schütte-style proof-theoretic analyses of Peano arithmetic. Alternatively a proof- and recursion-theoretic analysis of Gödel's 1958 functional interpretation of Heyting arithmetic can be employed for proving (*), cf. for example [Tait (1965), Buchholz (1980), Weiermann (1995)]. A proof of (*) which does not rely on metamathematical considerations – like primitive recursive stipulations of codes of infinite proof-trees – has been given in [Buchholz (1987), Buchholz and Wainer (1987)]. A proof of (*) using the slow growing hierarchy is given in [Arai (1991)]. A local predicativity style proof – which generalizes uniformly to theories of proof-theoretic strength less than or equal to KPM, cf. [Rathjen (1991)] – of (*) has been given in [Weiermann (1993), Blankertz and Weiermann (1995)]. Other proofs for (*) which are based on model theory can be found, for example, in [Hájek and Pudlák (1993)]. Buchholz (1992) introduced the technique of operator controlled derivations which allows a

* This paper is in its final form and no similar paper has been or is being submitted elsewhere.

simplified and conceptually improved exposition of Pohlers' local predicativity. One aim of the present paper is to give a contribution to the following question (Buchholz, private communication, 1993): Is it possible to use operator controlled derivations to give a proof for (*) – and generalizations of (*) – which is technically smooth? In this paper appropriate operators on subsets of the natural numbers are introduced via the Buchholz-Cichon-Weiermann (1994) approach to subrecursive hierarchies. It turns out that these operators work smoothly – i.e. virtually no auxiliary computations are needed – during the embedding and collapsing procedure. In the critical step of the argument (Reduction lemma and Cut-elimination) it is shown up how cut reduction directly corresponds to composition and diagonalization of the majorization functions involved. Only here an operator analysis is needed but nevertheless the critical arguments can be carried out in some few lines, cf. lemma 2.1 (vi) and (viii). Another aim of this paper is to present a method which can presumably be employed for giving as a direct corollary from Rathjen's proof-theoretic analyses – in which adaptations of operator controlled derivations are used – a classification of the provably recursive functions of $KP + \Pi^0_3 - Reflection$, $\Pi^1_2 - (CA)$ and related systems, cf. [Rathjen (1994),(1995)] and also [Arai (1995)].
The paper is self-contained. It only requires knowledge of basic facts about the ordinals up to ε_0 and elementary level facts about cut elimination in Tait's calculus for predicate logic.

2. Proof of the main Theorem

The set of non logical constants of PA includes the set of function symbols for primitive recursive functions and the relation symbol $=$.
(In the sequel $\underline{0}$ denotes the constant symbol for zero and S denotes the successor function symbol.) The logical operations include $\wedge, \vee, \forall, \exists$. We have an infinite list of variables $x_0, x_1, \ldots$ The set of *PA-terms* (which are denoted in the sequel by $r, s, t \ldots$) is the smallest set which contains the variables and constants and is closed under function application, i.e. if f is a k-ary function symbol and $t_1, \ldots, t_k$ are terms, so is $ft_1 \ldots t_k$. If $t(\mathrm{x})$ is a PA-term with $FV(t) \subseteq \{\mathrm{x}\}$ then $t^{\mathbb{N}}$ denotes the represented function in the standard structure $\mathbb{N}$. The set of *PA-formulas* (which are in the sequel denoted by A, B, F) is the smallest set which includes $s = t$, $\neg s = t$ (*prime formulas*) and is closed under conjunction, disjunction and quantification. The notation $\neg A$, for A arbitrary, is an abbreviation for the formula obtained via the de Morgan laws; $A \to B$ abbreviates $\neg A \vee B$. We denote finite sets of PA-formulas by $\Gamma, \Delta, \ldots$ As usual Γ, A stands for $\Gamma \cup \{A\}$ and Γ, Δ stands for $\Gamma \cup \Delta$.
The formal system PA is presented in a Tait calculus. PA includes the *logical axioms* $\Gamma, \neg A, A$, the *equality axioms* $\Gamma, t = t$ and $\Gamma, t \neq s, \neg A(t), A(s)$, the *successor function axioms* $\Gamma, \forall x(\neg \underline{0} = \mathrm{S}x)$ and $\Gamma, \forall x \forall y(\mathrm{S}x = \mathrm{S}y \to x = y)$,

the defining equations for primitive recursive function symbols (cf. [Pohlers (1989)]) and the *induction scheme* $\Gamma, A(\underline{0}) \wedge \forall x(A(x) \to A(\mathrm{S}x)) \to \forall x A(x)$. The *derivation relation* $\vdash$ for PA is defined as follows:

