§6. An inductive definition of K

The definition of K given in 5.17 is $\Sigma_\omega(V_{\Omega+1})$, and therefore much too complicated for some purposes. In this section we shall give an inductive definition of K whose logical form is as simple as possible. Assuming that K^c has no Woodin cardinals, we shall show that $K \cap HC$ is $\Sigma_1(L_{\omega_1}(\mathbb{R}))$ in the codes; Woodin has shown that in general no simpler definition is possible.

The following notion is central to our inductive definition of K.

Definition 6.1. *Let $\mathcal{M}$ be a proper premouse such that $\mathcal{M} \models ZF - \{Powerset\}$ and $\mathcal{J}_\alpha^\mathcal{M}$ is S-sound. We say $\mathcal{M}$ is (α, S)-strong iff there is an $(\omega, \Omega+1)$ iterable weasel which witnesses that $\mathcal{J}_\alpha^\mathcal{M}$ is S-sound, and whenever W is a weasel which witnesses that $\mathcal{J}_\alpha^\mathcal{M}$ is S-sound, and Σ is an $(\omega, \Omega+1)$ iteration strategy for W, then there is a length $\theta+1$ iteration tree $\mathcal{T}$ on W which is a play by Σ and such that $\forall\gamma < \theta(\nu(E_\gamma^\mathcal{T}) \geq \alpha)$, and a $Q \trianglelefteq W_\theta^\mathcal{T}$, and a fully elementary $\pi : \mathcal{M} \to Q$ such that $\pi \restriction \alpha =$ identity.*

We shall see that it is possible to define "(α, S)-strong" by induction on α. First, let us notice:

Lemma 6.2. *Let W be an $(\omega, \Omega+1)$ iterable weasel which witnesses that $\mathcal{J}_\alpha^W$ is S-sound; then W is (α, S) strong.*

Proof. Let R be a weasel which witnesses $\mathcal{J}_\alpha^W$ is S-sound, and let Σ be an $\Omega+1$ iteration strategy for R. Let Γ be an $\Omega+1$ iteration strategy for W, and let $(\mathcal{T}, \mathcal{U})$ be the successful coiteration of R with W determined by (Σ, Γ). Let Q be the common last model of $\mathcal{T}$ and $\mathcal{U}$, and let $\pi : W \to Q$ be the iteration map given by $\mathcal{U}$. By Lemma 5.1, $\pi \restriction \alpha =$ identity. □

Lemma 6.2 admits the following slight improvement. Let W witness that $\mathcal{J}_\alpha^W$ is S-sound, and let Σ be an $(\omega, \Omega+1)$ iteration strategy for W. Let $\mathcal{T}$ be an iteration tree played by Σ such that $\forall\gamma < \theta(\nu(E_\gamma^\mathcal{T}) \geq \alpha)$, where $\theta+1 = lh\ \mathcal{T}$; then $W_\theta^\mathcal{T}$ is (α, S) strong. [Proof: Let R be any weasel witnessing $\mathcal{J}_\alpha^W$ is S-sound. Comparing R with W, we get an iteration tree $\mathcal{U}$ on R and a map $\pi : W \to R_\eta^\mathcal{U}$, where $\eta = lh\,\mathcal{U} - 1$. By 5.1, $\text{crit}(\pi) \geq \alpha$. Let $\sigma : W_\theta^\mathcal{T} \to (R_\eta^\mathcal{U})_\theta^{\pi\mathcal{T}}$ be the copy map. Then σ and $\mathcal{U}^\frown\pi\mathcal{T}$ are as required in 6.1 for R.] This shows that we obtain a definition of (α, S) strength equivalent to 6.1 if we replace "whenever W is a weasel" by "there is a weasel W" in 6.1. It also shows that there are (α, S) strong weasels other than those described in 6.2. For example, suppose W witnesses that $\mathcal{J}_\alpha^W$ is S-sound, and E is an extender on the W sequence which is total on W and such that $\text{crit}(E) < \alpha \leq \nu(E)$. Setting $R = \text{Ult}(W, E)$, we have that R is (α, S) strong, but R does not witness that $\mathcal{J}_\alpha^R$ is S-sound.

In view of the fact that $K(S)$ is independent of S, one might expect the same to be true of (α, S)-strength. This is indeed the case.

Lemma 6.3. *Suppose $K(S)$ and $K(T)$ exist, and $\alpha \leq OR \cap K(S) \cap K(T)$; then for any $\mathcal{M}$, $\mathcal{M}$ is (α, S) strong iff $\mathcal{M}$ is (α, T) strong.*

Proof. Suppose $\mathcal{M}$ is (α, S)-strong. Let $\mathcal{R}$ witness that $\mathcal{J}_\alpha^{\mathcal{M}}$ is S-sound, and W witness that $\mathcal{J}_\alpha^{\mathcal{M}}$ is T-sound. Let Σ be an $(\omega, \Omega+1)$ iteration strategy for W, and Γ an $(\omega, \Omega+1)$ iteration strategy for R. From the proof of 5.16, we get iteration trees $\mathcal{T}$ and $\mathcal{U}$ on W and R which are plays of two rounds of $\mathcal{G}^*(W, (\omega, \Omega+1))$ and $\mathcal{G}^*(R, (\omega, \Omega+1))$ according to Σ and Γ respectively, and such that $\mathcal{T}$ and $\mathcal{U}$ have a common last model Q. The proof of 5.16 also shows that the iteration maps $\sigma : W \to Q$ and $\tau : R \to Q$ satisfy $\alpha \leq \min(\text{crit}(\sigma), \text{crit}(\tau))$. Since $\alpha \leq \text{crit}(\sigma)$, $\nu(E_\gamma^{\mathcal{T}}) \geq \alpha$ for all $\gamma + 1 < lh\, \mathcal{T}$.

Now Σ yields an $(\omega, \Omega+1)$-iteration strategy Σ^* for Q, and the strategy of copying via τ and using Σ^* on the copied tree is an $(\omega, \Omega+1)$-iteration strategy for R; call it Σ^{**}.

