5. Related Lindström Extensions

In this chapter FP+C is shown to be more expressive than the natural extensions of fixed-point logic by cardinality Lindström quantifiers.

- Section 5.1 introduces a structural padding technique that is suitable for the proof of this separation result. More generally, this technique serves to expose weaknesses of quantifier extensions in the case that these quantifiers do not have the right scaling properties with respect to certain extensions of structures.

- This technique is applied in Section 5.2 to show that $\mathrm{FP}(\mathcal{Q}_{\mathrm{card}})$ cannot express all FP+C-definable boolean queries. The same applies to $\mathrm{FP}(\mathcal{Q}^{\sim}_{\mathrm{card}})$ with quantifiers for all cardinality properties based on the counting of equivalence classes. In fact the separation even establishes that not all of FP^* can be captured by these quantifier extensions.

- In Section 5.3 we apply the padding technique to derive corollaries concerning the weakness of two other quantifier classes. The classes of all properties of rigid structures and that of all properties of sparse structures, respectively, are shown to fall short of FP^* and in particular of PTIME.

In the previous chapter FP+C has been characterized as the natural extension of fixed-point logic that incorporates expressive means for dealing with cardinalities and corresponding arithmetic. Recall that a main feature of the formalization was the introduction of a second, arithmetical sort. This type of a *functorial extension* — based partly on the manipulation of the structures under consideration — is intuitively different from the established formalism for extensions in abstract model theory, namely that of Lindström extensions or extensions through generalized quantifiers. Can this difference in appearance be substantiated in more rigorous terms? There is some sense in which this cannot be achieved: it is a known fact that the Lindström approach to extensions of logics is sufficiently general to describe any reasonable extension of first-order logic, more precisely any extension with the appropriate closure properties. No doubt therefore FP+C is equivalent with a Lindström extension of first-order logic, and also with a Lindström extension of fixed-point logic. As FP+C is a logic with recursive syntax and semantics these Lindström extensions can trivially be chosen to use recursive families of quantifiers. That one is forced to consider extensions by infinite families of quantifiers follows with an argument of Dawar and Hella [DH94] that applies to show that FP+C cannot be equivalent with a finite Lindström extension (see Theorem 5.9 below). The standard modelling of a logic $\mathcal{L} \supseteq$ FP with the right closure properties as a Lindström extension essentially turns each individual $\mathcal{L}$-definable class into a quantifier. Clearly this is unsatisfactory: the resulting presentation of FP+C as a Lindström extension of FP is quite artificial. It is not at all clear, however, which kinds of Lindström extensions should be considered *natural*. Two different types of criteria come to mind.

Syntactic criteria. One may consider certain *uniform sequences* of quantifiers. These are meant to adjoin the same structural property in varying context. Uniform sequences as considered for instance in [Daw95a] consist of all powers of a given quantifier and capture one structural property across all arities, or as applied to interpreted structures in any power. Compare Section 1.6.2. The usual way in which the transitive closure operator is adjoined to first-order logic to get transitive closure logic provides a natural example. Transitive closures are made definable for binary relations interpreted in any power of the universe.

While FP+C cannot be a finite extension of FP it is conceivable that it is obtained as an extension by finitely many uniform sequences of quantifiers. Indeed, it follows from Dawar's work that a class or logic, that is recursively presented (in some sufficiently strong sense; compare remarks in connection with Definition 1.7) and has natural closure properties, is equivalent with an extension of FP and even of first-order logic by just a single uniform series of quantifiers. In the general construction the quantifier giving rise to such a sequence embodies an enumeration of all queries that are to be captured. In special cases, as for instance for FP itself one may also abstract such a quantifier from typical and natural problems that are complete under appropriate logical reductions, cf. [Dah87, Gro95]. Whether such natural problems exist for FP+C, relative either to FP or to first-order, remains open.

Semantic criteria. One may also impose purely semantic conditions on the quantifiers adjoined. The investigations of this chapter are of this kind. In connection with fixed-point with counting there is an obvious issue in this line:

> Can FP+C be obtained as an extension of FP by cardinality Lindström quantifiers, i.e. by quantifiers whose semantics is entirely defined in terms of cardinalities of predicates?

Indeed, FP with the class of all PTIME cardinality Lindström quantifiers is the *natural a priori candidate* to capture a counting extension of FP in the Lindström formalism. Compare Definitions 1.52 and 1.54 for (quotient) cardinality quantifiers.

The main point of this chapter is that even the extension of FP by all cardinality Lindström quantifiers does not comprise all of FP+C, in fact not even all of FP*: $\mathrm{FP}(\mathcal{Q}_{\mathrm{card}}) \not\supseteq \mathrm{FP}^*$. Admitting further all quantifiers that capture cardinality properties in quotient interpretations — for the counting of equivalence classes rather than tuples — does not help either, even $\mathrm{FP}(\mathcal{Q}^{\sim}_{\mathrm{card}}) \not\supseteq \mathrm{FP}^*$.

Theorem 5.1. $\mathrm{FP{+}C} \not\subseteq \mathrm{FP}(\mathcal{Q}_{\mathrm{card}})$. *In particular the extension of* FP *by all* PTIME *cardinality Lindström quantifiers is strictly weaker than* FP+C. *These separations also hold for* $\mathrm{FP}(\mathcal{Q}^{\sim}_{\mathrm{card}})$, *the extension by all quotient cardinality Lindström quantifiers.*

It follows with Lemma 1.55 that the extension $\mathrm{FP}(\mathcal{Q}_{\mathrm{mon}})$ of fixed-point logic by the class of all monadic Lindström quantifiers does not contain PTIME, and that similarly all quantifiers obtained from monadic ones through generalized interpretations cannot suffice. The latter extension is in fact equivalent with $\mathrm{FP}(\mathcal{Q}^{\sim}_{\mathrm{card}})$ by Remark 1.56. We mention in this context the work of Kolaitis and Väänänen [KVä95] on extensions of the $L^k_{\infty\omega}$ by monadic quantifiers that bind single formulae (simple monadic quantifiers). Using sophisticated combinatorial techniques they obtain interesting separation results within the realm of monadic quantifiers, for instance that the Härtig quantifier is not expressible in any extension of $L^{\omega}_{\infty\omega}$ by finitely many simple monadic quantifiers.

The present results are obtained with a technique that resembles so-called *padding arguments* in complexity theory. Intuitively the situation of Theorem 5.1 can be understood through the following. With FP+C the results of counting operations can be processed recursively, and this FP-recursion (over the arithmetical sort) is full PTIME recursion in terms of the size of the universe. The FP-recursion captured by any sentence in an extension of FP by $C^{\omega}_{\infty\omega}$-definable quantifiers, on the other hand, is polynomially bounded in the size of the quotient of the k-th power of the universe with respect to $\equiv^{C^k}$ for some k. The latter is the size of the ***relational part*** of I_{C^k}. (This situation is reminiscent of that exhibited by FP; there a gap between the size of $\mathfrak{A}$ and of $I_{L^k}(\mathfrak{A})$ accounts for the complexity behaviour described in the second theorem of Abiteboul and Vianu.)

