2. The Games and Their Analysis

This chapter serves to review the Ehrenfeucht-Fraïssé style analysis of the logics $L^k_{\infty\omega}$ and $C^k_{\infty\omega}$ by means of the corresponding *pebble games*. Emphasis is on the games and their algebraic analysis rather than on the more syntactic descriptions in terms of Hintikka formulae and Scott sentences. The main result of this algebraic analysis is a definable *ordering with respect to types*. We obtain ordered representations of the quotients $\mathrm{Tp}^{\mathcal{L}}(\mathfrak{A}; k) = A^k / \equiv^{\mathcal{L}}$ for $\mathcal{L} = L^k_{\infty\omega}$ or $C^k_{\infty\omega}$ on finite relational structures $\mathfrak{A}$.

- Section 2.1 contains the definition of the games and the statement and proofs of the corresponding Ehrenfeucht-Fraïssé theorems which here are due to Barwise [Bar77], Immerman [Imm82], and Immerman and Lander [IL90], respectively. We present some typical examples that apply the game characterizations to derive non-expressibility results. Most notably a construction due to Cai, Fürer and Immerman proves that the logics $C^k_{\infty\omega}$ form a strict hierarchy with respect to k.

  A refined analysis of the games shows that $\equiv^{C^k_{\infty\omega}}$ and $\equiv^{C^k_{\omega\omega}}$, and similarly $\equiv^{L^k_{\infty\omega}}$ and $\equiv^{L^k_{\omega\omega}}$, coincide in restriction to finite structures.

- In Section 2.2 we review the colour refinement technique for graphs and discuss some variants and their definability properties.

- Ideas related to the colour refinement are employed in Section 2.3 to introduce the ordered quotients with respect to $C^k_{\infty\omega}$- or $L^k_{\infty\omega}$-types through a fixed-point process for the classification of game positions.

2.1 The Pebble Games for $L^k_{\infty\omega}$ and $C^k_{\infty\omega}$

The setting for the games is the usual one for comparison games. There are two players denoted **I** and **II** for *first* and *second player*. The game is played on a pair of finite structures $\mathfrak{A}$ and $\mathfrak{A}'$ of the same finite relational vocabulary τ. In the *k-pebble game* there are k marked pebbles for each of the two structures. Let both sets of pebbles be numbered $1, \ldots, k$. A *stage* of the game, or an instantaneous description of a game situation, is determined by a placement of the pebbles on elements of the corresponding structures.

Formally a stage is given by a tuple $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$, with $\overline{a} \in A^k$ and $\overline{a}' \in A'^k$ denoting the current positions of the pebbles. A *position* describes a pebble placement over one of the structures. The position over $\mathfrak{A}$ for instance in stage $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$ is $(\mathfrak{A}, \overline{a})$. Formally a position is an element of $\mathrm{fin}[\tau; k]$: a structure with a designated k-tuple of elements. A stage in the game is a pair of positions, or an element of $\mathrm{fin}[\tau; k] \times \mathrm{fin}[\tau; k]$.

In each *round* of the game exactly one pair of corresponding pebbles is repositioned in the respective structures. This repositioning is governed by an exchange of moves between the two players. The game for L^k and that for C^k differ with respect to the rules for this exchange.

The single round in the L^k-game.

I chooses a pebble index $j \in \{1, \dots, k\}$ and moves the corresponding pebble in one of the structures to an arbitrary element of that structure, for instance to $b \in A$.

II responds by moving the corresponding pebble over the opposite structure to an arbitrary element of that structure, here to some $b' \in A'$.

If this exchange is carried out in stage $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$ then the resulting stage after this round is $(\mathfrak{A}, \overline{a}\frac{b}{j}; \mathfrak{A}', \overline{a}'\frac{b'}{j})$. We write $\overline{a}\frac{b}{j}$ for the tuple $\overline{a}$ with j-th component replaced by b.

The single round in the C^k-game.

I chooses a pebble index $j \in \{1, \dots, k\}$ and a subset of the universe of one of the structures, say $B \subseteq A$.

II must choose a subset of exactly the same size in the opposite structure, here some $B' \subseteq A'$ with $|B'| = |B|$.

I now places the j-th pebble within the subset designated by **II**, here on some $b' \in B'$.

II responds by moving the corresponding pebble over the opposite structure to any element within the subset designated by **I**, here to some $b \in B$.

If this exchange is carried out in stage $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$ then the resulting stage is $(\mathfrak{A}, \overline{a}\frac{b}{j}; \mathfrak{A}', \overline{a}'\frac{b'}{j})$.

In both cases the game may continue as long as player **II** can maintain the following condition:

(W) The mapping associating the pebbled elements in $\mathfrak{A}$ with those in $\mathfrak{A}'$ must be a partial isomorphism, i.e. $\mathrm{atp}_{\mathfrak{A}}(\overline{a}) = \mathrm{atp}_{\mathfrak{A}'}(\overline{a}')$ for the current positions $(\mathfrak{A}, \overline{a})$ and $(\mathfrak{A}', \overline{a}')$.

I wins the game as soon as **II** violates this condition, and also if **II** cannot move according to the rules as may happen in the C^k-game owing to different sizes of the two structures.

Player **II** has a *winning strategy in the infinite game on* $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$ if **II** has a strategy to maintain condition (W) indefinitely in the game starting from stage $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$. Similarly we say that **II** has a *winning strategy* for i rounds in the game on $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$ if (W) can be maintained by **II** for at least i rounds starting from $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$. More formal characterizations are developed in an inductive fashion below.

Intuitively the ability of player **II** to respond to challenges of **I** is a measure for the similarity of the underlying positions. In each individual round **II** must preserve atomic indistinguishability of the resulting positions (W), otherwise the game is lost. The ability to maintain (W) for longer sequences of rounds and in response to any manoeuvres of **I** requires a higher degree of similarity of the initial positions. The point of the above rules for single rounds is that they make the games adequate for $L^k_{\infty\omega}$ and $C^k_{\infty\omega}$, respectively. The following two important theorems state that the degree of indistinguishability corresponding to the existence of a strategy precisely is equality of types in the respective logic.

Theorem 2.1 (Barwise, Immerman). *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be finite structures of the same finite relational vocabulary. Player* **II** *has a winning strategy in the infinite L^k-game on $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$ if and only if the the positions $(\mathfrak{A}, \overline{a})$ and $(\mathfrak{A}', \overline{a}')$ cannot be distinguished in $L^k_{\infty\omega}$, i.e. if $(\mathfrak{A}, \overline{a}) \equiv^{L^k_{\infty\omega}} (\mathfrak{A}', \overline{a}')$.*

Theorem 2.2 (Immerman, Lander). *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be finite structures of the same finite relational vocabulary. Player* **II** *has a winning strategy in the infinite C^k-game on $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$ if and only if the the positions $(\mathfrak{A}, \overline{a})$ and $(\mathfrak{A}', \overline{a}')$ cannot be distinguished in $C^k_{\infty\omega}$, i.e. if $(\mathfrak{A}, \overline{a}) \equiv^{C^k_{\infty\omega}} (\mathfrak{A}', \overline{a}')$.*

From the analysis of the games it will further follow that the conditions in Theorems 2.1 and 2.2 are also equivalent with indistinguishability in the *finitary* logics $L^k_{\omega\omega}$ and $C^k_{\omega\omega}$.