(Ax) $\vdash \Gamma$ if Γ is an axiom of PA.
($\wedge$) $\vdash \Gamma, A_i$ for all $i \in \{0,1\}$ imply $\vdash \Gamma, A_0 \wedge A_1$.
($\vee$) $\vdash \Gamma, A_i$ for some $i \in \{0,1\}$ implies $\vdash \Gamma, A_0 \vee A_1$.
($\forall$) $\vdash \Gamma, A(y)$ implies $\vdash \Gamma, \forall x A(x)$ if y does not occur in $\Gamma, \forall x A(x)$.
($\exists$) $\vdash \Gamma, A(t)$ implies $\vdash \Gamma, \exists x A(x)$.
(cut) $\vdash \Gamma, A$ and $\vdash \Gamma, \neg A$ imply $\vdash \Gamma$.

Definition 2.1 (Rank of a formula).

- $\mathrm{rk}(A) := 0$, *if A is a prime formula*
- $\mathrm{rk}(A \vee B) := \mathrm{rk}(A \wedge B) := \max\{\mathrm{rk}(A), \mathrm{rk}(B)\} + 1$
- $\mathrm{rk}(\forall x A(x)) := \mathrm{rk}(\exists x A(x)) := \mathrm{rk}(A) + 1$

We consider only ordinals less than ε_0. These ordinals are denoted by $\alpha, \beta, \gamma, \xi, \eta$. Finite ordinals are denoted by $k, m, n, \ldots$ $\#$ denotes the natural sum of ordinals (cf. [Schütte 1977] or [Pohlers 1989]). For each $\alpha < \varepsilon_0$ let $\mathrm{N}(\alpha)$ be the number of occurences of ω in the Cantor normal form representation of α. Thus $\mathrm{N}(0) = 0$, and $\mathrm{N}(\alpha) = n + \mathrm{N}(\alpha_1) + \cdots + \mathrm{N}(\alpha_n)$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} > \alpha_1 \geq \ldots \geq \alpha_n$. N satisfies the following conditions:

(N1) $\mathrm{N}(0) = 0$
(N2) $\mathrm{N}(\alpha \# \beta) = \mathrm{N}(\alpha) + \mathrm{N}(\beta)$
(N3) $\mathrm{N}(\omega^\alpha) = \mathrm{N}(\alpha) + 1$
(N4) $\mathrm{card}\ \{\alpha < \varepsilon_0 : \mathrm{N}(\alpha) < k\} < \omega$ for every $k \in \mathbb{N}$

Furthermore we see that $\mathrm{N}(\alpha + n) = \mathrm{N}(\alpha) + n$ for $n < \omega$.
Abbreviation: $\mathrm{N}\alpha := \mathrm{N}(\alpha)$.

Definition 2.2. *Let $\Phi(x) := 3^{x+1}$ and $f^{(2)}(x) := f(f(x))$ denote iteration.*

$$\mathrm{F}_\alpha(x) := \max\ \Big(\{2^x\} \cup \{\mathrm{F}_\gamma{}^{(2)}(x) \mid \gamma < \alpha \ \ \&\ \ \mathrm{N}\gamma \leq \Phi(\mathrm{N}\alpha + x)\}\Big)$$

This expression is well defined due to (N4).

Remark: It follows immediately from [Wainer (1970)] and [Buchholz, Cichon et Weiermann (1994)] that this hierarchy is equivalent to the ordinal recursive functions.

Definition 2.3. *Throughout the whole article we assume* $\mathrm{F} : \mathbb{N} \to \mathbb{N}$ *to be a monotone function. Given $\Theta \in \mathbb{N}$ we denote by* $\mathrm{F}[\Theta]$ *the function defined by*

$$\mathrm{F}[\Theta] : \mathbb{N} \to \mathbb{N};\ x \mapsto \mathrm{F}(\max\{x, \Theta\})$$

We write $\mathrm{F}[k_1, \ldots, k_n]$ *instead of* $\mathrm{F}[\max\{k_1, \ldots, k_n\}]$ *and use the abbreviation* $\quad \mathrm{F} \leq \mathrm{F}' \quad :\Leftrightarrow \quad \forall x \in \mathbb{N} \quad \mathrm{F}(x) \leq \mathrm{F}'(x)$.