In the case of cardinality Lindström quantifiers this gap can be manifested unconditionally to obtain the desired separation. The structures employed in these arguments are trivial extensions of ordered structures, with an increase in the size without any gain in internal relational structure, just as in padding arguments. FP with cardinality Lindström quantifiers is shown to have not the right *scaling properties* with respect to such extension.

5.1 A Structural Padding Technique

We consider functors that scale finite structures in size without otherwise adding structural complexity. Taking the disjoint sum with a pure set is a typical example. This operation increases the size but as for definable predicates, nothing is gained. We formalize this as follows. Consider a functor

$$\Gamma\colon \mathrm{fin}[\tau] \times \omega \longrightarrow \mathrm{fin}[\tau].$$

The second argument of Γ will serve as a scaling parameter for the desired extensions. The main example below is that of $\Gamma(\mathfrak{A}, n)$ being a trivial product of $\mathfrak{A}$ with the pure set $n = \{0, \ldots, n-1\}$. Assume further that for each r there is an encoding scheme that maps r-ary predicates R over $\Gamma(\mathfrak{A}, n)$ that are closed under automorphisms of $\Gamma(\mathfrak{A}, n)$ to tuples of predicates $[R]$ on $\mathfrak{A}$. We want to regard $[R]$ as an encoding or a *pull-back* for the values of global relations over the $\Gamma(\mathfrak{A}, n)$.

Definition 5.2. *A* good encoding scheme for Γ *is a mapping* $[\ ]$ *sending automorphism invariant* R *on* $\Gamma(\mathfrak{A}, n)$ *to tuples* $[R] = (R^1, \ldots, R^l)$ *on* $\mathfrak{A}$, *such that* $(\mathfrak{A}, [R])$ *and* n *determine* $(\Gamma(\mathfrak{A}, n), R)$ *up to isomorphism, and such that*

- *(i)* $[\ ]$ *is monotone:* $R_1 \subseteq R_2$ *implies* $R_1^i \subseteq R_2^i$ *for* $i = 1, \ldots l$.
- *(ii)* $[\ ]$ *is compatible with first-order definability in the following sense: if* R *is first-order definable from some global relations* $R_1, \ldots, R_k$ *over the* $\Gamma(\mathfrak{A}, n)$, *then the encoding relations* $[R]$ *for* R *are first-order definable over the* $\mathfrak{A}$ *from the encodings* $[R_i]$ *of the* R_i.

More precisely, (ii) means, that for first-order formula $\varphi(X_1, \ldots, X_k, \overline{x})$ there are first-order formulae φ_*^i such that for all sufficiently large n and for all $R_1, \ldots, R_k$ that are automorphism closed over $\Gamma(\mathfrak{A}, n)$:

$$[\varphi[\Gamma(\mathfrak{A}, n), R_1, \ldots, R_k]] = \left(\varphi_*^i[\mathfrak{A}, [R_1], \ldots, [R_k]]\right)_{i=1,\ldots,l}.$$

Note the uniformity with respect to n that is expressed in this notion.

We shall below need to extend the notion of good encodings to allow for parameters in the $\Gamma(\mathfrak{A}, n)$, see Definition 5.4.

Consider two examples: the disjoint sum and the trivial product with the pure set n.

$$(\mathfrak{A}, n) \longmapsto \mathfrak{A} \dot\cup n, \text{ and}$$
$$(\mathfrak{A}, n) \longmapsto \mathfrak{A} \otimes n.$$

- $\mathfrak{A} \dot\cup n$: if $\mathfrak{A} = (A, R_1^{\mathfrak{A}}, \ldots, R_s^{\mathfrak{A}})$ then $\mathfrak{A} \dot\cup n = (A \dot\cup n, R_1^{\mathfrak{A}}, \ldots, R_s^{\mathfrak{A}})$ is the disjoint union with the set n.
- $\mathfrak{A} \otimes n$: the universe of $\mathfrak{A} \otimes n$ is the product $A \times n$. Let $\pi_1\colon A \times n \to A$ and $\pi_2\colon A \times n \to n$ denote the natural projections to the factors as well as their extensions to higher powers as for instance in $\pi_1\colon (A \times n)^r \to A^r$. Then $\mathfrak{A} \otimes n = (A \times n, R_1^{\mathfrak{A}} \otimes n, \ldots, R_s^{\mathfrak{A}} \otimes n)$, where $R_i^{\mathfrak{A}} \otimes n = \pi_1^{-1}(R_i^{\mathfrak{A}})$.

Good encoding schemes are available for both functors. Consider the trivial product with n. Clearly a tuple $\overline{b} \in (A \times n)^r$ is described up to automorphisms of $\mathfrak{A} \otimes n$ by the pair $(\pi_1(\overline{b}), \mathrm{eq}(\pi_2(\overline{b})))$ consisting of its projection to

$\mathfrak{A}$ and the equality type of its projection to n. Let R be an r-ary predicate over $\mathfrak{A} \otimes n$ that is closed under automorphisms of $\mathfrak{A} \otimes n$. Then R is faithfully encoded by the tuple

$$[R] = (R^e)_{e \in \mathrm{Eq}(r)}$$
$$\text{where} \quad R^e = \{\pi_1(\bar{b}) \mid \bar{b} \in R, \mathrm{eq}(\pi_2(\bar{b})) = e\}.$$

Actually R is easily reconstructed from the R^e as

$$R = \{\bar{b} \mid \pi_1(\bar{b}) \in R^e \text{ for } e = \mathrm{eq}(\pi_2(\bar{b}))\}.$$

Monotonicity and compatibility with first-order transformations can be checked immediately. For instance, if $R = \{\bar{x} \mid \exists y R_1 \bar{x} y\}$, then R^e is the union over all sets $\{\bar{x} \mid \exists y (\bar{x} y \in R_1^{e'})\}$ where e' extends e to $r+1$ variables.

For trivial sums with n, a similar decomposition of predicates with respect to equality types of those parts of tuples that lie outside A would be a natural encoding. The universe A of $\mathfrak{A}$ is not definable as a subset of $A \dot{\cup} n$, however, so that the decomposition should be applied with respect to the parts lying outside the field of the $R_i^{\mathfrak{A}}$. We leave out the details, since in the explicit arguments of this chapter we choose to work with trivial products.

Since good encodings uniformly translate first-order manipulations on global relations to first-order manipulations on their encodings we have the following pull-back for fixed-point logic.