Corollary 2.3. *Let τ be finite and relational. The following are equivalent for all $(\mathfrak{A}, \overline{a}), (\mathfrak{A}', \overline{a}') \in \mathrm{fin}[\tau; k]$:*

(i) Player **II** *has a strategy in the infinite L^k-game on $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$.*
(ii) $(\mathfrak{A}, \overline{a}) \equiv^{L^k_{\infty\omega}} (\mathfrak{A}', \overline{a}')$, i.e. $\mathrm{tp}^{L^k_{\infty\omega}}_{\mathfrak{A}}(\overline{a}) = \mathrm{tp}^{L^k_{\infty\omega}}_{\mathfrak{A}'}(\overline{a}')$.
(iii) $(\mathfrak{A}, \overline{a}) \equiv^{L^k_{\omega\omega}} (\mathfrak{A}', \overline{a}')$, i.e. $\mathrm{tp}^{L^k_{\omega\omega}}_{\mathfrak{A}}(\overline{a}) = \mathrm{tp}^{L^k_{\omega\omega}}_{\mathfrak{A}'}(\overline{a}')$.

In particular any $L^k_{\infty\omega}$-type over $\mathrm{fin}[\tau]$ is fully determined by its $L^k_{\omega\omega}$-part.

Corollary 2.4. *Let τ be finite and relational. The following are equivalent for all $(\mathfrak{A}, \overline{a}), (\mathfrak{A}', \overline{a}') \in \mathrm{fin}[\tau; k]$:*

(i) Player **II** *has a strategy in the infinite C^k-game on $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$.*
(ii) $(\mathfrak{A}, \overline{a}) \equiv^{C^k_{\infty\omega}} (\mathfrak{A}', \overline{a}')$, i.e. $\mathrm{tp}^{C^k_{\infty\omega}}_{\mathfrak{A}}(\overline{a}) = \mathrm{tp}^{C^k_{\infty\omega}}_{\mathfrak{A}'}(\overline{a}')$.
(iii) $(\mathfrak{A}, \overline{a}) \equiv^{C^k_{\omega\omega}} (\mathfrak{A}', \overline{a}')$, i.e. $\mathrm{tp}^{C^k_{\omega\omega}}_{\mathfrak{A}}(\overline{a}) = \mathrm{tp}^{C^k_{\omega\omega}}_{\mathfrak{A}'}(\overline{a}')$.

Each $C^k_{\infty\omega}$-type is fully determined by its $C^k_{\omega\omega}$-part over $\mathrm{fin}[\tau]$.

With these equivalences proved, we shall simply speak of the L^k-type and C^k-type, and write for instance $\mathrm{tp}^{L^k}_{\mathfrak{A}}(\overline{a})$ and $\mathrm{tp}^{C^k}_{\mathfrak{A}}(\overline{a})$ for these; and also $\equiv^{L^k}$ and $\equiv^{C^k}$ for the corresponding notions of L^k- and C^k-equivalence.

The following section is devoted to applications of the game characterizations. In the consecutive sections we shall then present a detailed theoretical treatment for the case of the C^k-game. In Section 2.1.2 a direct and straightforward proof of Theorem 2.2 is presented. Section 2.1.3 presents a deeper analysis of the C^k-game, proving among other things Corollary 2.4. The analogous treatment for L^k is easily obtained along the same lines through obvious simplifications; this is summed up in Section 2.1.4.

2.1.1 Examples and Applications

We present examples that employ Theorems 2.1 and 2.2 to show inexpressibility in $L^k_{\infty\omega}$ or $C^k_{\infty\omega}$.

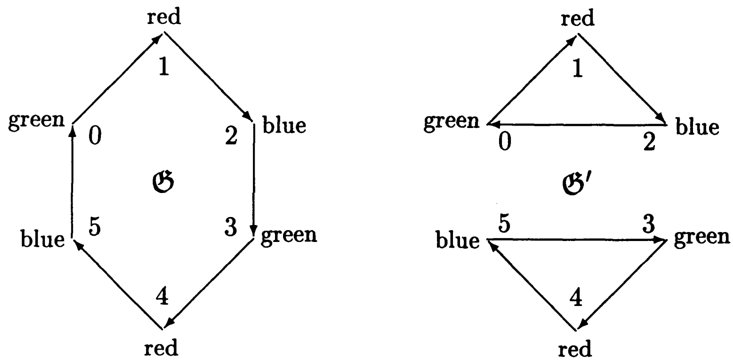
Example 2.5. As a trivial application of the L^k-game we find the following. Any two k-tuples $\overline{a}$ and $\overline{a}'$ over two plain sets A and A' of size at least k are $L^k_{\infty\omega}$-equivalent if and only if they have the same equality type: $\mathrm{eq}(\overline{a}) = \mathrm{eq}(\overline{a}') \Rightarrow \mathrm{tp}^{L^k}_{A}(\overline{a}) = \mathrm{tp}^{L^k}_{A'}(\overline{a}')$ if $|A|, |A'| \geqslant k$. It follows that $L^k_{\infty\omega}$ cannot distinguish between any two plain sets that have at least k elements. In particular the $L^k_{\infty\omega}$ form a strict hierarchy in expressiveness: $L^1_{\infty\omega} \subsetneq L^2_{\infty\omega} \subsetneq \cdots \subseteq L^{\omega}_{\infty\omega}$. The same applies to the corresponding fragments of first-order logic: $L^1_{\omega\omega} \subsetneq L^2_{\omega\omega} \subsetneq \cdots \subseteq L_{\omega\omega}$.

The following simple and elegant example is taken from [IL90].

Example 2.6 (Immerman, Lander). Consider the following two coloured directed graphs with six nodes each. $\mathfrak{G} = (\{0, \ldots, 5\}, E, U_r, U_b, U_g)$. The colours are interpreted $U_g = \{0, 3\}$ for *green*, $U_r = \{1, 4\}$ for *red* and $U_b = \{2, 5\}$ for *blue*. The edge relation E of $\mathfrak{G}$ connects the nodes $0, \ldots, 5$ in cyclic fashion. $\mathfrak{G}'$ is the same as $\mathfrak{G}$ as far as its universe and the colours are concerned. With respect to its edge relation E', however, $\mathfrak{G}'$ splits into two disjoint cycles $0, 1, 2$ and $3, 4, 5$ respectively. Compare the sketches in Figure 2.1. Note that these two graphs realize exactly the same atomic 2-types, $\mathrm{Atp}(\mathfrak{G}; 2) = \mathrm{Atp}(\mathfrak{G}'; 2)$. Furthermore we observe that each of these atomic 2-types is realized exactly twice in each structure.

We claim that $\mathfrak{G}$ and $\mathfrak{G}'$ are indistinguishable in $C^2_{\infty\omega}$. In this special case it can be shown that player **II** actually has a strategy to maintain atomic equivalence of positions. By Theorem 2.2 this implies that (a_1, a_2) from $\mathfrak{G}$ and (a'_1, a'_2) from $\mathfrak{G}'$ are C^2-equivalent if they satisfy the same atomic type. $\mathfrak{G} \equiv^{C^2} \mathfrak{G}'$ follows by Lemma 1.34 since $\mathrm{Atp}(\mathfrak{G}; 2) = \mathrm{Atp}(\mathfrak{G}'; 2)$ now implies $\mathrm{Tp}^{C^2}(\mathfrak{G}; 2) = \mathrm{Tp}^{C^2}(\mathfrak{G}'; 2)$. Before exhibiting a strategy for maintaining atomic equivalence, let us state the following consequences.

Fig. 2.1



(i) The transitive closure of a binary relation is not definable in $C^2_{\infty\omega}$. If the transitive closure of the binary relation E were definable by some formula $\varphi(x,y)$ of $C^2_{\infty\omega}[E]$ then the $C^2_{\infty\omega}[E]$-sentence $\chi := \forall x \forall y \varphi(x,y)$ would distinguish $\mathfrak{G}$ from $\mathfrak{G}'$.