According to 6.1, there is an iteration tree $\mathcal{V}$ on R having last model $\mathcal{P}$ which is a play by Σ^{**}, and such that $\forall\gamma(\gamma + 1 < lh\ \mathcal{V} \Rightarrow \nu(E_\gamma^{\mathcal{V}}) \geq \alpha)$, and an embedding $\pi : \mathcal{M} \to \mathcal{P}'$ for some $\mathcal{P}' \trianglelefteq \mathcal{P}$ such that $\pi \restriction \alpha =$ identity. Let $\tau^* : \mathcal{P} \to \mathcal{L}$, where $\mathcal{L}$ is the last model of the copied tree $\tau\mathcal{V}$ on Q, be the copy map; thus $\tau^* \restriction \alpha = \tau \restriction \alpha =$ identity. Let $\mathcal{L}' \trianglelefteq \mathcal{L}$ correspond to $\mathcal{P}'$. Then $\mathcal{L}'$ is an initial segment of the last model of $\mathcal{T}^\frown\tau\mathcal{V}$, which is a play by Σ; moreover $\tau^* \circ \pi$ maps $\mathcal{M}$ into $\mathcal{L}'$ and $(\tau^* \circ \pi) \restriction \alpha =$ identity.

This shows that $\mathcal{M}$ is (α, T)-strong, as desired. □

Definition 6.4. *Let $\mathcal{M}$ be a proper premouse, and let $\alpha < \Omega$. We say $\mathcal{M}$ is α-strong iff for some S, $\mathcal{M}$ is (α, S)-strong.*

We proceed to the inductive definition of "α-strong". The definition is based on a certain iterability property: roughly speaking, $\mathcal{M}$ is α-strong just in case $\mathcal{M}$ is jointly iterable with any $\mathcal{N}$ which is β-strong for all $\beta < \alpha$. In order to describe this iterability property we must introduce iteration trees whose "base" is not a single model, but rather a family of models. Such systems were called "psuedo-iteration trees" in [FSIT]. Here we shall simply call them iteration trees, and distinguish them from the iteration trees considered so far by means of their bases.

Definition 6.5. *A* simple phalanx *is a pair $(\langle \mathcal{M}_\beta \mid \beta \leq \alpha \rangle, \langle \lambda_\beta \mid \beta < \alpha \rangle)$ such that for all $\beta \leq \alpha$, $\mathcal{M}_\beta$ is an ω-sound proper premouse, and*

(1) $\beta \leq \gamma \leq \alpha \Rightarrow (\mathcal{M}_\gamma \models$ *"λ_β is a cardinal" and* $\rho_\omega(\mathcal{M}_\gamma) \geq \lambda_\beta)$,

(2) $\beta < \gamma \leq \alpha \Rightarrow \mathcal{M}_\beta$ *agrees with* $\mathcal{M}_\gamma$ *below* λ_β, *and*

(3) $\beta < \gamma < \alpha \Rightarrow \lambda_\beta < \lambda_\gamma$.

We have added the qualifier "simple" in 6.5 because we shall introduce a more general kind of phalanx in §9. Since we shall consider only simple phalanxes in this section, we shall drop the "simple" when referring to them.

If $\mathcal{B} = (\langle \mathcal{M}_\beta \mid \beta \leq \alpha \rangle, \langle \lambda_\beta \mid \beta < \alpha \rangle)$ is a phalanx, then we set $lh\ \mathcal{B} = \alpha + 1$, $\mathcal{M}_\beta^{\mathcal{B}} = \mathcal{M}_\beta$ for $\beta \leq \alpha$, and $\lambda(\beta, \mathcal{B}) = \lambda_\beta$ for $\beta < \alpha$.

A phalanx of length 1 is just a premouse. Iteration trees on phalanxes are the obvious generalization of iteration trees on premice; the main point is that we use $\lambda(\beta, \mathcal{B})$ to tell us when to apply an extender to $\mathcal{M}^{\mathcal{B}}_{\beta}$, just as we used $\nu(E^{\mathcal{T}}_{\beta})$ in the special case of a tree on a premouse. We shall have $\beta T \gamma$ for $\beta < \gamma < lh\ \mathcal{B}$, but this is only a notational convenience, and it would be more natural to think of a tree with $lh\ \mathcal{B}$ many roots. Since we only need normal, ω-maximal trees, we shall only define these.

Definition 6.6. *Let $\mathcal{B}$ be a phalanx of length $\alpha + 1$, and $\theta > \alpha + 1$. An (ω-maximal, normal) iteration tree of length θ on $\mathcal{B}$ is a system $\mathcal{T} = \langle E_\beta \mid \alpha+1 \leq \beta + 1 < \theta \rangle$ with associated tree order T, models $\mathcal{M}_\beta$ for $\beta < \theta$, and $D \subseteq \theta$ and embeddings $i_{\eta\beta} : \mathcal{M}_\eta \to \mathcal{M}_\beta$ defined for $\eta T \beta$ with $(\alpha \cup D) \cap (\eta, \beta]_T = \emptyset$, such that*

(1) $\mathcal{M}_\beta = \mathcal{M}^{\mathcal{B}}_\beta$ for all $\beta \leq \alpha$, and for β, $\gamma \leq \alpha$, $\beta T \gamma$ iff $\beta < \gamma$;

(2) $\forall \beta < \alpha (\lambda(\beta, \mathcal{B}) < lh\ E_\alpha)$, and for $\alpha + 1 \leq \beta + 1 < \gamma + 1 < \theta$, $lh\ E_\beta < lh\ E_\gamma$;

(3) for $\alpha + 1 \leq \beta + 1 < \theta$: T-pred $(\beta + 1)$ is the least ordinal γ such that $\gamma < \alpha$ and $crit(E_\beta) < \lambda(\gamma, \mathcal{B})$, or $\alpha \leq \gamma$ and $crit(E_\beta) < \nu(E_\alpha)$. Moreover, letting $\gamma = T$-pred $(\beta + 1)$ and $\kappa = crit(E_\beta)$,

$$\mathcal{M}_{\beta+1} = Ult_k(\mathcal{M}^*_\gamma, E^{\mathcal{T}}_\beta),$$

where $\mathcal{M}^*_\gamma$ is the longest initial segment of $\mathcal{M}_\gamma$ containing only subsets of κ measured by E_β, and k is largest such that $\kappa < \rho_k(\mathcal{M}^*_\gamma)$. Also, $\beta + 1 \in D$ iff $\mathcal{M}_\gamma \neq \mathcal{M}^*_\gamma$, and if $\beta + 1 \notin D$ then $i_{\gamma,\beta+1}$ is the canonical embedding from $\mathcal{M}_\gamma$ into $Ult_k(\mathcal{M}_\gamma, E_\beta)$, and $i_{\eta,\beta+1} = i_{\gamma,\beta+1} \circ i_{\eta\gamma}$ for $\eta T \gamma$ such that $D \cap (\eta, \gamma]_T = \emptyset$;

(4) if $\alpha < \beta < \theta$ and β is a limit, then $D \cap [0, \beta)_T$ is finite, $[0, \beta)_T$ is cofinal in β, and $\mathcal{M}_\beta$ is the direct limit of the $\mathcal{M}_\gamma$ for $\gamma \in [0, \beta)_T$ such that $\gamma \geq \alpha \cup \sup(D)$. Moreover, $i_{\gamma\beta} : \mathcal{M}_\gamma \to \mathcal{M}_\beta$ is the direct limit map for all $\gamma \geq \alpha \cup \sup(D)$.