Lemma 5.3. *If there is a good encoding scheme for $\Gamma : \mathrm{fin}[\tau] \times \omega \to \mathrm{fin}[\tau]$, then* FP *over the $\Gamma(\mathfrak{A}, n)$ is captured by* FP *over the $\mathfrak{A}$ themselves. This means, in the case of boolean queries, that for any sentence $\varphi \in \mathrm{FP}[\tau]$ there is a sentence $\varphi_* \in \mathrm{FP}[\tau]$ such that for all sufficiently large n*

$$\Gamma(\mathfrak{A}, n) \models \varphi \iff \mathfrak{A} \models \varphi_*.$$

Proof. Inductively it suffices to show that also FP-applications can be simulated at the level of the encodings $[R]$. Consider the formula $\mathrm{FP}_{X,\bar{x}} \varphi(X, \bar{x})$ where we assume that $\bar{x}$ contains all free first-order variables of φ (compare Lemma 1.28). Suppose that $\varphi_*^1, \ldots, \varphi_*^l$ are such that for all automorphism invariant P over $\Gamma(\mathfrak{A}, n)$ (with sufficiently large n)

$$\big[\varphi[\Gamma(\mathfrak{A}, n), P]\big] = \big(\varphi_*^1[\mathfrak{A}, [P]], \ldots, \varphi_*^l[\mathfrak{A}, [P]]\big).$$

Then the encoding tuple $[\mathrm{FP}_{X,\bar{x}} \varphi(X, \bar{x})]$ for $\mathrm{FP}_{X,\bar{x}} \varphi(X, \bar{x})$ is obtained over $\mathfrak{A}$ as the simultaneous fixed point determined by the system $\varphi_*^1, \ldots, \varphi_*^l$ (when appropriately initialized to $[\emptyset]$). Compare Example 1.27 for fixed-point systems, and the proof of Lemma 2.22 about initialization. □

If Γ scales the size of the $\Gamma(\mathfrak{A}, n)$ with n then the lemma implies that the power of FP does not correctly scale with the size of the $\Gamma(\mathfrak{A}, n)$, since FP-recursion on $\Gamma(\mathfrak{A}, n)$ collapses to FP-recursion on $\mathfrak{A}$ in a manner independent of n. Our aim is to extend this phenomenon to quantifier extensions of FP.

Consider a Lindström quantifier Q of type $\sigma = \{R_1, \dots, R_k\}$. Without loss of generality we may assume that applications of Q are in the following normal form:

$$\psi(\overline{z}) = Q\big(\overline{x}^{(i)}; \varphi_i(\overline{z}, \overline{x}^{(i)})\big)_{i=1,\dots,k}.$$

While the $\varphi_i[\Gamma(\mathfrak{A}, n)]$ are invariant under automorphisms and therefore covered by our encoding scheme, this need not be true of the predicates

$$\varphi_i[\Gamma(\mathfrak{A}, n), \overline{c}] = \big\{\overline{b} \in \Gamma(\mathfrak{A}, n) \mid \Gamma(\mathfrak{A}, n) \models \varphi_i[\overline{c}, \overline{b}]\big\}$$

for fixed parameters $\overline{c}$. But Q is applied to predicates of this type in the evaluation of ψ over $\Gamma(\mathfrak{A}, n)$. Note that the resulting predicate

$$\psi[\Gamma(\mathfrak{A}, n)] = \big\{\overline{c} \in \Gamma(\mathfrak{A}, n) \mid \Gamma(\mathfrak{A}, n) \models \psi[\overline{c}]\big\},$$

however, will again be automorphism invariant over $\Gamma(\mathfrak{A}, n)$.

In order to deal with the intermediate predicates $\varphi_i[\Gamma(\mathfrak{A}, n), \overline{c}]$ we consider an extension of our encoding schemes that covers such *fibres* of automorphism closed predicates. For predicates R and parameter tuple $\overline{c}$ let $R|\overline{c}$ denote the *fibre of R over $\overline{c}$*:

$$R|\overline{c} = \big\{\overline{b} \mid R\overline{c}\overline{b}\big\}.$$

Definition 5.4. *A* good encoding scheme with parameters *for Γ extends a good encoding scheme to a mapping $[\;]$ that encodes parameter defined fibres of automorphism invariant R over $\Gamma(\mathfrak{A}, n)$ through tuples of predicates $[R]_{\overline{c}} = \big(R^1_{\overline{c}}, \dots, R^l_{\overline{c}}\big)$, such that*

(i) $(\mathfrak{A}, [R]_{\overline{c}})$ and n determine $\big(\Gamma(\mathfrak{A}, n), R|\overline{c}, \overline{c}\big)$ up to isomorphism.

(ii) the $[R]_{\overline{c}}$ are uniformly first-order interdefinable with $[R]$ over $\mathfrak{A}$: there is an l-tuple of first-order formulae $\overline{\chi}$ such that

(a) $\big\{[R]_{\overline{c}} \mid \overline{c} \in \Gamma(\mathfrak{A}, n)\big\} = \big\{\overline{\chi}[\mathfrak{A}, [R], \overline{a}] \mid \overline{a} \in \mathfrak{A}\big\}$.

(b) for automorphism closed P over $\Gamma(\mathfrak{A}, n)$, $[P]$ is first-order definable from the set of those $\overline{a}$ for which $\overline{\chi}[\mathfrak{A}, [R], \overline{a}] \in \big\{[R]_{\overline{c}} \mid \overline{c} \in P\big\}$.

For $\Gamma(\mathfrak{A}, n) = \mathfrak{A} \otimes n$ such an extension of the encoding scheme considered above is obtained as follows. For R of arity $t + r$ we used $[R] = (R^e)_{e \in \mathrm{Eq}(t+r)}$. This extends to cover encodings of $R|\overline{c}$ with parameter tuples $\overline{c}$ of arity t, if we choose for $[R]_{\overline{c}}$ the tuple of predicates

$$R^e_{\overline{c}} := \Big\{\pi_1(\overline{c}, \overline{b}) \;\Big|\; \overline{b} \in R|\overline{c}, \mathrm{eq}\big(\pi_2(\overline{c}, \overline{b})\big) = e\Big\},$$

for $e \in \mathrm{Eq}(t + r)$. $R^e_{\overline{c}}$ can be non-empty only for those e that extend $\mathrm{eq}\big(\pi_2(\overline{c})\big)$ to $t + r$ variables. Note for (ii) above that, for such e, each $R^e_{\overline{c}}$ is first-order interdefinable with $\pi_1(\overline{c})$ and the fibre of R^e at $\pi_1(\overline{c})$. Therefore, $[R]_{\overline{c}}$ is first-order definable in terms of $[R]$, $\pi_1(\overline{c})$ and $\mathrm{eq}(\pi_2(\overline{c}))$.

For quantifier applications $\psi(\overline{z}) = Q(\overline{x}^{(i)}; \varphi_i(\overline{z}, \overline{x}^{(i)}))_{i=1,\dots,k}$ it remains to capture the semantics of Q over $\Gamma(\mathfrak{A}, n)$ in terms of the encodings of the fibres $\varphi_i[\Gamma(\mathfrak{A}, n), \overline{c}] = \varphi_i[\Gamma(\mathfrak{A}, n)] | \overline{c}$ over the base structure $\mathfrak{A}$.

Assume that this is possible. Then one can pass from the encodings of the $\varphi_i[\Gamma(\mathfrak{A}, n)]$ to the encodings of all $\varphi_i[\Gamma(\mathfrak{A}, n), \overline{c}]$, through a first-order variation of the parameters in the $\overline{\chi}$ according to (ii) (a) in Definition 5.4. If it can be determined in terms of these, whether $\big(\Gamma(\mathfrak{A}, n), (\varphi_i[\Gamma(\mathfrak{A}, n), \overline{c}])\big)$ is in Q, then (ii) (b) serves to obtain the encoding of $\psi[\Gamma(\mathfrak{A}, n)]$ from the collection of those choices of parameters for which this is the case.