(ii) Transitivity of a binary relation is not $C^2_{\infty\omega}$-definable and the class of all equivalence relations is not $C^2_{\infty\omega}$-definable. C^2-equivalence of $\mathfrak{G}$ and $\mathfrak{G}'$ directly implies C^2-equivalence also of those structures obtained from $\mathfrak{G}$ and $\mathfrak{G}'$ by removing the colours and replacing the edge relation E by its reflexive and symmetric closure, which is atomically definable from E. From $\mathfrak{G}'$ we thereby obtain an equivalence relation, not from $\mathfrak{G}$. Note that transitivity and the class of equivalence relations are first-order definable with 3 variables.

Let us return to the claim that **II** can maintain atomic equivalence. A strategy for player **II** is extracted from the following observation. Let $a \in \mathfrak{G}$ and $a' \in \mathfrak{G}'$ be of the same colour. Then there is a unique bijection π from $\mathfrak{G}$ to $\mathfrak{G}'$ that maps a to a' and preserves colours as well as edges that are incident with a or a'. This is checked directly; if without loss of generality we consider the case $a = a'$, then the identical mapping on $\{0, \dots, 5\}$ is as desired.

Suppose now that in the current stage $(\mathfrak{G}, a_1, a_2; \mathfrak{G}', a'_1, a'_2)$ of the game $\text{atp}_{\mathfrak{G}}(a_1, a_2) = \text{atp}_{\mathfrak{G}'}(a'_1, a'_2)$. We want to show that **II** can defend this property against any challenge by player **I**. Assume without loss of generality that player **I** chooses to play with the second pebble. Let π be chosen with respect to a_1 and a'_1 as above. Let then **II** play according to π: if for instance **I** proposes $B \subseteq \{0, \dots, 5\}$ as a subset of $\mathfrak{G}$ then **II** responds with $B' = \pi(B)$ and upon any choice for $b' \in B'$ by **I** player **II** may answer with $\pi^{-1}(b') \in B$. The defining condition on π guarantees that $\text{atp}_{\mathfrak{G}}(a_1, b) = \text{atp}_{\mathfrak{G}'}(a'_1, b')$.

The next example gives an account of the essential features of the construction by Cai, Fürer and Immerman of non-isomorphic but $C^k_{\infty\omega}$-equivalent finite graphs [CFI89]. We shall later also apply the result of these considerations — Theorem 2.9 below — to show that the counting extension of fixed-point logic does not capture PTIME. See Corollary 4.23 of Chapter 4.

The construction uses certain highly symmetric graphs with a parity-sensitive automorphism group. These "gadgets" were first employed by Immerman in [Imm81] to prove lower bounds on the number of variables needed for expressing certain reachability properties in graphs (without counting quantifiers).

Example 2.7 (Immerman and Cai, Fürer, Immerman). Main building blocks for the construction are the following gadgets. Fix some $m \geqslant 2$. Let $\mathcal{P}(m)$ denote the power set of the set $m = \{0, \dots, m-1\}$. We identify $\mathcal{P}(m)$ with the set of functions $s: m \to \{0,1\}$. Let $\mathfrak{H}$ be the following undirected graph with node set $H = I \dot{\cup} O$ where $I = \mathcal{P}(m)$, $O = m \times \{0,1\}$. The names I and O stand for *inner* and *outer nodes*, respectively. The edge relation of $\mathfrak{H}$ encodes the rôle of the inner nodes as subsets over m: $s \in I = \mathcal{P}(m)$ is joined exactly with all pairs $(u, s(u)) \in O$ for $u \in m$. For each $u \in m$ we refer to the two nodes $(u,0),(u,1)$ as a *pair of corresponding outer nodes.* The outer nodes of $\mathfrak{H}$ will serve as ports for gluing several copies of $\mathfrak{H}$ together. The crucial properties of the resulting graphs exploit the behaviour under automorphisms of $\mathfrak{H}$ that exchange pairs of corresponding outer nodes. Each $t \subseteq m$ induces an automorphism γ_t of $\mathfrak{H}$ that is determined by its behaviour on outer nodes

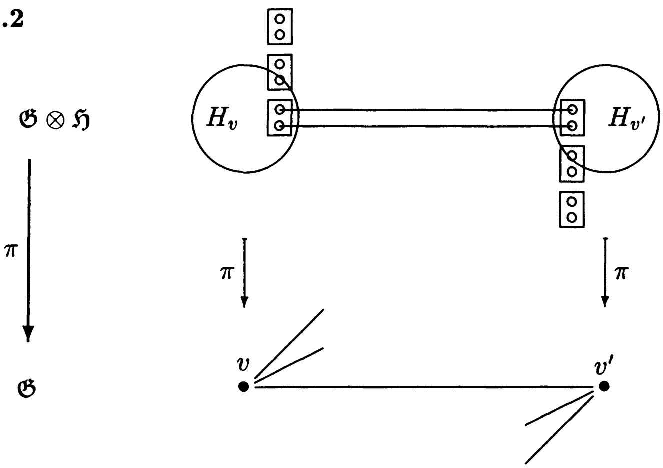
$$\gamma_t: (u,i) \longmapsto \big(u, i \oplus t(u)\big)$$

where $\oplus$ is addition modulo 2. Note that γ_t preserves the set of inner nodes and also each pair of corresponding outer nodes set-wise. On the outer nodes it swaps exactly those pairs of corresponding outer nodes $(u,0),(u,1)$ for which $u \in t$. Inner nodes are mapped according to $s \mapsto s \oplus t$ where $\oplus$ applied to the functions s and t is pointwise addition modulo 2.

We now split the set I of inner nodes into two disjoint subsets $I^i := \big\{s \subseteq m \;\big|\; |s| \equiv i \bmod 2\big\}$, for $i = 0,1$. Note that γ_t preserves the subsets I^i if and only if $|t|$ is even. For odd $|t|$ on the other hand γ_t induces a bijection between I^0 and I^1.

Let $\mathfrak{G} = (V, E, \leqslant)$ be any symmetric connected graph that is regular of degree m and linearly ordered by $\leqslant$. Let $\mathfrak{G} \otimes \mathfrak{H}$ be the result of substituting a copy of $\mathfrak{H}$ for each node of $\mathfrak{G}$ and joining outer nodes by a pair of edges in the natural fashion. In detail let $\mathfrak{G} \otimes \mathfrak{H} = \big(\widehat{V}, \widehat{E}, \preccurlyeq\big)$. $\widehat{V} = V \times H$ and $\preccurlyeq$ is the pre-ordering induced by $\leqslant$ on this product. $\widehat{E}$ consists of all edges from the respective copies of $\mathfrak{H}$ together with the following new links between outer nodes. If $(v, v') \in E$ with v' being the u-th neighbour of v in $\mathfrak{G}$ and v being the u'-th neighbour of v' (with respect to $\leqslant$) we include edges between $\big(v,(u,0)\big)$ and $\big(v',(u',0)\big)$ as well as between $\big(v,(u,1)\big)$ and $\big(v',(u',1)\big)$. We refer to these extra edges as *connecting edges.* Each edge of $\mathfrak{G}$ thus gets replaced by a pair of connecting edges. This is sketched in Figure 2.2. We denote by $\pi\colon \mathfrak{G} \otimes \mathfrak{H} \to \mathfrak{G}$ the natural projection to the first factor. Let $H_v := \pi^{-1}(v)$ denote the subset of nodes of $\mathfrak{G} \otimes \mathfrak{H}$ that belong to that copy of $\mathfrak{H}$ that is substituted for v.