In the situation of 6.6, we set $\theta = lh\ \mathcal{T}$, $\mathcal{M}_\beta = \mathcal{M}^{\mathcal{T}}_\beta$, $E_\beta = E^{\mathcal{T}}_\beta$, and so forth. For $\beta < \theta$, we let $\mathrm{root}^{\mathcal{T}}(\beta)$ be the largest $\gamma < lh\ \mathcal{B}$ such that $\gamma T \beta$.

If $\mathcal{B}$ is a phalanx, then $\mathcal{G}^*(\mathcal{B}, \theta)$ is the obvious generalization of the length θ normal iteration game on premice: I and II build an iteration tree on $\mathcal{B}$, with I extending the tree at successor steps and II at limit steps. If at some move $\alpha < \theta$, I produces an illfounded ultrapower or II does not play a cofinal wellfounded branch, then I wins, and otherwise II wins. A winning strategy for II in $\mathcal{G}^*(\mathcal{B}, \theta)$ is a *θ-iteration strategy for $\mathcal{B}$*, and $\mathcal{B}$ is *θ-iterable* just in case there is such a strategy.

We wish to state an iterability theorem for phalanxes which are generated from iterates of K^c.

Definition 6.7. *Let $\mathcal{R}$ be a proper premouse and Σ an $(\omega, \Omega + 1)$ iteration strategy for $\mathcal{R}$. We say that a phalanx $\mathcal{B}$ is $(\Sigma, \mathcal{R})$-generated iff for* *all $\beta < lh\ \mathcal{B}$, there is an almost normal iteration tree $\mathcal{T}$ on $\mathcal{R}$ which is a play according to Σ such that $\mathcal{M}_\beta \trianglelefteq \mathcal{P}$, where $\mathcal{P}$ is the last model of $\mathcal{T}$, and such that (i) if $\beta + 1 < lh\ \mathcal{B}$, then $\lambda(\beta, \mathcal{B})$ is a cardinal of $\mathcal{R}$ and $\forall\gamma\ (\gamma + 1 < lh\ \mathcal{T} \Rightarrow \nu(E_\gamma^{\mathcal{T}}) \geq \lambda(\beta, \mathcal{B}))$, and (ii) if $\beta + 1 = lh\ \mathcal{B}$, then $\forall\gamma\ \forall\alpha < \beta(\gamma + 1 < lh\ \mathcal{T} \Rightarrow \nu(E_\gamma^{\mathcal{T}}) \geq \lambda(\alpha, \mathcal{B}))$.*

Recall that if K^c has no Woodin cardinals, then there is a unique $(\omega, \Omega+1)$ iteration strategy for K^c (namely, choosing the unique cofinal wellfounded branch).

Definition 6.8. *Suppose $K^c \models$ "There are no Woodin cardinals"; then a phalanx $\mathcal{B}$ is K^c-generated iff $\mathcal{B}$ is (Σ, K^c) generated, where Σ is the unique $(\omega, \Omega + 1)$ iteration strategy for K^c.*

Our iterability proof for K^c in §9 will actually show:

Theorem 6.9. *Suppose $K^c \models$ "There are no Woodin cardinals"; then every K^c-generated phalanx $\mathcal{B}$ such that $lh\ \mathcal{B} < \Omega$ is $\Omega + 1$-iterable.*

Proof. Deferred to §9. □

We shall actually only characterize α strength inductively in the case α is a cardinal of K. In this case we have the following little lemma.

Lemma 6.10. *Suppose $K^c \models$"There are no Woodin cardinals", and let α be a cardinal of K. Suppose $\alpha < OR^{\mathcal{M}}$, and $\mathcal{M}$ is α strong. Then α is a cardinal of $\mathcal{M}$.*

Proof. There is a weasel W which witnesses that $\mathcal{J}_\alpha^W = \mathcal{J}_\alpha^K$ is S-sound, and an elementary $\pi : K \to W$ with $\text{crit}(\pi) \geq \alpha$. Since α is a cardinal of K, α is a cardinal of W. But then α is a cardinal of $\mathcal{P}$, whenever $\mathcal{P}$ is an initial segment of a model on an iteration tree $\mathcal{T}$ on W such that $lh(E_\gamma^{\mathcal{T}}) \geq \alpha$ for all $\gamma + 1 < lh\ \mathcal{T}$. We have $\sigma : \mathcal{M} \to \mathcal{P}$ with $\text{crit}(\sigma) \geq \alpha$, for some such $\mathcal{P}$, and this implies that α is a cardinal of $\mathcal{M}$. □

We can now prove the main result of this section.

Theorem 6.11. *Suppose K^c has no Woodin cardinals. Let $\mathcal{M}$ be a proper premouse, and let $\alpha < OR^{\mathcal{M}}$ be such that α is a cardinal of K and $\mathcal{J}_\alpha^{\mathcal{M}} = \mathcal{J}_\alpha^K$; then the following are equivalent:*

(1) $\mathcal{M}$ is α strong,

(2) if $((\mathcal{N}, \mathcal{M}), \langle\alpha\rangle)$ is a phalanx such that $\mathcal{N}$ is β strong for all K-cardinals $\beta < \alpha$, then $((\mathcal{N}, \mathcal{M}), \langle\alpha\rangle)$ is $\Omega + 1$ iterable.

Proof. We show first (2)⇒(1). Let W witness that $\mathcal{J}_\alpha^{\mathcal{M}}$ is S-sound, and let Σ be an $\Omega + 1$ iteration strategy for W. By 6.2, W is β strong for all $\beta < \alpha$, and so our hypothesis (2) gives us an $\Omega + 1$ iteration strategy Γ for the phalanx $((W, \mathcal{M}), \langle\alpha\rangle)$. We now compare $\mathcal{M}$ with W, using Σ to form an iteration tree $\mathcal{T}$ on W and Γ to form an iteration tree $\mathcal{U}$ on $((W, \mathcal{M}), \langle\alpha\rangle)$. The trees $\mathcal{T}$

and $\mathcal{U}$ are determined by iterating the least disagreement, starting from $\mathcal{M}$ vs. W, as well as by the rules for iteration trees and the iteration strategies. (See 8.1 of [FSIT] for an example of such a coiteration.)