Quantifiers, and in particular cardinality quantifiers, cannot be expected to display an independence of the scaling parameter n as expressed for FP in Lemma 5.3. But the n-dependence of quantifiers Q can be isolated in a non-uniform way.

We now fix some Γ and a good encoding scheme $R \mapsto [R]$, with parameter extensions $R, \overline{c} \mapsto [R]_{\overline{c}}$ for Γ. Let Q be a Lindström quantifier of type $\sigma = \{R_1, \dots, R_k\}$. Introduce a series of quantifiers Q_n^Γ where

$$Q_n^\Gamma = \left\{ (A, [R_1']_{\overline{c}}, \dots, [R_k']_{\overline{c}}) \;\middle|\; \begin{array}{l} (\Gamma(\mathfrak{A}, n), R_1'|\overline{c}, \dots, R_k'|\overline{c}) \restriction \sigma \in Q, \\ \text{the } R_i' \simeq\text{-closed on } \Gamma(\mathfrak{A}, n) \end{array} \right\}.$$

Here the R_i' are of arity $t + r_i$ if the arity of R_i is r_i and if parameter tuples of arity t are considered. The type of the Q_n^Γ is that obtained from the encoding scheme $R, \overline{c} \mapsto [R]_{\overline{c}}$ applied to the R_i'. This type accordingly depends on the arity of parameter tuples $\overline{c}$ that are admitted; we suppress this dependence in our notation.

Let Q_* stand for a quantifier symbol of appropriate type, i.e. a syntactic object that behaves just like one of the Q_n^Γ. With the arguments from above, the following extension of Lemma 5.3 is obtained:

Lemma 5.5. *For any sentence $\varphi \in \mathrm{FP}(Q)[\tau]$ there is a sentence $\varphi_*(Q_*) \in \mathrm{FP}(Q_*)[\tau]$ such that for all sufficiently large n*

$$\Gamma(\mathfrak{A}, n) \models \varphi \iff \mathfrak{A} \models \varphi_*(Q_n^\Gamma),$$

where $\varphi_*(Q_n^\Gamma)$ is the sentence $\varphi_*(Q_*)$ with the semantics of Q_n^Γ for the dummy quantifier Q_*.

The claim applies similarly to families $\mathcal{Q}$ of quantifiers. A separation of $\mathrm{FP}(\mathcal{Q})$ from a logic $\mathcal{L}$ can be achieved if it can be shown that the complexity of the quantifiers Q_n^Γ falls short of the complexity attainable in $\mathcal{L}$ on the $\Gamma(\mathfrak{A}, n)$ for large n. Since we pass from formulae $\varphi \in \mathrm{FP}(\mathcal{Q})$ to a family of formulae $\varphi_*(Q_n^\Gamma)$ with an a priori non-uniform dependence of the semantics of the Q_n^Γ on n, these arguments are adapted to non-uniform complexity considerations.

We apply this strategy to $\mathcal{L} = \mathrm{FP}^*$ in a context in which FP^* over the $\Gamma(\mathfrak{A}, n)$ captures full PTIME. We shall also use that $|\Gamma(\mathfrak{A}, n)| \geqslant n$ and that $\mathfrak{A}$ itself and n (as a number in the second sort) are uniformly FP^*-interpretable over $\Gamma(\mathfrak{A}, n)^*$.

We show that this is the case for trivial products $\Gamma : (\mathfrak{A}, n) \mapsto \mathfrak{A} \otimes n$, if $\mathfrak{A}$ is linearly ordered. Let $<$ be the symbol for the linear ordering on $\mathfrak{A} \in \mathrm{ord}[\tau]$. Denote by $\leqslant^{\mathfrak{A}}$ the corresponding ordering in the sense of $\leqslant$ on $\mathfrak{A}$. Then $\leqslant^{\mathfrak{A}}\otimes n$ is a pre-ordering on $A \times n$ whose equivalence relation is $=^{\mathfrak{A}}\otimes n$, the quotient interpretation of equality on A over the product $A \times n$. Note that $=^{\mathfrak{A}}\otimes n$ and $\leqslant^{\mathfrak{A}}\otimes n$ are definable on $\mathfrak{A} \otimes n$ from the given $<^{\mathfrak{A}}\otimes n$ according to $(a, a') \in \leqslant^{\mathfrak{A}} \otimes n \Leftrightarrow \mathfrak{A} \otimes n \models \neg a' < a$ and $(a, a') \in =^{\mathfrak{A}}\otimes n \Leftrightarrow \mathfrak{A} \otimes n \models \neg a' < a \wedge \neg a < a'$. It follows that $\mathfrak{A} \in \mathrm{ord}[\tau]$ is interpreted over $\mathfrak{A} \otimes n$ as a quotient with respect to $=^{\mathfrak{A}}\otimes n$ — even in a first-order definable manner.

It follows further that $\mathfrak{A}$ and n and an ordered version of $\mathfrak{A} \otimes n$ are FP^*-interpretable over the second sort of $(\mathfrak{A} \otimes n)^*$, whence FP^* captures PTIME over the $\mathfrak{A} \otimes n$ for $\mathfrak{A} \in \mathrm{ord}[\tau]$.

To make a comparison between the complexity of queries over the $\Gamma(\mathfrak{A}, n)$ and that of their non-uniform description over the $\mathfrak{A}$ precise, we introduce the notion of a *pull-back* with respect to a function γ. This function γ serves to couple the scaling parameter n of Γ to the size of $\mathfrak{A}$.

Definition 5.6. *Let* $\Gamma : \mathrm{fin}[\tau] \times \omega \to \mathrm{fin}[\tau]$ *and* $\gamma : \omega \to \omega$. *If* $\mathcal{K}$ *is a boolean query on* $\mathrm{fin}[\tau]$ *then the following class is the* pull-back *of* $\mathcal{K}$ *under* Γ *and* γ:

$$\mathcal{K}_{\Gamma,\gamma} := \Big\{ \mathfrak{A} \in \mathrm{fin}[\tau] \;\Big|\; \Gamma(\mathfrak{A}, \gamma(|A|)) \in \mathcal{K} \Big\}.$$

A pull-back *of a quantifier* Q *of type* $\sigma = \{R_1, \dots, R_k\}$ *with respect to* Γ *and* γ *(and associated encoding scheme) is a quantifier*

$$Q^{\Gamma}_{\gamma} = \left\{ (A, [R'_1]_{\bar{c}}, \dots, [R'_k]_{\bar{c}}) \;\middle|\; \begin{array}{l} (\Gamma(\mathfrak{A}, \gamma(|A|)), R'_1|\bar{c}, \dots, R'_k|\bar{c}) \upharpoonright \sigma \in Q, \\ \text{the } R'_i \simeq\text{-closed on } \Gamma(\mathfrak{A}, \gamma(|A|)) \end{array} \right\}.$$

Lemma 5.7. *Assume that* Γ *is such that* $|\Gamma(\mathfrak{A}, n)| \geqslant n$ *and that* $\mathfrak{A}$ *and* n *are uniformly* FP^**-interpretable over* $\Gamma(\mathfrak{A}, n)^*$ *for* $\mathfrak{A} \in \mathrm{ord}[\tau]$. *Then there is for every recursive query* $\mathcal{K}_0$ *on* $\mathrm{ord}[\tau]$ *a class* $\mathcal{K}$ *which is* FP^**-definable over* $\{\Gamma(\mathfrak{A}, n) \mid \mathfrak{A} \in \mathrm{ord}[\tau]\}$, *such that* $\mathcal{K}_0$ *is the pull-back of* $\mathcal{K}$ *under* Γ *and* γ, *for all sufficiently fast growing* γ.