Remark: $\mathrm{F}[n,m] = \mathrm{F}[n][m]$ and $\mathrm{F}[n][n] = \mathrm{F}[n]$.

Lemma 2.1. *(i)* $x < \mathrm{F}_\alpha(x) < \mathrm{F}_\alpha(x+1)$
(ii) $\mathrm{N}\alpha < \mathrm{F}_\alpha(x) < \mathrm{F}_\alpha{}^{(2)}(x) \leq \mathrm{F}_{\alpha+1}(x)$, *in particular* $n < \mathrm{F}_n(x)$
(iii) $\alpha < \gamma \;\&\; \mathrm{N}\alpha \leq \Phi(\mathrm{N}\gamma + \Theta) \quad\Rightarrow\quad \mathrm{F}_\alpha[\Theta] \leq \mathrm{F}_\gamma[\Theta]$
(iv) For every n-ary primitive recursive function f there exists a $p \in \mathbb{N}$ *such that* $\forall \mathbf{k} \in \mathbb{N}^n \quad f(\mathbf{k}) < \mathrm{F}_{\omega\cdot p}(\max \mathbf{k})$.
(v) $\mathrm{F}_\gamma[\Theta] \leq \mathrm{F}_{\gamma \# \beta}[\Theta]$
(vi) $k < \mathrm{F}_\gamma(\Theta) \quad\Rightarrow\quad \mathrm{F}_\gamma[\Theta][k] \leq \mathrm{F}_{\gamma+1}[\Theta]$
(vii) $\mathrm{N}\alpha, \mathrm{N}\beta < \mathrm{F}_\gamma(\Theta) \quad\Rightarrow\quad \mathrm{N}(\alpha+\beta), \mathrm{N}\omega^\alpha < \mathrm{F}_{\gamma+1}(\Theta)$
(viii) $\alpha_0 < \alpha \;\&\; \mathrm{N}\alpha_0 < \mathrm{F}_\gamma(\Theta) \quad\Rightarrow\quad \mathrm{F}_{\gamma \# \alpha_0+2}[\Theta] \leq \mathrm{F}_{\gamma \# \alpha+1}[\Theta]$

Proof. (i)-(vii) are simple. In (vi) for example $k < \mathrm{F}_\gamma(\Theta)$ implies for all $x \in \mathbb{N}$ $\mathrm{F}_\gamma[\Theta][k](x) = \mathrm{F}_\gamma(\max\{\Theta, k, x\}) \leq \mathrm{F}_\gamma{}^{(2)}(\max\{\Theta, x\}) \leq \mathrm{F}_{\gamma+1}(\max\{\Theta, x\})$.
(viii) If $\alpha = \alpha_0 + n$ for some $n \geq 1$ the claim follows from (v). So we can assume

$$(1) \qquad \gamma \# \alpha_0 + 2 < \gamma \# \alpha$$

and the premise $\mathrm{N}\alpha_0 < \mathrm{F}_\gamma(\Theta)$ implies

$$(2) \qquad \mathrm{N}(\gamma \# \alpha_0 + 2) \leq \mathrm{N}\gamma + \mathrm{F}_\gamma(\Theta) + 1 \leq \Phi(\mathrm{N}(\gamma \# \alpha) + \mathrm{F}_{\gamma \# \alpha}(\Theta))$$

For all $x \in \mathbb{N}$ we obtain from (1) and (2) by part (iii) and (ii)

$$\mathrm{F}_{\gamma \# \alpha_0+2}[\Theta](x) = \mathrm{F}_{\gamma \# \alpha_0+2}(\max\{\Theta, x\}) <^{(iii)} \mathrm{F}^{(2)}_{\gamma \# \alpha}(\max\{\Theta, x\}) \leq^{(ii)} \mathrm{F}_{\gamma \# \alpha+1}[\Theta](x).$$

Definition 2.4 (F-controlled derivations).
$\mathrm{F}\overset{\alpha}{\underset{r}{\vdash}} \Gamma$ *holds iff* $\mathrm{N}\alpha < \mathrm{F}(0)$ *and one of the following is true:*