Let $lh\ \mathcal{U} = \theta + 1$ and $lh\ \mathcal{T} = \gamma + 1$. We claim that $\text{root}^{\mathcal{U}}(\theta) = 1$. For otherwise $\text{root}^{\mathcal{U}}(\theta) = 0$, and the universality of W implies that $\mathcal{M}_\theta^{\mathcal{U}} = \mathcal{M}_\gamma^{\mathcal{T}}$, and that $i_{0\theta}^{\mathcal{U}}$ and $i_{0\gamma}^{\mathcal{T}}$ exist. Moreover, the rules for $\mathcal{U}$ guarantee that $\text{crit}(i_{0\theta}^{\mathcal{U}}) < \alpha$. Since W has the S-hull and S-definability properties at all $\beta < \alpha$, we then get the usual contradiction involving the common fixed points of $i_{0\theta}^{\mathcal{U}}$ and $i_{0\gamma}^{\mathcal{T}}$.

Thus $\text{root}^{\mathcal{U}}(\theta) = 1$. Since W is universal, $i_{1,\theta}^{\mathcal{U}}$ exists, and maps $\mathcal{M}$ into some initial segment of $\mathcal{M}_\gamma^{\mathcal{T}}$. By the rules for $\mathcal{U}$, $\text{crit}(i_{1,\theta}^{\mathcal{U}}) \geq \alpha$. Thus $\mathcal{T}$ and $i_{1,\theta}^{\mathcal{U}}$ witness that $\mathcal{M}$ is (α, S) strong.

We now prove (1)⇒(2). Let us consider first the case α is a successor cardinal of K, say $\alpha = (\beta^+)^K = (\beta^+)^{\mathcal{M}}$ where β is a cardinal of K. Let $(\langle \mathcal{N}, \mathcal{M}\rangle, \alpha)$ be a phalanx such that $\mathcal{N}$ is β-strong. We shall show $(\langle \mathcal{N}, \mathcal{M}\rangle, \alpha)$ is $\Omega+1$ iterable by embedding it into a K^c-generated phalanx, and then using 6.9.

Note that $\mathcal{M}$ and $\mathcal{N}$ agree below α, and since $\mathcal{M}$ is α- strong, $\mathcal{J}_\alpha^{\mathcal{M}}$ is A_0-sound. Let W be a weasel which witnesses that $\mathcal{J}_\alpha^{\mathcal{M}}$ is A_0-sound. By Definition 6.1, there are (finite compositions of normal) iteration trees $\mathcal{T}_0$ and $\mathcal{T}_1$ on W, having last models $\mathcal{P}_0$ and $\mathcal{P}_1$ respectively, such that $\forall\gamma[(\gamma + 1 < lh\ \mathcal{T}_0 \Rightarrow \nu(E_\gamma^{\mathcal{T}_0}) \geq \beta)$ and $(\gamma + 1 < lh\ \mathcal{T}_0 \Rightarrow \nu(E_\gamma^{\mathcal{T}_1}) \geq \alpha)]$, and there are fully elementary embeddings τ_0 and τ_1 such that

$$\tau_0 : \mathcal{N} \longrightarrow \mathcal{J}_{\eta_0}^{\mathcal{P}_0} \quad \text{and} \quad \tau_0 \restriction \beta = \text{identity}\,,$$

and

$$\tau_1 : \mathcal{M} \longrightarrow \mathcal{J}_{\eta_1}^{\mathcal{P}_1} \quad \text{and} \quad \tau_1 \restriction \alpha = \text{identity}\,.$$

The proof of 5.10 shows that we may assume our A_0-soundness witness W is chosen so that there is an elementary $\sigma : W \longrightarrow K^c$. Since α is a cardinal of K, we may also assume that α is a cardinal of W. Let $\sigma\mathcal{T}_0$ and $\sigma\mathcal{T}_1$ be the copied versions of $\mathcal{T}_0$ and $\mathcal{T}_1$ on K^c. Since W has no Woodin cardinals (because K^c has none), the trees $\mathcal{T}_0$ and $\mathcal{T}_1$ are simple. This implies that the copying construction does not break down, and that $\sigma\mathcal{T}_0$ and $\sigma\mathcal{T}_1$ are according to the unique $(\omega, \Omega + 1)$ iteration strategy for K^c. If E is an extender used in $\sigma\mathcal{T}_0$, then $\nu(E) \geq \sigma(\beta)$, and if E is used in $\sigma\mathcal{T}_1$, then $\nu(E) \geq \sigma(\alpha)$. Let

$$\psi_0 : \mathcal{P}_0 \longrightarrow Q_0 \quad \text{and} \quad \psi_1 : \mathcal{P}_1 \longrightarrow Q_1$$

be the copy maps, where Q_0 and Q_1 are the last models of $\sigma\mathcal{T}_0$ and $\sigma\mathcal{T}_1$ respectively. We have $\psi_0 \restriction \beta = \sigma \restriction \beta$ and $\psi_1 \restriction \alpha = \sigma \restriction \alpha$. Let, for $i \in \{0, 1\}$,

$$\mathcal{R}_i = \begin{cases} Q_i & \text{if} \quad \mathcal{P}_i = \mathcal{J}_{\eta_i}^{\mathcal{P}_i}\,, \\ \mathcal{J}_{\psi_i(\eta_i)}^{Q_i} & \text{otherwise}\,. \end{cases}$$

We claim that $(\langle \mathcal{R}_0, \mathcal{R}_1\rangle, \langle\sigma(\alpha)\rangle)$ is a K^c-generated phalanx, the trees by which it is generated being $\sigma\mathcal{T}_0$ and $\sigma\mathcal{T}_1$. For this, we must look more closely at the extenders used in $\mathcal{T}_0$. We claim that if E is used in $\mathcal{T}_0$, then $lh\ E > \alpha$. For if some E such that $lh\ E < \alpha$ is used in $\mathcal{T}_0$, then there is a $B \subseteq \beta$ such that $B \in \mathcal{J}_\alpha^W$ and $B \notin \mathcal{P}_0$. Since $\mathcal{M}, \mathcal{N}$, and W agree below α, $B \in \mathcal{N}$, so $\tau_0(B) \in \mathcal{P}_0$, so $\tau_0(B) \cap \beta = B \in \mathcal{P}_0$, a contradiction. Also, $lh\ E \neq \alpha$ for all E on the W sequence, since α is a cardinal of W. Thus $lh\ E > \alpha$ for all E used in $\mathcal{T}_0$. Since α is a cardinal of W, this means $\nu(E) \geq \alpha$ for all E used in $\mathcal{T}_0$. That implies that $\nu(E) \geq \sigma(\alpha)$ for all E used in $\sigma\mathcal{T}_0$. The remaining clauses in the definition of "K^c-generated phalanx" hold obviously of $(\langle \mathcal{R}_0, \mathcal{R}_1\rangle, \langle \sigma(\alpha)\rangle)$.