Fig. 2.2



Let $I^i_v \subseteq H_v$ denote the respective subsets of the set of inner nodes within H_v, $i = 0, 1$. Consider automorphisms of $\mathfrak{G}\otimes\mathfrak{H}$ with respect to their behaviour on the sets I^i_v. If $v_0, \ldots, v_l$ is a simple path in $\mathfrak{G}$ then there is an automorphism γ of $\mathfrak{G} \otimes \mathfrak{H}$ with the following properties: γ fixes all H_v for $v \neq v_0, \ldots, v_l$ pointwise, γ preserves the subsets $I^i_{v_j}$ for $j = 1, \ldots, l-1$ and exchanges $I^0_{v_j}$ with $I^1_{v_j}$ for $j = 0, l$. Such γ is pieced together from automorphisms γ_t of the individual embedded $\mathfrak{H}$. For the copy of $\mathfrak{H}$ over v_j choose t to be the subset of m that contains u if the given path connects v_j to its u-th neighbour in $\mathfrak{G}$. Thus $|t|$ is even for all inner nodes of the path and odd for the end points of the path.

For $U \subseteq V$ let $(\mathfrak{G}\otimes\mathfrak{H})_U$ be the subgraph of $\mathfrak{G}\otimes\mathfrak{H}$ that results from deleting all inner nodes in I^0_v for $v \in U$ and those in I^1_v for $v \notin U$. Since $\mathfrak{G}$ is connected, it follows from the above automorphism argument that all the $(\mathfrak{G} \otimes \mathfrak{H})_U$ fall into at most two classes up to isomorphisms. If the symmetric difference between U_1 and U_2 is even, then $(\mathfrak{G} \otimes \mathfrak{H})_{U_1} \simeq (\mathfrak{G} \otimes \mathfrak{H})_{U_2}$. We claim that otherwise indeed $(\mathfrak{G} \otimes \mathfrak{H})_{U_1}$ and $(\mathfrak{G} \otimes \mathfrak{H})_{U_2}$ are non-isomorphic. This can be seen by means of the following numerical invariant on the $(\mathfrak{G}\otimes\mathfrak{H})_U$. Suppose a given graph is isomorphic to some $(\mathfrak{G} \otimes \mathfrak{H})_U$. Note that the projection π to $\mathfrak{G}$ and in particular therefore the node sets $\pi^{-1}(v)$, the sets of inner nodes in $\pi^{-1}(v)$, and the pairs of connecting edges between outer nodes of different copies of $\mathfrak{H}$ are well defined in terms of the given graph. Let $S \subseteq \widehat{E}$ be any set of edges that contains exactly one member from each pair of connecting edges and let N be any set of inner nodes that contains exactly one member from each $\pi^{-1}(v)$. Call a connecting edge incident with an inner node if there is an edge that joins that node with one of the end-points of the given edge. Let i be the result of counting modulo 2 the number of edges in S that are incident with N. We check that i is independent of the choices made. Replacing any edge in S by its partner edge changes the incidence with N in exactly two places. Replacing an inner node of $\pi^{-1}(v)$ by another one changes incidence with S in an even number of places, since either both nodes are in I_v^0 or both are in I_v^1. It is immediate, however, that $i = 0$ on $(\mathfrak{G} \otimes \mathfrak{H})_\emptyset$ and $i = 1$ on $(\mathfrak{G} \otimes \mathfrak{H})_{\{v\}}$ for any single node v.

For definite representatives of the two isomorphism types put $(\mathfrak{G} \otimes \mathfrak{H})^0 := (\mathfrak{G} \otimes \mathfrak{H})_\emptyset$ and $(\mathfrak{G} \otimes \mathfrak{H})^1 := (\mathfrak{G} \otimes \mathfrak{H})_{\{v_0\}}$ where v_0 is the $\leqslant$-least node of $\mathfrak{G}$. We use such representatives in the simple case that $\mathfrak{G}$ is a complete graph to obtain the desired separation result. Let $\mathfrak{K}_{m+1}$ be the ordered complete graph over $m+1$ nodes:

$$\mathfrak{K}_{m+1} = \Big(\{1, \ldots, m+1\}, \{(k,l) | k \neq l\}, \leqslant\Big).$$

Denote the above graph $\mathfrak{H}$ with node set $\mathcal{P}(m) \dot\cup m \times \{0,1\}$ by $\mathfrak{H}_m$ to indicate the dependence on m.

Lemma 2.8. *Let* $\mathfrak{A} = (\mathfrak{K}_{m+1} \otimes \mathfrak{H}_m)^0$ *and* $\mathfrak{A}' = (\mathfrak{K}_{m+1} \otimes \mathfrak{H}_m)^1$. *Then for* $m \geqslant 2$*:*

$$\mathfrak{A} \equiv^{C^m} \mathfrak{A}' \quad \textit{but} \quad \mathfrak{A} \not\equiv^{L^{m+1}} \mathfrak{A}'.$$

Proof. It is instructive to consider first the case $m = 2$. An inspection of the construction in this simple case shows that $\mathfrak{A}$ is the disjoint union of two cycles of length 9, each grouped into three groups of 3 consecutive vertices that belong to the same class of the pre-ordering. $\mathfrak{A}'$ is a single cycle of length 18 with a corresponding grouping into 6 blocks of three vertices each. If we replace the classes of the pre-ordering by three monadic predicates U_r, U_b and U_g for colours red, blue and green as in Example 2.6 then the relation between $\mathfrak{A}$ and $\mathfrak{A}'$ is the same as between the graphs $\mathfrak{G}$ and $\mathfrak{G}'$ in Example 2.6, only each node of the graphs there is replaced by a path of length 3 to obtain the present ones. The claim for $m = 2$ therefore essentially follows from the considerations in Example 2.6.

We turn to the general case. Let the natural projections from $\mathfrak{A}$ and $\mathfrak{A}'$ to $\mathfrak{K}_{m+1}$ be denoted π and π', respectively. Note that membership in $\pi^{-1}(j)$ (respectively $\pi'^{-1}(j)$) is definable in $L^2_{\omega\omega}$, since $\pi^{-1}(j)$ consists of the j-th class with respect to $\preccurlyeq$. Concrete formulae are obtained exactly as in Example 1.9. It follows that in order not to lose, player **II** must necessarily respect π and π' as well as the properties of being an inner node in $\pi^{-1}(j)$ or of being an end point of a connecting edge between $\pi^{-1}(i)$ and $\pi^{-1}(j)$ for any $1 \leqslant i, j \leqslant m+1$. This is true for both the C^k- and the L^k-games.

We first employ the L^{m+1}-game to show that $\mathfrak{A}$ and $\mathfrak{A}'$ are not L^{m+1}-equivalent. By the above considerations, player **I** can force **II** into positions such that the j-th pebbles are placed on inner nodes $a_j \in \pi^{-1}(j)$ and $a'_j \in \pi'^{-1}(j)$ for $1 \leqslant j \leqslant m+1$. For each $j \neq 1$ consider the pair of corresponding outer nodes in $\pi^{-1}(1)$ in $\mathfrak{A}$ that belong to connecting edges between $\pi^{-1}(1)$ and $\pi^{-1}(j)$. Note that exactly one node of this pair has distance 2 from a_j, the other one has distance greater than 2. Let v_j be the one with distance 2. By the construction of $\mathfrak{A}$ it is clear that the number of v_j that are direct neighbours to a_1 is even. Choosing nodes v'_j for $2 \leqslant j \leqslant m+1$ in $\mathfrak{A}'$ in the same manner, we find that the number of v'_j that are direct neighbours to a'_1 must be odd. There is therefore at least one index $j \geqslant 2$ such that v_j is a neighbour of a_1 while v'_j is not a neighbour of a'_1 or vice versa. Assume without loss of generality the former is true of $j = 2$. Let player **I** move pebble 3 in $\mathfrak{A}$ to v_2. **II** must move pebble 3 to a neighbour of a'_1 in $\mathfrak{A}'$ in order not to lose immediately. If **II** places this pebble not on one of the outer nodes in $\pi^{-1}(1)$ belonging to a connecting edge to $\pi^{-1}(2)$ then **II** loses within one more round. Choosing the one of these outer nodes that is a neighbour of a'_1 and therefore different from v'_2 **II** still loses in one more round, since now pebbles 2 and 3 are placed at distance 2 in $\mathfrak{A}$ and at distance greater 2 in $\mathfrak{A}'$.