Sketch of Proof. Let $\mathcal{K}_0 \subseteq \mathrm{ord}[\tau]$ be recursive. It follows that there is a function γ whose graph is in PTIME and such that membership of $\mathfrak{A}$ in $\mathcal{K}_0$ is decidable in time $\gamma(|A|)$. For instance $\gamma(m)$ could be the step counter for the consecutive simulation of some algorithm for $\mathcal{K}_0$ on all $\mathfrak{A}$ over universe m. Put

$$\mathcal{K} := \{ \Gamma(\mathfrak{A}, n) \mid \mathfrak{A} \in \mathcal{K}_0, n \geqslant \gamma(|A|) \}.$$

We observe first that the class $\{\Gamma(\mathfrak{A}, n) \mid \mathfrak{A} \in \text{ord}[\tau], n \geqslant \gamma(|A|)\}$ is FP*-definable over $\{\Gamma(\mathfrak{A}, n) \mid \mathfrak{A} \in \text{ord}[\tau]\}$: n and $|A|$ are available over the second sort by the assumptions on Γ, and the graph of γ is in PTIME.

$\mathfrak{A}$ is FP*-interpreted over $\Gamma(\mathfrak{A}, n)^*$ by assumption on Γ, and $\mathfrak{A} \in \mathcal{K}_0$ is decidable in time $\gamma(|A|)$ by the choice of γ. It follows that $\mathcal{K}$ is FP*-definable over $\{\Gamma(\mathfrak{A}, n) \mid \mathfrak{A} \in \text{ord}[\tau]\}$.

But $\mathcal{K}_0 = \mathcal{K}_{\Gamma,\gamma}$ by construction. Observe that γ may be replaced with any other function γ' that grows at least as fast as γ: $\mathcal{K}_0 = \mathcal{K}_{\Gamma,\gamma'}$ for any γ' such that $\gamma'(m) \geqslant \gamma(m)$ for all m. □

So the pull-backs of FP*-definable queries are of arbitrarily high complexity. In the next section we shall see that in contrast the pull-backs with respect to $\Gamma(\mathfrak{A}, n) = \mathfrak{A} \otimes n$ of FP($\mathcal{Q}_{\text{card}}$)-definable queries are in PTIME/$_{\text{poly}}$ — polynomial time with non-uniform polynomial advice.

Recall the definition of PTIME/$_{\text{poly}}$ from complexity theory. A class $\mathcal{K}$ of ordered τ-structures is in PTIME/$_{\text{poly}}$ if there is an *advice function* T defined on ω with values that are polynomially bounded in size and such that membership of $\mathfrak{A}$ in $\mathcal{K}$ can be decided in polynomial time upon input $(\mathfrak{A}, T(|A|))$. PTIME/$_{\text{poly}}$ may equivalently be characterized by computability in polynomial size families of boolean circuits. In any case, standard diagonalization techniques based on counting arguments show that PTIME/$_{\text{poly}}$ is strictly contained in the class of all recursive sets. See for instance [Weg87].

A Lindström quantifier is in PTIME/$_{\text{poly}}$ if there is a polynomially size bounded advice function T such that the class of pairs

$$\{(\mathfrak{A}; T(|A|)) \mid \mathfrak{A} \in Q\}$$

is in PTIME. Think of $T(n)$ as a polynomial size table encoding the semantics of Q over size n structures. FP(Q)-definable queries can obviously be evaluated in PTIME/$_{\text{poly}}$ if Q is in PTIME/$_{\text{poly}}$.

Putting the results of the above considerations together we obtain the following general statement. It will be applied below to the functor $\Gamma: (\mathfrak{A}, n) \mapsto \mathfrak{A} \otimes n$.

Proposition 5.8. *Let $\Gamma: \text{fin}[\tau] \times \omega \to \text{fin}[\tau]$ be a functor that admits a good encoding scheme with parameters. Assume that Γ is such that $|\Gamma(\mathfrak{A}, n)| \geqslant n$ and that $\mathfrak{A}$ and n are uniformly* FP***-interpreted over $\Gamma(\mathfrak{A}, n)^*$ for $\mathfrak{A} \in \text{ord}[\tau]$. Suppose further that for all quantifiers $Q \in \mathcal{Q}$ and for sufficiently fast growing $\gamma: \omega \to \omega$ the pull-backs Q^{Γ}_{γ} are in* PTIME/$_{\text{poly}}$. *Then* $\text{FP}^* \not\leq \text{FP}(\mathcal{Q})$.

Proof. Under the assumptions on Γ we may apply Lemma 5.7 to find that any recursive query on ord[τ] is the pull-back of some query that is FP*-definable over $\{\Gamma(\mathfrak{A}, n) \mid \mathfrak{A} \in \text{ord}[\tau]\}$. The complexity of pull-backs of FP($\mathcal{Q}$)-definable queries is at most PTIME/$_{\text{poly}}$ by Lemma 5.5 and the assumptions on $\mathcal{Q}$. □

5.2 Cardinality Lindström Quantifiers

Before applying the techniques prepared in the previous section to the proof of Theorem 5.1, we show in an aside that no finite collection of generalized quantifiers can capture FP+C. The argument is an adaptation of the proof by Dawar and Hella [DH94] that no finite extension of FP captures PTIME.

Theorem 5.9 (Dawar, Hella). *For any finite set $\mathcal{Q}$ of* PTIME *Lindström quantifiers:* FP+C $\not\subseteq$ FP($\mathcal{Q}$).

Proof. We consider FP+C and FP($\mathcal{Q}$) over pure sets ($\tau = \emptyset$) and show that over these FP+C $\not\subseteq$ FP($\mathcal{Q}$). Because over pure sets FP+C $\equiv$ FP* we even show that FP* $\not\subseteq$ FP($\mathcal{Q}$). Obviously FP* captures PTIME over pure sets.

Consider now definability in FP($\mathcal{Q}$) for finite $\mathcal{Q}$ over pure sets. By invariance under automorphisms, any predicate definable over pure sets has to be quantifier free equality definable, or a union of equality types. In each bounded arity k there are only finitely many equality types, so that it follows (with an argument strictly analogous to that in Corollary 1.32) that over the empty vocabulary FP($\mathcal{Q}$) $\equiv L_{\omega\omega}(\mathcal{Q})$.