By 6.9 we have an $\Omega + 1$ iteration strategy Σ for $(\langle \mathcal{R}_0, \mathcal{R}_1\rangle, \langle \sigma(\alpha)\rangle)$. We can use Σ and a simple copying construction to get an $\Omega+1$ iteration strategy Γ for $(\langle \mathcal{N}, \mathcal{M}\rangle, \langle \alpha\rangle)$. We shall describe this construction now; it involves a small wrinkle on the usual copying procedure, and it shows why it is necessary that $\mathcal{M}$ be α-strong, and not just β-strong.

Our strategy Γ is to insure that if $\mathcal{T}$ is the iteration tree on $(\langle \mathcal{N}, \mathcal{M}\rangle, \langle \alpha\rangle)$ representing the current position in $\mathcal{G}^*((\langle \mathcal{N}, \mathcal{M}\rangle, \langle \alpha\rangle), \Omega+1)$, then as we built $\mathcal{T}$ we constructed an iteration tree $\mathcal{U}$ on $(\langle \mathcal{R}_0, \mathcal{R}_1\rangle, \langle \sigma(\alpha)\rangle)$ such that $\mathcal{U}$ is a play by Σ and has the same tree order as $\mathcal{T}$, together with embeddings

$$\pi_\gamma : \mathcal{M}_\gamma^{\mathcal{T}} \to \mathcal{M}_\gamma^{\mathcal{U}}$$

defined for all $\gamma < lh\ \mathcal{T}$, satisfying:

(a) for $\eta < \gamma < lh\ \mathcal{T}$, $\pi_\eta \restriction \nu_\eta = \pi_\gamma \restriction \nu_\eta$, where

$$\nu_\eta = \begin{cases} \beta & \text{if } \eta = 0, \\ \nu(E_\eta^{\mathcal{T}}) & \text{if } \eta > 0 \text{ and } E_\eta^{\mathcal{T}} \text{ is} \\ & \text{of type III}, \\ lh(E_\eta^{\mathcal{T}}) & \text{otherwise;} \end{cases}$$

moreover, $E_\eta^{\mathcal{U}} = \pi_\eta(E_\eta^{\mathcal{T}})$;

(b) for all $\gamma < lh\ \mathcal{T}$ such that $\gamma \geq 2$, π_γ is a $(\deg^{\mathcal{T}}(\gamma), X)$ embedding, where $X = (i_{\eta\gamma}^{\mathcal{T}} \circ i_\eta^*)''(\mathcal{M}_\eta^*)^{\mathcal{T}}$, for η the least ordinal such that $i_{\eta\gamma}^{\mathcal{T}} \circ i_\eta^*$ exists; for $\gamma \in \{0, 1\}$, π_γ is fully elementary;

(c) for $\eta < \gamma < lh\ \mathcal{T}$, if $i_{\eta\gamma}^{\mathcal{T}}$ exists, then $i_{\eta\gamma}^{\mathcal{U}}$ exists and $\pi_\gamma \circ i_{\eta\gamma}^{\mathcal{T}} = i_{\eta\gamma}^{\mathcal{U}} \circ \pi_\eta$.

These are just the usual copying conditions, except that the agreement-of-embeddings ordinal ν_0 is β, rather than α.

We have $\mathcal{M}_0^{\mathcal{T}} = \mathcal{N}$, $\mathcal{M}_1^{\mathcal{T}} = \mathcal{M}$, $\mathcal{M}_0^{\mathcal{U}} = \mathcal{R}_0$, and $\mathcal{M}_1^{\mathcal{U}} = \mathcal{R}_1$ to begin with, and we set

$$\pi_0 = \psi_0 \circ \tau_0 \text{ and } \pi_1 = \psi_1 \circ \tau_1 .$$

Since $\pi_0 \restriction \beta = \pi_1 \restriction \beta$ and π_0, π_1 are fully elementary, our induction hypotheses (a) - (d) hold.

[To see π_0 and π_1 are fully elementary, notice that $\mathcal{M}$ and $\mathcal{N}$ satisfy ZF-Powerset, and τ_0 and τ_1 are fully elementary according to 6.1. If $\mathcal{J}_{\eta_i}^{\mathcal{P}_i} = \mathcal{P}_i$, this means $\mathcal{P}_i \models$ ZF-Powerset, so $\deg^{\mathcal{T}_i}(\xi_i) = \omega$, where $\mathcal{P}_i = \mathcal{M}_{\xi_i}^{\mathcal{T}_i}$, and thus ψ_i is fully elementary ($i \in \{0, 1\}$). On the other hand, if $\mathcal{J}_{\eta_i}^{\mathcal{P}_i}$ is a proper initial segment of $\mathcal{P}_i$, then $\psi_i \restriction \mathcal{J}^{\mathcal{P}_i}_{\eta_i}$ is obviously fully elementary. So in any case π_0 and π_1 are fully elementary.]

Now suppose we are at a limit step λ in the construction of $\mathcal{T}$ and $\mathcal{U}$. Σ chooses a cofinal wellfounded branch b of $\mathcal{U} \restriction \lambda$, and we let Γ choose b as its cofinal wellfounded branch of $\mathcal{T} \restriction \lambda$. It is cofinal because $\mathcal{T}$ and $\mathcal{U}$ have the same tree order, and wellfounded because we have an embedding $\pi : \mathcal{M}^{\mathcal{T}}_b \rightarrow \mathcal{M}^{\mathcal{U}}_b$ given by

$$\pi(i^{\mathcal{T}}_{\gamma b}(x)) = i^{\mathcal{U}}_{\gamma b}(\pi_\gamma(x))$$

defined for $\gamma \in b$ sufficiently large. Setting $\pi_\lambda = \pi$, we can easily check (a) - (d).