(Ax)	$\Gamma \cap \Delta(\mathbb{N}) \neq \emptyset$			
$(\vee)$	$A_0 \vee A_1 \in \Gamma$	$\& \; \exists i \in \{0,1\}$	$\exists \alpha_0 < \alpha$	$\mathrm{F}\overset{\alpha_0}{\underset{r}{\vdash}} \Gamma, A_i$
$(\wedge)$	$A_0 \wedge A_1 \in \Gamma$	$\& \; \forall i \in \{0,1\}$	$\exists \alpha_0 < \alpha$	$\mathrm{F}\overset{\alpha_0}{\underset{r}{\vdash}} \Gamma, A_i$
$(\exists)$	$\exists x A(x) \in \Gamma$	$\& \; \exists n < \mathrm{F}(0)$	$\exists \alpha_0 < \alpha$	$\mathrm{F}\overset{\alpha_0}{\underset{r}{\vdash}} \Gamma, A(\underline{n})$
$(\forall)$	$\forall x A(x) \in \Gamma$	$\& \; \forall n \in \mathbb{N}$	$\exists \alpha_n < \alpha$	$\mathrm{F}[n]\overset{\alpha_n}{\underset{r}{\vdash}} \Gamma, A(\underline{n})$
(cut)	$\mathrm{rk}(A) < r$	$\&$	$\exists \alpha_0 < \alpha$	$\mathrm{F}\overset{\alpha_0}{\underset{r}{\vdash}} \Gamma, (\neg)A$

The abbreviations used in (Ax) and (cut) are the following:

- $\Delta(\mathbb{N}) := \{A \mid A \text{ is prime formula and } \mathbb{N} \models A\}$
- $\mathrm{F}\overset{\alpha_0}{\underset{r}{\vdash}} \Gamma, (\neg)A \quad :\Leftrightarrow \quad \mathrm{F}\overset{\alpha_0}{\underset{r}{\vdash}} \Gamma, A \;\&\; \mathrm{F}\overset{\alpha_0}{\underset{r}{\vdash}} \Gamma, \neg A$

So the F-controlled derivations are just like usual PA-derivations but with ω-rule and some information about the $\exists$-witnesses and the derivation length. The first one is the essential aid for collapsing (lemma 2.6) while the latter is used to apply lemma 2.1 (viii) in the cut-elimination procedure 2.5.

Lemma 2.2 (Monotonicity).
For $r \leq s$ & $\alpha \leq \beta$ & $\mathrm{N}\beta < \mathrm{F}'(0)$ & $\mathrm{F} \leq \mathrm{F}'$ *the following holds*

$$\mathrm{F}\big|\!\frac{\alpha}{r}\, \Gamma \quad\Rightarrow\quad \mathrm{F}'\big|\!\frac{\beta}{s}\, \Gamma, \Gamma'$$

Lemma 2.3 (Inversion).

(i) $\mathrm{F}\big|\!\frac{\alpha}{r}\, \Gamma, A \;\&\; \neg A \in \Delta(\mathbf{N}) \quad\Rightarrow\quad \mathrm{F}\big|\!\frac{\alpha}{r}\, \Gamma$

(ii) $\mathrm{F}\big|\!\frac{\alpha}{r}\, \Gamma, A_0 \wedge A_1 \quad\Rightarrow\quad \forall i \in \{0,1\} \;\; \mathrm{F}\big|\!\frac{\alpha}{r}\, \Gamma, A_i$

(iii) $\mathrm{F}\big|\!\frac{\alpha}{r}\, \Gamma, \forall x A(x) \quad\Rightarrow\quad \forall n \in \mathbf{N} \;\; \mathrm{F}[n]\big|\!\frac{\alpha}{r}\, \Gamma, A(\underline{n})$

Lemma 2.4 (Reduction). *Let* A *be a formula* $A_0 \vee A_1$ *or* $\exists x B(x)$ *of rank* $\leq r$.

$$\mathrm{F}_\gamma[\Theta]\big|\!\frac{\alpha}{r}\, \Gamma, \neg A \;\&\; \mathrm{F}_\gamma[\Theta]\big|\!\frac{\beta}{r}\, \Delta, A \quad\Rightarrow\quad \mathrm{F}_{\gamma+1}[\Theta]\big|\!\frac{\alpha+\beta}{r}\, \Gamma, \Delta$$