Consider a single quantifier $Q \in \mathcal{Q}$ and without loss of generality assume that its type consists of a single relation R of arity r (tuples of predicates can be encoded into single predicates by first-order means, and corresponding transformations of Q do not affect polynomiality).

Let $L^k_{\omega\omega}(Q)$ be that syntactic fragment of $L_{\omega\omega}(Q)$ which uses only first-order variables $x_1, \ldots, x_k$. In $L^k_{\omega\omega}(Q)$ over pure sets, Q can only be applied to r-ary predicates that are quantifier free equality definable (with parameters) in at most k variables. Up to logical equivalence there is a finite list of quantifier free equality formulae $\chi_j(\overline{x}, \overline{x}')$ in variables $x_1, \ldots, x_k$ that provide such definitions. Let the χ_j be of the form

$$\chi_j(\overline{x}, \overline{x}') = \theta_j(\overline{x}) \wedge \eta_j(\overline{x}, \overline{x}'),$$

with $\overline{x}$ and $\overline{x}'$ disjoint, $\overline{x}'$ of arity r, and with θ_j specifying a complete equality type in the parameters $\overline{x}$. Then the semantics of Q in $L^k_{\omega\omega}(Q)$ is exhaustively described over each individual set n by a finite table $T(n)$ that encodes the behaviour of Q on the $\chi_j[n]$. Let $T(n)$ be the finite list of indices j for which

$$\big(n, \{\overline{m}' \mid n \models \chi_j[\overline{m}, \overline{m}']\}\big) \in Q \quad \text{for } \mathrm{eq}(\overline{m}) = \theta_j.$$

There are only finitely many possibilities $T_1, \ldots, T_l$ for this entire table. For any fixed value T_i the quantifier Q in $L^k_{\omega\omega}(Q)$ becomes uniformly first-order definable over all n with $T(n) = T_i$. A formula $\xi(\overline{x}) = Q\big(\overline{x}'; \varphi(\overline{x}, \overline{x}')\big)$ is equivalent over all n with $T(n) = T_i$ with the disjunction

$$\xi_i(\overline{x}) := \bigvee_{j \in T_i} \big(\theta_j(\overline{x}) \wedge \forall \overline{x}'(\varphi \leftrightarrow \chi_j)\big).$$

It follows that Q can be eliminated (in $L^k_{\omega\omega}(Q)$ over pure sets) at the cost of introducing cardinality quantifiers Q_i of type $\emptyset$ according to

$$Q_i := \{A \mid T(|A|) = T_i\}.$$

For then, $\xi(\overline{x})$ as above becomes equivalent with $\bigvee_i (Q_i \wedge \xi_i(\overline{x}))$. This carries through inductively to eliminate all occurrences of Q.

If the complexity of the original Q is in PTIME of degree d, and if $d \geqslant r$, then the tables $T(n)$ can be computed in PTIME of degree d, too. This is because a standard representation of each $(n, \{\overline{m}' \mid n \models \chi_j[\overline{m}, \overline{m}']\})$ may be constructed in time $O(n^r)$. Therefore, also the Q_i are in PTIME of degree d.

Thus, for any finite set $\mathcal{Q}$ of PTIME quantifiers there is some d, such that over the empty vocabulary $L^k_{\omega\omega}(\mathcal{Q}) \equiv L^k_{\omega\omega}(\mathcal{Q}')$ for some finite set $\mathcal{Q}'$ of quantifiers of type $\emptyset$ whose complexity is in PTIME of degree bounded by d. For d we may take the maximal degree in a set of polynomials that bound the complexities of the $Q \in \mathcal{Q}$. Note that $\mathcal{Q}'$ depends on k, but the bound d does not.

That the quantifiers in $\mathcal{Q}'$ are of type $\emptyset$ means that their semantics only depends on the size of the universe. Let $\varphi \in L^k_{\omega\omega}(\mathcal{Q}')$ be a sentence. Then there is some m such that $n \models \varphi \Leftrightarrow n' \models \varphi$ for all $n, n' \geqslant m$ which satisfy the same $Q \in \mathcal{Q}'$. Asymptotically therefore, and over the empty vocabulary, any boolean query in $L^k_{\omega\omega}(\mathcal{Q}')$ is equivalent with a boolean combination of quantifiers $Q \in \mathcal{Q}'$, and therefore its complexity is of degree bounded by d. Now, since over pure sets

$$\mathrm{FP}(\mathcal{Q}) \equiv L_{\omega\omega}(\mathcal{Q}) \equiv \bigcup_k L^k_{\omega\omega}(\mathcal{Q}),$$

FP($\mathcal{Q}$) can only define boolean queries whose complexity is of a constantly bounded degree. It is obvious on the other hand that no such restriction applies to FP* over pure sets, because there are numerical properties of arbitrarily high polynomial degree in PTIME. □

5.2.1 Proof of Theorem 5.1

Plain cardinality Lindström quantifiers. Consider first the case of $\mathrm{FP}(\mathcal{Q}_{\mathrm{card}})$. $\Gamma: (\mathfrak{A}, n) \mapsto \mathfrak{A} \otimes n$ is now fixed. We want to show the following for all $\gamma: \omega \to \omega$.

(∗) For any $Q \in \mathcal{Q}_{\mathrm{card}}$, the quantifiers Q^Γ_γ — the pull-backs of Q from $\mathfrak{A} \otimes \gamma(|A|)$ to $\mathfrak{A}$ — can be encoded in polynomially size bounded tables $T(|A|)$. In other words: each Q^Γ_γ is in $\mathrm{PTIME}/_{\mathrm{poly}}$.

Recall Definition 5.6 for the Q^Γ_γ. By Proposition 5.8, (∗) suffices to prove that part of Theorem 5.1 that deals with ordinary cardinality Lindström quantifiers.

For the proof of (*) first observe that the Q^Γ_γ for $Q \in \mathcal{Q}_{\mathrm{card}}$ are themselves in $\mathcal{Q}_{\mathrm{card}}$. Recall that we write π_i for the projections to the factors in $\mathfrak{A}\otimes n$. The extended encoding scheme $R, \bar{c} \mapsto [R]_{\bar{c}}$ for the fibres of $t{+}r$-ary automorphism closed predicates R with parameter tuples $\bar{c}$ of arity t, takes for $[R]_{\bar{c}}$ the tuple of predicates

$$R^e_{\bar{c}} = \{\pi_1(\bar{c},\bar{b}) \mid \bar{b} \in R|\bar{c}, \mathrm{eq}(\pi_2(\bar{c},\bar{b})) = e\}, \quad e \in \mathrm{Eq}(t+r).$$

If $R|\bar{c}$ is non-empty, then the $R^e_{\bar{c}}$ determine $\pi_1(\bar{c})$ and $\mathrm{eq}(\pi_2(\bar{c}))$. $\bar{c}$ itself is then determined up to an arbitrary choice of $\pi_2(\bar{c})$ that realizes $\mathrm{eq}(\pi_2(\bar{c}))$. Up to this choice, the fibre $R|\bar{c}$ can be recovered from the encoding as

$$R|\bar{c} = \dot{\bigcup_e} \Big\{\bar{b} \in (A\times n)^r \;\Big|\; \pi_1(\bar{c},\bar{b}) \in R^e_{\bar{c}}, \mathrm{eq}(\pi_2(\bar{c},\bar{b})) = e\Big\}.$$

Therefore

$$\big|R|\bar{c}\big| = \sum_e \big|R^e_{\bar{c}}\big|\, \nu_e(n),$$

where ν_e is the counting function whose value on n is the number of realizations of e over n that extend any fixed realization of $\mathrm{eq}(\pi_2(\bar{c}))$.