Now suppose we are at step $\eta + 1$ in the construction of $\mathcal{T}$ and $\mathcal{U}$. Player I in $\mathcal{G}^*(\langle \mathcal{N}, \mathcal{M}\rangle, \langle\alpha\rangle)$ has just played $E^{\mathcal{T}}_\eta$, and thereby determined $\mathcal{T} \restriction \eta + 2$. We must determine $\mathcal{U} \restriction \eta + 2$ together with $\pi_{\eta+1}$. In the case that T-pred$(\eta + 1) \neq 0$, we can simply quote the shift lemma, Lemma 5.2 of [FSIT], and obtain the desired $\mathcal{M}^{\mathcal{U}}_{\eta+1}$ and $\pi_{\eta+1}$. We omit further detail, and go on to the case T-pred$(\eta + 1) = 0$. [Unfortunately, the agreement-of-embeddings hypothesis for the copying construction was mis-stated in [FSIT], because squashed ultrapowers were overlooked. We only get $\pi_\eta \restriction \nu(E^{\mathcal{T}}_\eta) = \pi_\gamma \restriction \nu(E^{\mathcal{T}}_\eta)$, for $\eta < \gamma$, in the case $E^{\mathcal{T}}_\eta$ is type III, rather than $\pi_\eta \restriction (lh(E^{\mathcal{T}}_\eta) + 1) = \pi_\gamma \restriction (lh\ E^{\mathcal{T}}_\eta + 1)$ as claimed in [FSIT] (after 5.2, in the definition of $\pi\mathcal{T}$). This weaker agreement causes no new problems, however.]

Let $\kappa = \text{crit}(E^{\mathcal{T}}_\eta)$, so that $\kappa < \alpha$ and hence $\kappa \leq \beta$. To simply quote the Shift lemma we would need that $\pi_0 \restriction (\kappa^+)^{\mathcal{M}^{\mathcal{T}}_\eta} = \pi_\eta \restriction (\kappa^+)^{\mathcal{M}^{\mathcal{T}}_\eta}$, and that is more than we know. Still, the proof of the Shift lemma works: set

$$\mathcal{M}^{\mathcal{U}}_{\eta+1} = \text{Ult}_\omega(\mathcal{R}_0, \pi_\eta(E^{\mathcal{T}}_\eta)).$$

From 6.6 (2), we get $\nu_1 \geq \alpha$, and our agreement hypothesis (a) then gives $\pi_1 \restriction \alpha = \pi_\eta \restriction \alpha$. Thus $\pi_\eta(\kappa) = \pi_1(\kappa) < \pi_1(\alpha)$. Also, $\pi_1(\alpha) = \sigma(\alpha)$. (Since $\tau_1 \restriction \alpha =$ identity and $\psi_1 \restriction \alpha = \sigma \restriction \alpha$, $\pi_1(\beta) = \sigma(\beta)$. But $\pi_1(\alpha)$ is the $\mathcal{R}_1$-successor cardinal of $\pi_1(\beta)$, and $\sigma(\alpha)$ is the K^c-successor cardinal of $\sigma(\beta)$, and since all extenders used in $\sigma\mathcal{T}_1$ have length $> \sigma(\alpha)$, these are the same.) Since $\mathcal{M}^{\mathcal{U}}_\eta$ agrees with $\mathcal{R}_1$, and hence $\mathcal{R}_0$, through $\sigma(\alpha) = \pi_1(\alpha)$, $\mathcal{M}^{\mathcal{U}}_\eta$ and $\mathcal{R}_0$ have the same subsets of $\pi_\eta(\kappa)$, and the ultrapower defining $\mathcal{M}^{\mathcal{U}}_{\eta+1}$ makes sense.

We can now define $\pi_{\eta+1} : \mathcal{M}^{\mathcal{T}}_{\eta+1} \rightarrow \mathcal{M}^{\mathcal{U}}_{\eta+1}$ by:

$$\pi_{\eta+1}([a, f]^{\mathcal{T}}_{E^{\mathcal{T}}_\eta}) = [\pi_\eta(a), \pi_0(f) \restriction [\pi_\eta(\kappa)]^{|a|}]^{\mathcal{R}_0}_{\pi_\eta(E^{\mathcal{T}}_\eta)}.$$

The shift lemma argument shows that $\pi_{\eta+1}$ is well defined, fully elementary, and has the desired agreement with π_η. To see this, recall that $\nu(E) \geq \alpha$ for all E used in $\sigma\mathcal{T}_0$. This implies that $\psi_0 \restriction \alpha = \sigma \restriction \alpha$, and thus ψ_0, ψ_1, π_1, and π_η all agree with σ on α. Now $\kappa \leq \beta$, and for any $A \subseteq \beta$ in $\mathcal{N}$,

$$\begin{aligned}\pi_0(A) \cap \pi_\eta(\beta) &= \pi_0(A) \cap \psi_0(\beta) \\ &= \psi_0(\tau_0(A)) \cap \psi_0(\beta) \\ &= \psi_0(\tau_0(A) \cap \beta) \\ &= \psi_0(A) \\ &= \pi_\eta(A)\,.\end{aligned}$$

Thus, for example, if $f = g$ on $A \subseteq [\kappa]^{|a|}$ with $A \in (E_\eta^{\mathcal{T}})_a$, then $\pi_0(f) = \pi_0(g)$ on $\pi_0(A)$, and hence $\pi_0(f) = \pi_0(g)$ on $\pi_0(A) \cap [\pi_\eta(\kappa)]^{|a|}$. But then $\pi_0(f) = \pi_0(g)$ on $\pi_\eta(A)$, and $\pi_\eta(A) \in (\pi_\eta(E_\eta^{\mathcal{T}}))_{\pi_\eta(a)}$. This shows that $\pi_{\eta+1}$ is well defined, and the other conditions on it can also be checked easily.

This completes the proof of (1)⇒(2) in the case that α is a successor cardinal of K. It is worth noting that we really used that $\mathcal{M}$ was α-strong, and not just β-strong. This guaranteed $\tau_1 \restriction \alpha = \mathrm{id}$, and thus $\pi_1 \restriction \alpha = \sigma \restriction \alpha$. That in turn gave $\pi_\eta \restriction \alpha = \psi_0 \restriction \alpha$, which was crucial. It is not true that if $\mathcal{M}$ is β-strong, where β is a cardinal of K, and $\mathcal{J}_\alpha^{\mathcal{M}} = \mathcal{J}_\alpha^K$ for $\alpha = (\beta^+)^K$, then $\mathcal{M}$ is α-strong.