It remains to exhibit a strategy for player **II** in the C^m-game on $\mathfrak{A}$ and $\mathfrak{A}'$. We show that **II** can maintain the following condition on the stages $(\mathfrak{A}, a_1, \ldots, a_m; \mathfrak{A}', a'_1, \ldots, a'_m)$:

$$(*) \qquad \begin{array}{l} \pi(\overline{a}) = \pi'(\overline{a}') \quad \text{and} \\ \Big(\mathfrak{A} \restriction \pi^{-1}\big(\pi(\overline{a})\big), \overline{a}\Big) \simeq \Big(\mathfrak{A}' \restriction \pi'^{-1}\big(\pi'(\overline{a}')\big), \overline{a}'\Big). \end{array}$$

We argue that this suffices for $\mathfrak{A} \equiv^{C^m} \mathfrak{A}'$. In any game position $(\mathfrak{A}, \overline{a})$ at least one $\pi^{-1}(j)$ remains unpebbled. Consider a position $\overline{a}$ over $\mathfrak{A}$ in which $\pi^{-1}(1)$ is unpebbled. By construction the identity mapping is an isomorphism between the induced subgraphs of $\mathfrak{A}$ and $\mathfrak{A}'$ on $\pi^{-1}(\{2, \ldots, m+1\})$:

$$\mathfrak{A} \restriction \pi^{-1}\big(\{2, \ldots, m+1\}\big) = \mathfrak{A}' \restriction \pi'^{-1}\big(\{2, \ldots, m+1\}\big).$$

Thus $(*)$ is seen to hold of $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a})$ if $\overline{a}$ is disjoint from $\pi^{-1}(1)$. In the general case there still is an isomorphism between $\mathfrak{A} \restriction \big(A \setminus \pi^{-1}(j)\big)$ and $\mathfrak{A}' \restriction \big(A' \setminus \pi'^{-1}(j)\big)$ for any j, because $\mathfrak{A}' = (\mathfrak{K}_{m+1} \otimes \mathfrak{H}_m)^1 = (\mathfrak{K}_{m+1} \otimes \mathfrak{H}_m)_{\{1\}} \simeq (\mathfrak{K}_{m+1} \otimes \mathfrak{H}_m)_{\{j\}}$. Therefore, for all $\overline{a}$ there is some $\overline{a}'$ such that $(*)$ holds of $(\mathfrak{A}, \overline{a})$ and $(\mathfrak{A}', \overline{a}')$, and vice versa. If **II** can maintain $(*)$, this implies that $\mathrm{Tp}^{C^m}(\mathfrak{A}; 2) = \mathrm{Tp}^{C^m}(\mathfrak{A}'; 2)$ and, with Lemma 1.34, that indeed $\mathfrak{A} \equiv^{C^m} \mathfrak{A}'$.

Assume now that $(*)$ is satisfied in the current position. Assume further that **I** chooses pebble 1 to play. Without loss of generality suppose that $\pi(a_2, \ldots, a_m) = \pi'(a'_2, \ldots, a'_m) \subseteq \{3, \ldots, m+1\}$ and that the given isomorphism is the identity mapping in restriction to $\pi^{-1}(\{3, \ldots, m+1\})$:

$$\begin{aligned}&\Big(\mathfrak{A}\restriction\pi^{-1}(\{3,\dots,m+1\}),a_2,\dots,a_m\Big)\\ =\;&\Big(\mathfrak{A}'\restriction\pi'^{-1}(\{3,\dots,m+1\}),a'_2,\dots,a'_m\Big).\end{aligned}$$

Consider any potential target position for pebble 1 over $\mathfrak{A}$ say. If a_1 is placed within $\pi^{-1}(\{3,\dots,m+1\})$ then we want a'_1 to be placed according to the given isomorphism (which happens to be the identity under our assumptions). The interesting case is that a_1 is moved to either $\pi^{-1}(1)$ or $\pi^{-1}(2)$. It follows from the considerations above that for $i=1,2$ there are isomorphisms γ_i between $\mathfrak{A}\restriction\pi^{-1}(\{1,\dots,m+1\}\setminus\{i\})$ and $\mathfrak{A}'\restriction\pi'^{-1}(\{1,\dots,m+1\}\setminus\{i\})$ such that γ_i restricts to the identity mapping over $\pi^{-1}(\{3,\dots,m+1\})$, and thus extends the given isomorphism between $\big(\mathfrak{A}\restriction\pi^{-1}(\{3,\dots,m+1\}),a_2,\dots,a_m\big)$ and $\big(\mathfrak{A}'\restriction\pi'^{-1}(\{3,\dots,m+1\}),a'_2,\dots,a'_m\big)$. Let now γ be the following bijection between $\mathfrak{A}$ and $\mathfrak{A}'$:

$$\gamma(v):=\begin{cases} v & \text{for } v\in\pi^{-1}(\{3,\dots,m+1\})\\ \gamma_2(v) & \text{for } v\in\pi^{-1}(1)\\ \gamma_1(v) & \text{for } v\in\pi^{-1}(2)\,.\end{cases}$$

Let **II** play according to γ: if **I** proposes $B\subseteq\mathfrak{A}$ say, then **II** answers $B'=\gamma(B)$ and upon a move of pebble 1 in $\mathfrak{A}'$ to $b'\in\gamma(B)$, **II** moves pebble 1 in $\mathfrak{A}$ to $\gamma^{-1}(b')$. $(*)$ is satisfied by construction in the resulting stage — the required isomorphism is provided by the corresponding restriction of γ. □

We thus have in particular the following theorem.

Theorem 2.9. *The logics $C^k_{\infty\omega}$ form a strict hierarchy with respect to k even for boolean queries on finite graphs:*

$$C^1_{\infty\omega}\subsetneq C^2_{\infty\omega}\subsetneq\dots\subsetneq C^k_{\infty\omega}\subsetneq C^{k+1}_{\infty\omega}\subsetneq\dots\subseteq C^\omega_{\infty\omega}.$$

It follows that $C^\omega_{\infty\omega}\subsetneq L_{\infty\omega}$ — not every query on finite structures is expressible in $C^\omega_{\infty\omega}$.

The second claim is provable from the first by diagonalization. A concrete graph query which is not in $C^\omega_{\infty\omega}$ is of course $\big\{(\mathfrak{K}_{m+1}\otimes\mathfrak{H}_m)^0\ \big|\ m\geqslant 2\big\}$, or rather the closure of this set under isomorphisms.

2.1.2 Proof of Theorem 2.2

The proof is given in two separate lemmas, one for each implication in the theorem.