Suppose for instance Q is of type $\{R_1\}$, R_1 of arity r, and based on the numerical relation $S \subseteq \omega^2$. Then, for automorphism invariant R and parameters $\bar{c}$,

$$(A\times n, R|\bar{c}) \in Q \quad \text{if} \quad (|A|n, |R|\bar{c}|) \in S$$

and

$$(A, [R]_{\bar{c}}) \in Q^\Gamma_\gamma \quad \text{if} \quad \Big(|A|\,\gamma(|A|), \textstyle\sum_e |R^e_{\bar{c}}|\, \nu_e(\gamma(|A|))\Big) \in S.$$

This latter condition constitutes a cardinality quantifier $\widehat{Q}$ of the type of the encoding $[R]_{\bar{c}}$ over the base structures $\mathfrak{A}$. The same applies without any changes to cardinality quantifiers Q of more complex types.

It is obvious, finally, that the semantics of cardinality quantifiers can be fully encoded in polynomial size tables. Let the arities in $\widehat{Q}$ be bounded by $\hat{r}$ and let $\widehat{S}$ be the numerical relation for $\widehat{Q}$. To evaluate $\widehat{Q}$ over a structure of size m, one need only know $\widehat{S} \restriction \{0,\ldots,m^{\hat{r}}\}$. This restriction of $\widehat{S}$ is naturally encoded in a polynomial size table. This finishes the proof of Theorem 5.1 as far as $\mathrm{FP}(\mathcal{Q}_{\mathrm{card}})$ is concerned.

The following discussion shows how to extend the argument to $\mathrm{FP}(\mathcal{Q}^{\sim}_{\mathrm{card}})$ where counting of equivalence classes is involved. This is based on a slightly more technical analysis of the encodings.

Quotient cardinality quantifiers. We claim that also for a quotient cardinality quantifier $Q \in \mathcal{Q}^{\sim}_{\text{card}}$ the pull-backs Q^{Γ}_{γ} of Q from $\mathfrak{A} \otimes \gamma(|A|)$ to $\mathfrak{A}$ are in $\text{PTIME}/_{\text{poly}}$, or encodable in polynomially size bounded tables $T(|A|)$.

In order not to get overburdened by technical details, let us consider the special case of a parameter free pull-back. This is the case of a pull-back quantifier Q^{Γ}_{γ} that captures the counting of equivalence classes over $\mathfrak{A}\otimes n$ with respect to an equivalence relation R that is interpreted without parameters. The technical lemma on equality defined equivalence relations, that governs this case, may be extended to the general case with parameters in order to prove the full claim.

In the parameter free case we deal as above with the encoding scheme that is based on the mapping

$$\begin{array}{rcl} \Pi\colon (A \times n)^r & \longrightarrow & A^r \times \text{Eq}(r) \\ \bar{b} & \longmapsto & \big(\pi_1(\bar{b}), \text{eq}(\pi_2(\bar{b}))\big). \end{array}$$

Lemma 5.10. *Let $R \subseteq (A \times n)^{2r}$ be closed under automorphisms of $\mathfrak{A} \otimes n$ and assume that R interprets an equivalence relation on the r-th power of $A \times n$. Let $\bar{a}_0 \in A^r$, $e_0 \in \text{Eq}(r)$. Then the following are satisfied:*

(i) the index of the restriction of R to $\Pi^{-1}(\bar{a}_0, e_0)$ is of the form $p(n)/r!$ where p is a polynomial of degree at most r and with coefficients in $\{0, \dots, (r!)^2\}$. These coefficients can be determined from the encoding $[R] = (R^e)_{e \in \text{Eq}(2r)}$ on $\mathfrak{A}$ in PTIME.

(ii) if $P \subseteq (A \times n)^r$ is an automorphism closed predicate on $\mathfrak{A} \otimes n$, then the index of the restriction of R to P is of the form $q(n)/r!$ for a polynomial q of degree at most r with coefficients in $\{0, \dots, (r!)^3|A|^r\}$. Again the coefficients are PTIME *computable from the encodings $[R]$ and $[P]$ on $\mathfrak{A}$.*

Proof. Assume n is much greater than r.
(i) For the first claim consider any quantifier free equality defined equivalence relation $\sim$ on the set $e_0[n] := \{\overline{m} \in n^r \mid \text{eq}(\overline{m}) = e_0\}$. Without loss of generality assume that e_0 is the equality type that forces all r components of the $\overline{m}$ to be distinct. Otherwise the claim is reduced to smaller r. Let $i \in \{1, \dots, r\}$ be called *free* under $\sim$ if there are $\overline{m}, \overline{m}' \in e_0[n]$ with $m'_i \notin \{m_1, \dots, m_r\}$ and $\overline{m} \sim \overline{m}'$. An easy automorphism argument that exploits transitivity and symmetry of $\sim$ shows that, if i is free in $\sim$, then $\overline{m} \sim \overline{m}\frac{m}{i}$ for all $\overline{m} \in e_0[n]$ and all $m \notin \{m_1, \dots, m_r\}$. In this case therefore, $\sim$ is reducible to an equivalence relation $\sim'$ on the remaining components that has the same index as $\sim$: if for instance r is free, let e'_0 be the restriction of e_0 to the first $r-1$ variables, and put for $\overline{m}, \overline{m}' \in e'_0[n]$

$$\overline{m} \sim' \overline{m}' \quad \text{if} \quad \overline{m}m \sim \overline{m}'m' \text{ for all } m \notin \overline{m} \text{ and } m' \notin \overline{m}'.$$

We may therefore assume without loss of generality that no i is free in $\sim$. This implies that $\overline{m} \sim \overline{m}'$ only if $\overline{m}' = \rho(\overline{m})$ for some permutation $\rho \in S_r$. Let

G be the normal subgroup (!) of S_r consisting of those ρ for which $\overline{m} \sim \rho(\overline{m})$. The index of $\sim$ on $e_0[n]$ is the product of the number $\binom{n}{r}$ of different r-element subsets of n with the index of G in S_r.