Proof by induction on β. The interesting case is that $A \equiv \exists x B(x)$ is the main formula of the last deduction step. For some $k < \mathrm{F}_\gamma(\Theta)$ and $\beta_0 < \beta$ we have the premise

$$\mathrm{F}_\gamma[\Theta]\big|\!\frac{\beta_0}{r}\, \Delta, A, B(\underline{k})$$

The induction hypothesis yields

$$\mathrm{F}_{\gamma+1}[\Theta]\big|\!\frac{\alpha+\beta_0}{r}\, \Delta, B(\underline{k})$$

By the inversion lemma 2.3 (iii) and the fact $\mathrm{F}_\gamma[\Theta][k] \leq \mathrm{F}_{\gamma+1}[\Theta]$ (lemma 2.1 (vi)) we can transform the first derivation into

$$\mathrm{F}_{\gamma+1}[\Theta]\big|\!\frac{\alpha}{r}\, \Gamma, \neg B(\underline{k})$$

$\mathrm{N}(\alpha+\beta) < \mathrm{F}_{\gamma+1}(\Theta)$ is true by lemma 2.1 (vii). Furthermore $\mathrm{rk}(B(\underline{k})) < \mathrm{rk}(\forall x B(x))$, so the claim can be obtained by a cut (using monotonicity before).

Lemma 2.5 (Cut-elimination). $\mathrm{F}_\gamma[\Theta]\big|\!\frac{\alpha}{r+1}\, \Gamma \quad\Rightarrow\quad \mathrm{F}_{\gamma \# \alpha+1}[\Theta]\big|\!\frac{\omega^\alpha}{r}\, \Gamma$

Proof by induction on α. In the interesting case of a cut we have for some $\alpha_0 < \alpha$

$$\mathrm{F}_\gamma[\Theta]\big|\!\frac{\alpha_0}{r+1}\, \Gamma, (\neg)A$$

for some A of rank $\leq r$. The induction hypothesis yields

$$\mathrm{F}_{\gamma \# \alpha_0+1}[\Theta]\big|\!\frac{\omega^{\alpha_0}}{r}\, \Gamma, (\neg)A$$

By reduction 2.4 (or lemma 2.3 (i) for prime formulas $(\neg)A$) we obtain

$$\mathrm{F}_{\gamma \# \alpha_0+2}[\Theta]\big|\!\frac{\omega^{\alpha_0}\cdot 2}{r}\, \Gamma$$

and the claim follows from lemma 2.1 (vii), (viii), $\omega^{\alpha_0}\cdot 2 < \omega^\alpha$ and monotonicity.

Lemma 2.6 (Collapsing).

$$\mathrm{F}\overset{\alpha}{\underset{0}{\vdash}} \exists y A(y) \;\&\; \mathrm{rk}(A) = 0 \quad \Rightarrow \quad \exists n < \mathrm{F}(0) \;\; \mathbb{N} \models A(\underline{n})$$

Proof by induction on α. Since we have a cut-free derivation the last deduction step was $(\exists)$ so there is an $n < \mathrm{F}(0)$ and an $\alpha_0 < \alpha$ such that

$$\mathrm{F}_\gamma[\Theta]\overset{\alpha_0}{\underset{0}{\vdash}} A(\underline{n}), \exists y A(y)$$

If $\mathbb{N} \models A(\underline{n})$ we are done otherwise we can use lemma 2.3 (i) and the induction hypothesis to proof the claim.

Definition 2.5. $A \sim A' \quad :\Leftrightarrow$ *There are a PA-formula B, pairwise distinct variables $x_1, ..., x_n$ and closed PA-terms $t_1, s_1, \ldots, t_n, s_n$ such that $t_i^{\mathrm{N}} = s_i^{\mathrm{N}}$ $(i = 1, ..., n)$ and $A \equiv B_{x_1,...,x_n}(t_1, ..., t_n)$, $A' \equiv B_{x_1,...,x_n}(s_1, ..., s_n)$.*