Lemma 2.10. *If $(\mathfrak{A},\bar a)\not\equiv^{C^k_{\infty\omega}}(\mathfrak{A}',\bar a')$ then player* **I** *can force a win in the game on $\big(\mathfrak{A},\bar a;\mathfrak{A}',\bar a'\big)$.*

Proof. Let $(\mathfrak{A}, \overline{a}) \not\equiv^{C^k_{\infty\omega}} (\mathfrak{A}', \overline{a}')$. There is some formula φ in $C^k_{\infty\omega}$ such that $\mathfrak{A} \models \varphi[\overline{a}]$ but $\mathfrak{A}' \models \neg\varphi[\overline{a}']$. Let ξ be the quantifier rank of φ. $\xi > 0$ unless **I** has already won. We prove that **I** can in one move force resulting positions that can be distinguished by a formula of quantifier rank $\zeta < \xi$. This suffices to give **I** a strategy, since by repeated application of such moves the ordinal valued quantifier rank of the distinguishing formula must reach 0 in finitely many steps — a win for **I**. Assume without loss of generality that φ is of the form $\exists^{\geqslant m} x_j \psi(\overline{x})$. Other cases reduce to this one through the symmetry of the claim and by replacing φ by one of its boolean constituents if necessary. If **I** chooses pebble index j and proposes a set $B := \{b \in A \mid \mathfrak{A} \models \psi[\overline{a}\frac{b}{j}]\}$ of cardinality m, then **II** cannot help but include at least one element b' in the response B' such that $\mathfrak{A}' \models \neg\psi[\overline{a}'\frac{b'}{j}]$. This is simply because by assumption on φ there are less than m positive examples available over $(\mathfrak{A}', \overline{a}')$. **I** need only choose such a b' from B' to force a resulting position in which ψ of quantifier rank less than ξ distinguishes the two tuples. □

Lemma 2.11. *Player* **II** *has a strategy to maintain* $\equiv^{C^k_{\infty\omega}}$*-equivalence of game positions.*

Proof. Assume $(\mathfrak{A}, \overline{a}) \equiv^{C^k_{\infty\omega}} (\mathfrak{A}', \overline{a}')$. It has to be shown that in response to any choices **I** can make during one round **II** can achieve $\equiv^{C^k_{\infty\omega}}$-equivalence in the resulting positions. From Lemma 1.39 we know that each $C^k_{\infty\omega}$-type α is isolated by some formula $\varphi_\alpha(\overline{x}) \in C^k_{\infty\omega}$. For each α and each j, the number

$$\nu^\alpha_j(\mathfrak{A}, \overline{a}) = \left|\{b \in A \mid \mathrm{tp}^{C^k_{\infty\omega}}_{\mathfrak{A}}(\overline{a}\tfrac{b}{j}) = \alpha\}\right| = \left|\{b \in A \mid \mathfrak{A} \models \varphi_\alpha[\overline{a}\tfrac{b}{j}]\}\right|$$

is determined by $\mathrm{tp}^{C^k_{\infty\omega}}_{\mathfrak{A}}(\overline{a})$: $\exists^{=m} x_j \varphi_\alpha(\overline{x})$ is in $\mathrm{tp}^{C^k_{\infty\omega}}_{\mathfrak{A}}(\overline{a})$ exactly for $m = \nu^\alpha_j(\mathfrak{A}, \overline{a})$. $(\mathfrak{A}, \overline{a}) \equiv^{C^k_{\infty\omega}} (\mathfrak{A}', \overline{a}')$ therefore implies that for all α and j the corresponding numbers must be equal for $(\mathfrak{A}, \overline{a})$ and $(\mathfrak{A}', \overline{a}')$: $\nu^\alpha_j(\mathfrak{A}, \overline{a}) = \nu^\alpha_j(\mathfrak{A}', \overline{a}')$. Suppose now that **I** chooses to play in the j-th component and proposes $B \subseteq A$ as a challenge. By the above equality **II** can choose $B' \subseteq A'$ such that for all α:

$$\left|\{b \in B \mid \mathrm{tp}^{C^k_{\infty\omega}}_{\mathfrak{A}}(\overline{a}\tfrac{b}{j}) = \alpha\}\right| = \left|\{b' \in B' \mid \mathrm{tp}^{C^k_{\infty\omega}}_{\mathfrak{A}'}(\overline{a}'\tfrac{b'}{j}) = \alpha\}\right|.$$

But now, no matter which $b' \in B'$ **I** chooses, **II** can make sure to answer with some $b \in B$ such that the resulting tuples, $\overline{a}\frac{b}{j}$ and $\overline{a}'\frac{b'}{j}$ again realize the same $C^k_{\infty\omega}$-type, so that $\equiv^{C^k_{\infty\omega}}$-equivalence is maintained. □

Before pursuing the analysis of the games, let us remark that unlike the standard treatment of the k-pebble games for $L^k_{\infty\omega}$ and $C^k_{\infty\omega}$ we have chosen to consider only positions with all k pebbles placed on their respective structures. The standard treatment allows to start the game with all pebbles outside the structures. Until the point where all pebbles have been placed player **I** may either choose to play a round using one of the pebbles already placed or one of those not yet used. Otherwise everything is unchanged. That choice has the advantage that the main theorems directly apply to naked structures and characterize the equivalence relations $\equiv^{\mathcal{L}}$ over $\text{fin}[\tau]$ rather than over $\text{fin}[\tau;k]$. The disadvantage is that the games are slightly less uniform during the initial phase in which only some of the pebbles have been placed and the formal treatment must make more or less awkward provisions for that. We do not really lose anything in our restriction to full positions, however, because by Lemma 1.34 $\mathfrak{A} \equiv^{\mathcal{L}} \mathfrak{A}'$ if and only if $\mathfrak{A}$ and $\mathfrak{A}'$ realize exactly the same $\mathcal{L}$-types. As we shall mostly study $\equiv^{\mathcal{L}}$ as an equivalence relation on $\text{fin}[\tau;k]$, we prefer to deal with the variant introduced above.

2.1.3 Further Analysis of the C^k-Game

An inductive analysis of strategies. Think of an arbitrary but fixed k throughout the following. The obvious dependence of various introduced notions on the value of k is mostly suppressed in the notation. Recall that $\text{fin}[\tau;k]$ is the class of all finite τ-structures with a k-tuple of designated elements.

Definition 2.12. *Let $\approx_0$ be the relation of atomic equivalence on* $\text{fin}[\tau;k]$*:*

$$(\mathfrak{A},\overline{a}) \approx_0 (\mathfrak{A}',\overline{a}') \quad \text{if} \quad \text{atp}_{\mathfrak{A}}(\overline{a}) = \text{atp}_{\mathfrak{A}'}(\overline{a}').$$

Recall that atomic equivalence is what is required in the winning condition for player **II**, (W): player **II** has not yet lost in stage $(\mathfrak{A},\overline{a};\mathfrak{A}',\overline{a}')$ if $(\mathfrak{A},\overline{a}) \approx_0 (\mathfrak{A}',\overline{a}')$. Obviously $\approx_0$ is an equivalence relation on positions. A strategy for **II** must specify possible moves for **II** that allow to stay within $\approx_0$ in response to any moves **I** might make. Inductively this task reduces to the specification of strategies for one additional round. Suppose the relation $\approx_i$ on pairs of positions captures the existence of a strategy for at least i moves. Then the corresponding relation $\approx_{i+1}$ must exactly contain all stages (pairs of positions) in which **II** has a strategy for a single round to enforce a resulting stage in $\approx_i$. What constitutes a strategy for the single round is governed by the rules of the game.

Lemma 2.13. *Let $\sim$ be an equivalence relation on* $\text{fin}[\tau;k]$*. Let $\sim'$ be the relation on* $\text{fin}[\tau;k]$ *that contains $(\mathfrak{A},\overline{a};\mathfrak{A}',\overline{a}')$ if and only if $(\mathfrak{A},\overline{a}) \sim (\mathfrak{A}',\overline{a}')$ and in a single round of the C^k-game on stage $(\mathfrak{A},\overline{a};\mathfrak{A}',\overline{a}')$ player* **II** *can force the resulting stage to be in $\sim$ again. Then $\sim'$ is definable as follows:*

$$\begin{aligned}(\mathfrak{A},\overline{a}) \sim' (\mathfrak{A}',\overline{a}') \quad &\textit{if}\\ &(\mathfrak{A},\overline{a}) \sim (\mathfrak{A}',\overline{a}')\\ &\textit{and for all } j \in \{1,\dots,k\} \textit{ and all } \alpha \in \text{fin}[\tau;k]/\!\sim\\ &\left|\{b \in A \mid (\mathfrak{A},\overline{a}\tfrac{b}{j}) \in \alpha\}\right| = \left|\{b' \in A' \mid (\mathfrak{A}',\overline{a}'\tfrac{b'}{j}) \in \alpha\}\right|.\end{aligned}$$

In particular $\sim'$ is also an equivalence relation on $\text{fin}[\tau;k]$.