The claim about the form of the index as a polynomial in n follows. A representation of this polynomial by its coefficients is PTIME computable over the encodings in $\mathfrak{A}$ because the above sequence of reductions is governed by even first-order definable properties of the given equivalence relation R.
(ii) For a preliminary observation let $M \subseteq A^r \times \mathrm{Eq}(r)$, $(\overline{a}, e) \notin M$. Then exactly one of the following holds:

(a) any $\overline{b} \in \Pi^{-1}(\overline{a}, e)$ is R-equivalent with some $\overline{b}_1 \in \Pi^{-1}(M)$.
(b) no $\overline{b} \in \Pi^{-1}(\overline{a}, e)$ is R-equivalent with any $\overline{b}_1 \in \Pi^{-1}(M)$.

Again, a simple automorphism argument proves this claim: for $\overline{b}, \overline{b}' \in \Pi^{-1}(\overline{a}, e)$ there is an automorphism of $\mathfrak{A} \otimes n$ which maps $\overline{b}$ to $\overline{b}'$ while leaving $\Pi^{-1}(M)$ invariant as a set. The distinction between cases (a) and (b) is first-order in terms of R, $\Pi^{-1}(M)$ and $\Pi^{-1}(\overline{a}, e)$. It is therefore first-order and in PTIME also in terms of $[R]$, M, $\overline{a}$ and e over $\mathfrak{A}$.

Let R and P be as required in the lemma. The index $\left|P^{\mathfrak{A}\otimes n}/R^{\mathfrak{A}\otimes n}\right|$ can be determined by going through all $(\overline{a}, e) \in A^r \times \mathrm{Eq}(r)$ in some arbitrarily fixed enumeration as $(\overline{a}, e)_i = (\overline{a}_i, e_i)$, and summing over the indices $\left|\Pi^{-1}(\overline{a}_i, e_i)/R\right|$ whenever $\overline{a}_i \in P^{e_i}$ and case (b) above applies to $(\overline{a}_i, e_i)$ with respect to $M = \left\{(\overline{a}, e)_{i'} \mid i' < i\right\}$. This proves claim (ii) of the lemma, since $\left|A^r \times \mathrm{Eq}(r)\right| \leqslant r!|A|^r$. □

With this lemma the quantifier free pull-back of a quotient cardinality quantifier is seen to be in PTIME/$_{\text{poly}}$ as follows. The lemma shows that the indices over $\mathfrak{A} \otimes n$ of $\simeq$-closed interpreted predicates with respect to $\simeq$-closed interpreted equivalences can be represented as polynomials in n, of constantly bounded degree and with a range for the coefficients that is polynomially bounded in $|A|$ ((ii) of the lemma). All these indices are therefore uniquely encodable as numbers to base n, of bounded length and with entries corresponding to the above ranges for the coefficients. The numerical predicate S of Q can therefore — to the extent that matters over structures $\mathfrak{A} \otimes n$ with $|A| = m$ — be encoded in tables of size polynomial in m, with entries to be understood as (tuples of) numbers expressed to base n.

5.3 Aside on Further Applications

Though not directly related to issues of fixed-point with counting, we present two other simple applications of the technique developed in this chapter. Namely we can prove that *sparse* and *rigid* quantifiers do not suffice to capture PTIME.

A Lindström quantifier Q is called rigid if all structures in its defining class are rigid, i.e. possess no non-trivial automorphisms.

A relational structure $\mathfrak{B}$ is called f-sparse if the number of elements of B that occur in any of the predicates in $\mathfrak{B}$ is at most $f(|B|)$. We call f sub-linear if $f(cn)/n \to 0$ for $n \to \infty$ for all c. Q is sparse if there is a sub-linear function f such that all structures in Q are f-sparse

Let $\mathcal{Q}_{\text{sparse}}$ and $\mathcal{Q}_{\text{rigid}}$ be the classes of all sparse or rigid Lindström quantifiers, respectively.

Theorem 5.11. *Neither* $\text{FP}(\mathcal{Q}_{\text{sparse}})$ *nor* $\text{FP}(\mathcal{Q}_{\text{rigid}})$ *comprise all* FP^*, *in particular* $\text{PTIME} \not\subseteq \text{FP}(\mathcal{Q}_{\text{sparse}}), \text{FP}(\mathcal{Q}_{\text{rigid}})$.

Sketch of Proof. The proof is straightforward if we consider once more the functor $\Gamma: (\mathfrak{A}, n) \mapsto \mathfrak{A} \otimes n$ and the associated pull-backs.

Consider rigid quantifiers first. Let $(\mathfrak{A} \otimes n, \bar{c})$ be such that n exceeds the arity of $\bar{c}$. Then $(\mathfrak{A} \otimes n, \bar{c})$ has non-trivial automorphisms and no structure that is interpreted with parameters $\bar{c}$ over $\mathfrak{A} \otimes n$ can be rigid. In other words, the pull-back of any rigid quantifier corresponds to the trivially false quantifier $Q_n^\Gamma = \emptyset$ for all sufficiently large n.

Consider now a sparse quantifier and its pull-backs involving parameter tuples $\bar{c}$ of arity t. For sufficiently large n, any relation that is interpreted over $(\mathfrak{A} \otimes n, \bar{c})$ either only contains subtuples of $\bar{c}$, or it contains a non-trivial orbit under the automorphism group of $(\mathfrak{A} \otimes n, \bar{c})$, which grows at least linearly with n. But for sub-linear f the bound $f(|A|n)$ grows slower than n, so that for sufficiently large n, Q can evaluate to true at most on those trivial structures whose relations consist of subtuples of $\bar{c}$. These are finitely bounded in the size of their relations and in number. In restriction to their fields, these relations can thus be distinguished up to isomorphism even in first-order. For sufficiently large n the entire information in the Q_n^Γ thus is, which of these trivial structures are in Q, when embedded in the universe of size $|A|n$. Therefore the Q_n^Γ reduce to cardinality quantifiers of type $\emptyset$.

We thus find that the pull-backs of $\text{FP}(\mathcal{Q}_{\text{rigid}})$-definable classes are FP-definable. The pull-backs of $\text{FP}(\mathcal{Q}_{\text{sparse}})$-definable classes are definable in the extension of FP with cardinality quantifiers of type $\emptyset$ if only the pull-back function γ is sufficiently fast growing. In particular the latter are in $\text{PTIME}/_{\text{poly}}$ once more. This proves the desired separations. □

Remarks. In a paper [Ott94] on *simple Lindström extensions* the above results (with the exception of the case of sparse quantifiers) have been presented under a slightly different angle. The emphasis there is on quantifiers that express simple properties in the sense that these properties themselves are robust with respect to certain trivial extensions and can be decided in terms of invariants of sub-exponential range. In the case of counting quantifiers such invariants consist of numerical functions that count tuples in predicates; their range is clearly polynomial. I have here chosen to stress the technical basis of the separation proofs rather than a notion of simplicity. This basis is the same really for the applications here and in [Ott94], apart from the small difference that here we work with trivial products rather than with trivial sums. This variation is motivated by the formally smoother encoding schemes available over trivial products. The new application to sparse quantifiers is also due to this change. It relies on the property of trivial products that the pull-backs of sparse relations are sparse themselves. This is not true for trivial sums. Conceivably the general technique applies to other natural classes of quantifiers that might require yet other scaling functors Γ.