The case α is a limit cardinal of K is similar. Let $\mathcal{N}$ be β-strong for all K-cardinals $\beta < \alpha$, and $\mathcal{J}_\alpha^{\mathcal{N}} = \mathcal{J}_\alpha^{\mathcal{M}}$. Let W witness that $\mathcal{J}_\alpha^{\mathcal{M}} = \mathcal{J}_\alpha^K$ is A_0-sound, and let $\sigma : W \to K^c$. For each K-cardinal $\beta \leq \alpha$ let $\mathcal{T}_\beta$ be an iteration tree on W with last model $\mathcal{P}_\beta$, and let $\tau_\beta : \mathcal{N} \to \mathcal{J}_{\eta_\beta}^{\mathcal{P}_\beta}$ with $\tau_\beta \restriction \beta = \mathrm{id}$ for $\beta < \alpha$. Let $\tau_\alpha : \mathcal{M} \to \mathcal{J}_{\eta_\alpha}^{\mathcal{P}_\alpha}$ with $\tau_\alpha \restriction \alpha = \mathrm{id}$. For $\beta \leq \alpha$, let $\sigma\mathcal{T}_\beta$ be the copied tree on K^c, Q_β its last model, $\psi_\beta : \mathcal{P}_\beta \to Q_\beta$ the copy map, and $\mathcal{R}_\beta = \mathcal{J}_{\psi_\beta(\eta_\beta)}^{Q_\beta}$ or $\mathcal{R}_\beta = Q_\beta$ as appropriate. Then $(\langle \mathcal{R}_\beta \mid \beta \leq \alpha \wedge \beta \text{ a cardinal of } K\rangle, \langle \sigma(\beta) \mid \beta < \alpha \wedge \beta \text{ a cardinal of } K\rangle)$ is a K^c-generated phalanx, and therefore $\Omega + 1$ iterable. But then we can win the iteration game $\mathcal{G}^*((\langle \mathcal{N}, \mathcal{M}\rangle, \langle \alpha\rangle), \Omega + 1)$ just as before; letting $\pi_\beta : \mathcal{N} \to \mathcal{T}_\beta$ be given by $\pi_\beta = \psi_\beta \circ \tau_\beta$, for $\beta \leq \alpha$, and defining the remaining π's inductively, we copy the evolving $\mathcal{T}$ on $(\langle \mathcal{N}, \mathcal{M}\rangle, \langle \alpha\rangle)$ by applying $\pi_\eta(E_\eta^{\mathcal{T}})$ to the model required by the rules for trees on $(\langle \mathcal{R}_\beta \mid \beta \leq \alpha \wedge \beta \text{ a } K\text{-cardinal}\rangle, \langle \sigma(\beta) \mid \beta < \alpha \wedge \beta \text{ a } K\text{-cardinal}\rangle)$. Since for $\beta \leq \alpha$, $\pi_\beta \restriction \beta = \psi_\beta \restriction \beta = \sigma \restriction \beta$, we have enough agreement to simply quote the shift lemma. Although $\mathcal{T}$ and its copy $\mathcal{U}$ have slightly different tree orders, this causes no problems.

This completes the proof of 6.11. □

To see that 6.11 gives on inductive definition of K, assuming K^c has no Woodin cardinals, suppose that α is a cardinal of K and we know which premice are α-strong. Then

$$\exists \beta < (\alpha^+)^K (\mathcal{P} = \mathcal{J}_\beta^K) \Leftrightarrow \exists \mathcal{M}(\mathcal{M} \text{ is } \alpha\text{-strong } \wedge \exists \beta < (\alpha^+)^{\mathcal{M}}(\mathcal{P} = \mathcal{J}_\beta^{\mathcal{M}}))\,.$$

(We get ⇒ from 6.2. We get ⇐ easily from the definition of "α-strong".)

We can determine $(\alpha^+)^K$ and $\mathcal{J}_{(\alpha^+)^K}^K$ using this equivalence. Using 6.11, we can then determine which premice are $(\alpha^+)^K$-strong. The limit steps in the inductive definition of "α is a cardinal of K" and "$\mathcal{M}$ is α-strong" are trivial modulo 6.11.

This definition still involves quantification over $V_{\Omega+1}$. In order to avoid that, we must show that if $\mathcal{M}$ is of size α, and 6.11 (2) fails, then there is an $\mathcal{N}$ of size α and an iteration tree $\mathcal{T}$ of size α on $(\langle \mathcal{N}, \mathcal{M}\rangle, \langle \alpha \rangle)$ witnessing the failure of iterability. (We shall actually get a countable $\mathcal{T}$.) This is a reflection result much like lemma 2.4.

Definition 6.12. *A premouse $\mathcal{M}$ is properly small iff $\mathcal{M} \models$ "There are no Woodin cardinals $\wedge$ there is a largest cardinal". A phalanx $\mathcal{B}$ is properly small iff $\forall \alpha < lh(\mathcal{B})$ ($\mathcal{M}_\alpha^{\mathcal{B}}$ is properly small).*

The uniqueness results of §6 of [FSIT] easily yield the following.

Lemma 6.13. *Let $\mathcal{B}$ be a properly small phalanx, and let $\mathcal{T}$ be an iteration tree on $\mathcal{B}$; then $\mathcal{T}$ is simple.*

Proof (Sketch). Suppose b and c are distinct branches of $\mathcal{T}$ with $\sup(b) = \lambda = \sup(c)$, b and c existing in some generic extension of V. If b and c do not drop, then $\delta(\mathcal{T} \restriction \lambda) < \mathrm{OR}^{\mathcal{M}_b^{\mathcal{T}}}$ and $\delta(\mathcal{T} \restriction \lambda) < \mathrm{OR}^{\mathcal{M}_c^{\mathcal{T}}}$ because $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_c^{\mathcal{T}}$ have a largest cardinal. (This is why we included this condition.) From 6.1 of [FSIT] we get that $\delta(\mathcal{T} \restriction \lambda)$ is Woodin in $\mathcal{M}_b^{\mathcal{T}}$ if $\mathrm{OR}^{\mathcal{M}_b^{\mathcal{T}}} \leq \mathrm{OR}^{\mathcal{M}_c^{\mathcal{T}}}$, and Woodin in $\mathcal{M}_c^{\mathcal{T}}$ otherwise. This contradicts the proper smallness of the premice in $\mathcal{B}$. If one of b and c drops, then we can argue to a contradiction as in the proof of 6.2 of [FSIT]. □

We thank Kai Hauser for pointing out that our original version of 6.13 was false. (We had omitted having a largest cardinal from the definition of properly small.)

By 6.13, a properly small phalanx can have at most one $\Omega + 1$ iteration strategy, that strategy being to choose the unique cofinal wellfounded branch.