Proof. i) Suppose first that the condition on the right hand side is satisfied by $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$. The proof that **II** can force $\sim$-equivalence in a single round is very similar to the proof of Lemma 2.11 above. Note that both, the rules for a round in the game and the condition in the lemma are symmetric with respect to the constituent positions $(\mathfrak{A}, \overline{a})$ and $(\mathfrak{A}', \overline{a}')$. Let **I** in the first part of the round choose j and $B \subseteq A$. Split B into disjoint subsets B_α for $\alpha \in \mathrm{fin}[\tau; k] /{\sim}$ through: $B_\alpha := \{b \in B \mid (\mathfrak{A}, \overline{a}\frac{b}{j}) \in \alpha\}$. By assumption, there exists for each B_α a subset $B'_\alpha \subseteq \{b' \in A' \mid (\mathfrak{A}', \overline{a}'\frac{b'}{j}) \in \alpha\}$ of exactly the same size as B_α. Note that the sets $\{b' \in A' \mid (\mathfrak{A}', \overline{a}'\frac{b'}{j}) \in \alpha\}$ are disjoint for different α. If **II** responds with $B' := \bigcup_\alpha B'_\alpha$ then, in the second exchange of moves in this round, **II** can force $\sim$-equivalence as desired: **I** chooses $b' \in B'_{\alpha_0}$ for some α_0; **II** need merely choose b from B_{α_0} to ensure $(\mathfrak{A}, \overline{a}\frac{b}{j}) \sim (\mathfrak{A}', \overline{a}'\frac{b'}{j})$ since both positions are in α_0.
ii) Suppose now that the condition on the right hand side is not satisfied. The interesting case is that this is not due to $\sim$-inequivalence. We show how **I** can force a successor stage that is not in $\sim$. By symmetry we may assume that for some j and α, $\left|\{b \in A \mid (\mathfrak{A}, \overline{a}\frac{b}{j}) \in \alpha\}\right| > \left|\{b' \in A' \mid (\mathfrak{A}', \overline{a}'\frac{b'}{j}) \in \alpha\}\right|$. Let **I** choose this j and $B := \{b \in A \mid (\mathfrak{A}, \overline{a}\frac{b}{j}) \in \alpha\}$. Whichever B' of the same size as B player **II** chooses, there has to be some $b' \in B'$ such that $(\mathfrak{A}', \overline{a}'\frac{b'}{j})$ is not in α. If **I** chooses such b' a resulting stage with $\sim$-inequivalent positions is forced. □

Definition 2.14. *Define a family of binary relations $\approx_i$ on* $\mathrm{fin}[\tau; k]$ *as follows:*

$(\mathfrak{A}, \overline{a}) \approx_i (\mathfrak{A}', \overline{a}')$ *if* *Player **II** has a strategy for at least i rounds in the C^k-game on $(\mathfrak{A}, \overline{a}; \mathfrak{A}', \overline{a}')$.*

Note that the above definition of $\approx_0$ as equality of atomic types is consistent with this new definition. Lemma 2.13 can be applied to generate inductively equivalence relations $\approx_i$ that capture the existence of a strategy for at least i moves. Obviously $\approx_{i+1}$ is obtained from $\approx_i$ through the refinement step described in Lemma 2.13, $\approx_{i+1} = (\approx_i)'$.

In particular it follows inductively from the condition in Lemma 2.13 that all the $\approx_i$ are equivalence relations on $\mathrm{fin}[\tau; k]$. For future reference we present the inductive description of the $\approx_i$ in detail.

Proposition 2.15. *Let the $\approx_i$ on* $\mathrm{fin}[\tau; k]$ *be defined through the existence of a strategy for player **II** for at least i rounds in the C^k-game. Then these are inductively definable in the following process:*

$(\mathfrak{A}, \overline{a}) \approx_0 (\mathfrak{A}', \overline{a}')$ *iff* $\mathrm{atp}_{\mathfrak{A}}(\overline{a}) = \mathrm{atp}_{\mathfrak{A}'}(\overline{a}')$
$(\mathfrak{A}, \overline{a}) \approx_{i+1} (\mathfrak{A}', \overline{a}')$ *iff*
$(\mathfrak{A}, \overline{a}) \approx_i (\mathfrak{A}', \overline{a}')$
and for all $j \in \{1, \ldots, k\}$ and all $\alpha \in \mathrm{fin}[\tau; k] /{\approx_i}$
$\left|\{b \in A \mid (\mathfrak{A}, \overline{a}\frac{b}{j}) \in \alpha\}\right| = \left|\{b' \in A' \mid (\mathfrak{A}', \overline{a}'\frac{b'}{j}) \in \alpha\}\right|.$

As a sequence of successively refined equivalence relations the $\approx_i$ possess a limit or roughest common refinement. Formally this limit $\approx$ is the intersection of all $\approx_i$ for $i \in \omega$:

$$\approx_i \xrightarrow{i\to\infty} \approx = \bigcap_i \approx_i .$$

We show that $\approx$ captures the existence of a strategy in the infinite game.

Lemma 2.16. *Let* $\approx := \bigcap_i \approx_i$. *Then*

$(\mathfrak{A},\overline{a}) \approx (\mathfrak{A}',\overline{a}')$ *iff* *Player* **II** *has a strategy in the infinite* C^k*-game on* $(\mathfrak{A},\overline{a};\mathfrak{A}',\overline{a}')$.

Proof. This is the first place in the analysis of the games where we use the finiteness of the underlying structures. Fix two structures $\mathfrak{A},\mathfrak{A}'$ and let $\approx^{\mathfrak{A}\mathfrak{A}'}$ and $\approx_i^{\mathfrak{A}\mathfrak{A}'}$ stand for the restrictions of $\approx$ and $\approx_i$ to positions over $\mathfrak{A}$ and $\mathfrak{A}'$. Thus $\approx^{\mathfrak{A}\mathfrak{A}'}$ is the limit of the decreasing sequence of subsets $\approx_i^{\mathfrak{A}\mathfrak{A}'}$ of the finite set $A^k \times A'^k$. It follows that $\approx_{i+1}^{\mathfrak{A}\mathfrak{A}'} = \approx_i^{\mathfrak{A}\mathfrak{A}'} = \approx^{\mathfrak{A}\mathfrak{A}'}$ for some i. But this means that for such i and in games over $\mathfrak{A}$ and $\mathfrak{A}'$ player **II** is guaranteed to have a strategy for at least $i+1$ rounds whenever there is a strategy for at least i rounds. The strategy in the infinite game now simply is to maintain $\approx_i^{\mathfrak{A}\mathfrak{A}'}$-equivalence: $\approx_i^{\mathfrak{A}\mathfrak{A}'}$-equivalence implies $\approx_{i+1}^{\mathfrak{A}\mathfrak{A}'}$-equivalence and this can by definition be used to enforce $\approx_i^{\mathfrak{A}\mathfrak{A}'}$-equivalence in each consecutive round. □

Equivalence of positions and equality of types. We can now show that the $\approx$-classes coincide with the $C^k_{\omega\omega}$-types as well as with the $C^k_{\infty\omega}$-types over fin$[\tau]$. This correspondence in particular yields a proof of Corollary 2.4. Recall form Definition 1.36 that the $C^k_{\infty\omega;i}$ consist of all those formulae of $C^k_{\infty\omega}$ whose quantifier rank is at most i. By what we already have, it suffices to show that $\approx_i$ is equivalence in $C^k_{\infty\omega;i}$ for all $i \in \omega$. For then, the following limit equations prove the claim:

$$\begin{array}{ccc}
\equiv^{C^k_{\omega\omega;i}} & \xrightarrow{i\to\infty} & \equiv^{C^k_{\omega\omega}} \\
\| & & \\
\equiv^{C^k_{\infty\omega;i}} & & \\
\| & & \\
\approx_i & \xrightarrow{i\to\infty} & \approx
\end{array}$$

The indicated limits are clear: $C^k_{\omega\omega} = \bigcup_i C^k_{\omega\omega;i}$ so that $C^k_{\omega\omega}$-equivalence is the limit of the equivalences with respect to the $C^k_{\omega\omega;i}$. $\approx = \bigcap_i \approx_i$ by the definition of $\approx$.