Lemma 2.7 (Tautology and embedding of mathematical axioms).
(i) $A \sim A' \quad \Rightarrow \quad \mathrm{F}_k\overset{k}{\underset{0}{\vdash}} A, \neg A'$ *where* $k := 2 \cdot \mathrm{rk}(A)$
(ii) For every formula A with $FV(A) \subseteq \{x\}$ there is a $k \in \mathbb{N}$ such that
$\mathrm{F}_k\overset{\omega+3}{\underset{0}{\vdash}} A(\underline{0}) \wedge \forall x(A(x) \rightarrow A(\mathrm{S}x)) \rightarrow \forall x A(x)$
(iii) For every other math. axiom of PA A there is a $k < \omega$ such that $\mathrm{F}_k\overset{k}{\underset{0}{\vdash}} A$

Proof. (i) as usual using $\mathrm{N}k < \mathrm{F}_k(0)$. (iii) If Γ contains only Π-formulas and $\overset{k}{\underset{0}{\vdash}} \Gamma$ denotes a usual cut-free PA-deriviation (without F-controlling) we can easily conclude $\mathrm{F}_k\overset{k}{\underset{0}{\vdash}} \Gamma$ by induction on k. Since the result $\overset{k}{\underset{0}{\vdash}} A$ for mathematical axioms A except induction is folklore the claim follows.
(ii) Let $k := \mathrm{rk}(A(\underline{0}))$. We show by induction on n

$$(*) \qquad \mathrm{F}_{2k}[n]\overset{2k+2n}{\underset{0}{\vdash}} \neg A(\underline{0}), \neg\forall x(A(x) \rightarrow A(\mathrm{S}x)), A(\underline{n})$$

$n = 0$: The tautology lemma 2.7 (i) yields $\mathrm{F}_{2k}\overset{2k}{\underset{0}{\vdash}} \neg A(\underline{0}), A(\underline{0})$. So the claim follows by monotonicity.
$n \mapsto n+1$: By tautology lemma we have $\mathrm{F}_{2k}\overset{2k}{\underset{0}{\vdash}} \neg A(\mathrm{S}\underline{n}), A(\underline{n+1})$. Connecting this with the derivation given by induction hypothesis by $(\wedge)$ yields
$\mathrm{F}_{2k}[n+1]\overset{2k+2n+1}{\underset{0}{\vdash}} \neg A(\underline{0}), \neg\forall x(A(x) \rightarrow A(\mathrm{S}x)), A(\underline{n}) \wedge \neg A(\mathrm{S}\underline{n}), A(\underline{n+1})$
So $(*)$ follows by an application of $(\exists)$. The claim follows from $(*)$ by an application of $(\forall)$ and three applications of $(\vee)$.

Lemma 2.8 (Embedding). *For every Γ satisfying $FV(\Gamma) \subseteq \{x_1, \ldots, x_m\}$ and $PA \vdash \Gamma$ there exist $\gamma < \omega^2, \alpha < \omega \cdot 2$, and $r < \omega$ such that*

$$\forall \mathbf{n} \in \mathbb{N}^m \quad \mathrm{F}_\gamma[\mathbf{n}]\overset{\alpha}{\underset{r}{\vdash}} \Gamma(\underline{\mathbf{n}})$$

Proof by induction on the derivation of Γ.
1. $A, \neg A \in \Gamma$ is an axiom. The tautology lemma yields the claim.
2. $\forall x A(x, \mathbf{x}) \in \Gamma(\mathbf{x})$ and Γ results from $\Gamma(\mathbf{x}), A(y, \mathbf{x})$ and $y \notin \{x_1, \ldots, x_m\}$ holds true. By induction hypothesis there are $\gamma < \omega^2, \alpha < \omega \cdot 2, r < \omega$ such that

$$\forall n \in \mathbb{N}\, \forall \mathbf{n} \in \mathbb{N}^m \quad \mathrm{F}_\gamma[n, \mathbf{n}] \left|\frac{\alpha}{r}\right. \Gamma(\underline{\mathbf{n}}), A(\underline{n}, \underline{\mathbf{n}})$$