Lemma 6.14. *Suppose K^c has no Woodin cardinals, and that α is a cardinal of K. Let $\mathcal{M}$ be a properly small premouse of cardinality α such that $\mathcal{J}_\alpha^{\mathcal{M}} = \mathcal{J}_\alpha^K$ but $\mathcal{M}$ is not α-strong. Then there is a properly small premouse $\mathcal{N}$ of cardinality α such that $\mathcal{J}_\alpha^{\mathcal{N}} = \mathcal{J}_\alpha^K$ and $\mathcal{N}$ α-strong, and a countable putative iteration tree $\mathcal{T}$ on $(\langle \mathcal{N}, \mathcal{M}\rangle, \langle \alpha \rangle)$ such that either $\mathcal{T}$ has a last, illfounded model, or $\mathcal{T}$ has limit length but no cofinal wellfounded branch.*

Proof. Let W be a weasel which witnesses that $\mathcal{J}_\alpha^K$ is A_0-sound. By 6.2, W is α-strong. From the proof of (2)⇒(1) in 6.11, we have that $(\langle W, \mathcal{M}\rangle, \langle \alpha \rangle)$ is not $\Omega + 1$ iterable. It follows that there is a putative iteration tree $\mathcal{U}$ of length $\leq \Omega$ on $(\langle W, \mathcal{M}\rangle, \langle \alpha \rangle)$ which is *bad*; i.e., has a last, illfounded model or is of limit length and has no cofinal wellfounded branch.

Since Ω is weakly compact, $lh\ \mathcal{U} < \Omega$. This means that for all sufficiently large successor cardinals μ of W, we can associate to $\mathcal{U}$ a tree $\mathcal{U}_\mu$ on $(\langle \mathcal{J}_\mu^W, \mathcal{M}\rangle, \langle \alpha \rangle)$. $\mathcal{U}_\mu$ has the same tree order and uses the same extenders as

$\mathcal{U}$; the models on $\mathcal{U}_\mu$ are initial segments of the models on $\mathcal{U}$. We claim that there is a μ such that $\mathcal{U}_\mu$ is bad. If $\mathcal{U}$ has successor length this is obvious, as the last model of $\mathcal{U}$ is the union over μ of the last models of the $\mathcal{U}_\mu$. Suppose $\mathcal{U}$ has limit length, and b_μ is a cofinal wellfounded branch of $\mathcal{U}_\mu$, for all $\mu < \Omega$ such that μ is a successor cardinal of W. Notice that if $\mu < \eta$, then b_η is a cofinal wellfounded branch of $\mathcal{U}_\mu$, and thus by 6.13, $b_\eta = b_\mu$. Letting b be the common value of b_μ for all appropriate μ, we then have that b is a cofinal wellfounded branch of $\mathcal{U}$, a contradiction.

Let $\mathcal{V} = \mathcal{U}_\mu$ and $\mathcal{P} = \mathcal{J}_\mu^W$, where μ is a successor cardinal of W large enough that $\mathcal{V}$ is a bad tree on $(\langle \mathcal{P}, \mathcal{M}\rangle, \langle\alpha\rangle)$. Note that $\mathcal{P}$ is α-strong, and $(\langle \mathcal{P}, \mathcal{M}\rangle, \langle\alpha\rangle)$ is properly small. Let $X \prec V_\eta$, for some η, with $\mathcal{V}, \mathcal{P}, \mathcal{M}, \alpha \in X$, and X countable. Let $\pi : R \cong X$ be the transitive collapse, and $\pi(\bar{\mathcal{V}}) = \mathcal{V}$, etc. Let $\lambda \in X \cap \Omega$ be such that $\mathcal{V}, \mathcal{P}, \mathcal{M}, \alpha \in V_\lambda$; then $V_\lambda^\sharp \in X$, and thus $R \models V_{\bar\lambda}^\sharp$ exists. Because π embeds $(V_{\bar\lambda}^\sharp)^R$ into $V_\lambda^\sharp$, we have $(V_{\bar\lambda}^\sharp)^R = (V_{\bar\lambda}^R)^\sharp$, and so $R[x]$ is correct for Π_2^1 assertions about x, whenever x is an R-generic real coding $V_{\bar\lambda}^R$. But now R satisfies "$\bar{\mathcal{V}}$ is a bad tree on $(\langle \bar{\mathcal{P}}, \bar{\mathcal{M}}\rangle, \langle\bar\alpha\rangle)$", and because $\bar{\mathcal{V}}$ is simple by 6.13, $R[x]$ must satisfy the same. Thus $\bar{\mathcal{V}}$ is indeed a bad tree on $(\langle \bar{\mathcal{P}}, \bar{\mathcal{M}}\rangle, \langle\bar\alpha\rangle)$.

Now let $X \prec Y \prec V_\eta$, where $(\alpha+1) \cup \mathcal{M} \subseteq Y$ and $|Y| \leq \alpha$. Let $\sigma : S \cong Y$ be the transitive collapse, and $\psi : R \to S$ be such that $\pi = \sigma \circ \psi$. Notice that $\psi(\bar{\mathcal{M}}) = \mathcal{M}$ and $\psi(\bar\alpha) = \alpha$. Let $\mathcal{N} = \psi(\bar{\mathcal{P}})$. Using ψ we can copy $\bar{\mathcal{V}}$ as a tree $\psi\bar{\mathcal{V}}$ on $(\langle \mathcal{N}, \mathcal{M}\rangle, \alpha)$, noting that because $\bar{\mathcal{V}}$ is simple, $\psi\bar{\mathcal{V}}$ can never have a wellfounded maximal branch. It follows that $\psi\bar{\mathcal{V}}$ is a bad tree on $(\langle \mathcal{N}, \mathcal{M}\rangle, \langle\alpha\rangle)$. Since $\sigma : \mathcal{N} \to \mathcal{P}$ and $\sigma \restriction (\alpha+1) =$ identity, $\mathcal{N}$ is α-strong. This completes the proof of 6.14. □

Clearly, if α is a cardinal of K and $\beta < (\alpha^+)^K$, then there is a properly small, α-strong $\mathcal{M}$ such that $\mathcal{J}_\beta^\mathcal{M} = \mathcal{J}_\beta^K$ and $\beta < (\alpha^+)^\mathcal{M}$. So in our inductive definition of K we need only consider properly small mice. Thus 6.11 and 6.14 together yield:

Theorem 6.15. *Suppose K^c has no Woodin cardinals; then there are formulae $\psi(v_0, v_1)$, $\varphi(v_0, v_1)$ in the language of set theory such that whenever G is V-generic/ $\mathbb{P}$, where $\mathbb{P} \in V_\Omega$, then $V[G]$ satisfies the following sentences:*

(1) $\forall x, y \in {}^\omega\omega\, \forall\alpha < \omega_1\ [(L_{\alpha+1}(\mathbb{R}) \models \varphi[x,y]) \Leftrightarrow \exists\delta \leq \alpha$ (x codes $\delta \wedge y$ codes a δ-strong, properly small premouse)];

(2) $\forall x, y \in {}^\omega\omega\ \forall\alpha < \omega_1\ [(L_{\alpha+1}(\mathbb{R}) \models \psi[x,y]) \Leftrightarrow \exists\delta \leq \alpha$($x$ codes $\delta \wedge y$ codes $\mathcal{J}_\delta^K$)].