We obtain the claim by an application of $(\forall)$ since $\mathrm{F}_\gamma[n, \mathbf{n}] = \mathrm{F}_\gamma[\mathbf{n}][n]$.
3. $\exists x A(x, \mathbf{x}) \in \Gamma(\mathbf{x})$ and Γ results from $\Gamma(\mathbf{x}), A(t(\mathbf{x}), \mathbf{x})$. We can assume that $x \notin \{x_1, \ldots . x_m\}$ and $FV(t) \subseteq \{x_1, \ldots . x_m\}$. By induction hypothesis there are $\gamma < \omega^2, \alpha_0 < \omega \cdot 2$ and $r_0 < \omega$ such that

$$(3) \qquad \forall \mathbf{n} \in \mathbb{N}^m \quad \mathrm{F}_\gamma[\mathbf{n}] \left|\frac{\alpha_0}{r_0}\right. \Gamma(\underline{\mathbf{n}}), A(t(\underline{\mathbf{n}}), \underline{\mathbf{n}})$$

The tautology lemma 2.7 (i) yields a $k < \omega$ such that

$$(4) \qquad \forall \mathbf{n} \in \mathbb{N}^m \quad \mathrm{F}_k \left|\frac{k}{0}\right. A(\underline{t^{\mathbf{N}}(\mathbf{n})}, \underline{\mathbf{n}}), \neg A(t(\underline{\mathbf{n}}), \underline{\mathbf{n}})$$

Since $\lambda \mathbf{x}.t^{\mathbf{N}}(\mathbf{x})$ is a primitive recursive function due to lemma 2.1 (iv) there is a $p < \omega$ satisfying $\forall \mathbf{x} \in \mathbb{N}^m\ t^{\mathbf{N}}(\mathbf{x}) < \mathrm{F}_{\omega \cdot p}(\max \mathbf{x})$. Choosing $p > k$ implies $\mathrm{F}_k \leq \mathrm{F}_k[\mathbf{n}] \leq \mathrm{F}_{\omega \cdot p}[\mathbf{n}]$ with aid of lemma 2.1 (iii). Letting $\alpha := \max\, \{\alpha_0, k\}$ and $r > \max\, \{r_0, \mathrm{rk}(A)\}$ we obtain

$$\forall \mathbf{n} \in \mathbb{N}^m \quad \mathrm{F}_{\gamma \# \omega \cdot p}[\mathbf{n}] \left|\frac{\alpha}{r}\right. \Gamma(\mathbf{n}), A(\underline{t^{\mathbf{N}}(\mathbf{n})}, \underline{\mathbf{n}}), (\neg) A(t(\underline{\mathbf{n}}), \underline{\mathbf{n}}).$$

from (3) and (4) by monotonicity 2.2. Applying a cut we get

$$\forall \mathbf{n} \in \mathbb{N}^m \quad \mathrm{F}_{\gamma \# \omega \cdot p}[\mathbf{n}] \left|\frac{\alpha + 1}{r}\right. \Gamma(\mathbf{n}), A(\underline{t^{\mathbf{N}}(\mathbf{n})}, \underline{\mathbf{n}})$$

and $(\exists)$ proves the claim since $t^{\mathbf{N}}(\mathbf{n}) < \mathrm{F}_{\gamma \# \omega \cdot p}(\mathbf{n})$.
4. Γ results from Γ, A and $\Gamma, \neg A$ by a cut. By induction hypothesis there are $\gamma < \omega^2, \alpha < \omega \cdot 2$ and $r < \omega$ such that

$$\forall \mathbf{n} \in \mathbb{N}^m \quad \mathrm{F}_\gamma[\mathbf{n}] \left|\frac{\alpha}{r}\right. \Gamma(\underline{\mathbf{n}}), (\neg) A(\underline{\mathbf{n}}, \underline{\mathbf{0}})$$

By choosing $r > \mathrm{rk}(A)$ we obtain the claim by an application of (cut).
5. The missing cases are covered by lemma 2.7 or easy (rules for $\vee$ and $\wedge$).

Theorem 2.1. *Let A be a prime formula such that $PA \vdash \forall x \exists y\, A(x, y)$ and $FV(A) \subseteq \{x, y\}$. Then there is a $\gamma < \varepsilon_0$ such that*

$$\forall x \in \mathbb{N}\ \exists y < F_\gamma(x) \quad \mathbb{N} \models A(\underline{x}, \underline{y})$$

Proof by embedding 2.8, iterated cut-elimination 2.5, inversion 2.3 (iii) and collapsing 2.6.

Remark: The methods of this paper yield also classifications of the provable recursive functions of the fragments $(I\Sigma_{n+1})$ of PA and of PA+TI($\prec \Gamma$). The extension to KPω has recently been carried out in full detail by the authors.

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