Coincidence between $C^k_{\omega\omega;i}$-equivalence and $C^k_{\infty\omega;i}$-equivalence follows from our preliminary analysis in Chapter 1, see Corollary 1.40. But from Theorem 2.2 and Lemma 2.16 we already know that $\approx$ is $C^k_{\infty\omega}$-equivalence. It follows that indeed on fin$[\tau; k]$ all three notions of equivalence

$$\equiv^{C^k_{\omega\omega}}, \equiv^{C^k_{\infty\omega}}, \text{ and } \approx$$

coincide. This is precisely the statement of Corollary 2.4. It remains to prove inductively the coincidence between $\approx_i$ and $C^k_{\infty\omega;i}$-equivalence.

Lemma 2.17. *The equivalence relation* $\approx_i$ *coincides with* $C^k_{\infty\omega;i}$*-equivalence on* fin$[\tau; k]$ *for all* $i \in \omega$.

Proof. By induction on i. The claim is true for $i = 0$ by definition. Recall from Proposition 2.15 how $\approx_{i+1}$ is characterized in terms of $\approx_i$:

$$\begin{aligned}(\mathfrak{A}, \overline{a}) \approx_{i+1} (\mathfrak{A}', \overline{a}') \quad \text{iff} \quad & \\ & (\mathfrak{A}, \overline{a}) \approx_i (\mathfrak{A}', \overline{a}') \\ & \text{and for all } j \in \{1, \dots, k\} \text{ and all } \alpha \in \text{fin}[\tau; k] /{\approx_i} \\ & \left|\{b \in A \mid (\mathfrak{A}, \overline{a}\tfrac{b}{j}) \in \alpha\}\right| = \left|\{b' \in A' \mid (\mathfrak{A}', \overline{a}'\tfrac{b'}{j}) \in \alpha\}\right|.\end{aligned}$$

It suffices to prove the following, which says that the $\equiv^{C^k_{\infty\omega;i}}$ are governed by the same rules:

$$\begin{aligned}(\mathfrak{A}, \overline{a}) \equiv^{C^k_{\infty\omega;i+1}} (\mathfrak{A}', \overline{a}') \quad \text{iff} \quad & \\ & (\mathfrak{A}, \overline{a}) \equiv^{C^k_{\infty\omega;i}} (\mathfrak{A}', \overline{a}') \\ & \text{and for all } j \in \{1, \dots, k\} \text{ and all } \alpha \in \text{fin}[\tau; k] /{\equiv^{C^k_{\infty\omega;i}}} \\ & \left|\{b \in A \mid (\mathfrak{A}, \overline{a}\tfrac{b}{j}) \in \alpha\}\right| = \left|\{b' \in A' \mid (\mathfrak{A}', \overline{a}'\tfrac{b'}{j}) \in \alpha\}\right|.\end{aligned}$$

The "only if"–part is clear, since by Lemma 1.39 each $\equiv^{C^k_{\infty\omega;i}}$-class α is isolated by a formula $\varphi_\alpha(\overline{x}) \in C^k_{\infty\omega;i}$. Therefore, if

$$\nu^\alpha_j(\mathfrak{A}, \overline{a}) := \left|\{b \in A \mid (\mathfrak{A}, \overline{a}\tfrac{b}{j}) \in \alpha\}\right|,$$

then $\exists^{=m} x_j \varphi_\alpha(\overline{x})$ is in the $C^k_{\infty\omega;i+1}$-type of $(\mathfrak{A}, \overline{a})$ for $m = \nu^\alpha_j(\mathfrak{A}, \overline{a})$. For the "if"–part it suffices to show that the numbers $\nu^\alpha_j(\mathfrak{A}, \overline{a})$, for all α and j isolate the $C^k_{\infty\omega;i+1}$-type of $(\mathfrak{A}, \overline{a})$. This, however, is clear: whether $\mathfrak{A} \models \exists^{\geqslant m} x_j \psi[\overline{a}]$ for $\psi \in C^k_{\infty\omega;i}$ is determined by $\sum \nu^\alpha_j(\mathfrak{A}, \overline{a})$ for those α that contain ψ. □

Since we only deal with finite structures we henceforth identify $\equiv^{C^k_{\infty\omega}}$ and $\equiv^{C^k_{\omega\omega}}$ and indistinguishably write $\equiv^{C^k}$. Correspondingly, the distinction between $C^k_{\infty\omega}$- and $C^k_{\omega\omega}$-types is dropped and we may simply speak of C^k-types over finite structures.

Referring back to the inductive generation of the $\approx_i$ as characterized in Proposition 2.15 and combining this with the insight that the limit of the $\approx_i$

is C^k-equivalence, we have the following rather algebraic characterization of $\equiv^{C^k}$ over $\text{fin}[\tau; k]$.

Remark 2.18. $\equiv^{C^k}$ *on* $\text{fin}[\tau; k]$ *is the roughest equivalence relation* $\approx$ *on* $\text{fin}[\tau; k]$ *that is at least as fine as atomic equivalence and satisfies the following fixed-point equation:*

$$\begin{aligned}&(\mathfrak{A}, \overline{a}) \approx (\mathfrak{A}', \overline{a}')\\ &\Longleftrightarrow\\ &\textit{for all } j \in \{1, \dots, k\} \textit{ and all } \alpha \in \text{fin}[\tau; k] \,/\!\approx\\ &\left|\left\{b \in A \mid (\mathfrak{A}, \overline{a}\tfrac{b}{j}) \in \alpha\right\}\right| = \left|\left\{b' \in A' \mid (\mathfrak{A}', \overline{a}'\tfrac{b'}{j}) \in \alpha\right\}\right|.\end{aligned}$$

The fixed-point equation directly corresponds with the equation that governs the refinement step $\approx_i \longmapsto \approx_{i+1}$ in Proposition 2.15.

2.1.4 The Analogous Treatment for L^k

Both, the proof of Theorem 2.1 and the analysis of the L^k-game that leads to Corollary 2.3, are carried out along exactly the same lines as for the C^k-game. The more transparent rules for the single round, however, lead to considerable simplifications. The inductive generation of the corresponding equivalence relations $\approx_i$ on game positions is formally much simpler, though strictly analogous in spirit. Instead of Proposition 2.15 we now find the following.

Proposition 2.19. *With the* $\approx_i$ *defined through the existence of a strategy for player* **II** *in the* L^k*-game, these are inductively definable as follows:*

$$\begin{aligned}&(\mathfrak{A}, \overline{a}) \approx_0 (\mathfrak{A}', \overline{a}') \quad \textit{iff} \quad \text{atp}_{\mathfrak{A}}(\overline{a}) = \text{atp}_{\mathfrak{A}'}(\overline{a}')\\ &(\mathfrak{A}, \overline{a}) \approx_{i+1} (\mathfrak{A}', \overline{a}') \quad \textit{iff}\\ &\qquad (\mathfrak{A}, \overline{a}) \approx_i (\mathfrak{A}', \overline{a}')\\ &\qquad \textit{and for all } j \in \{1, \dots, k\} \textit{ and all } \alpha \in \text{fin}[\tau; k] \,/\!\approx_i\\ &\qquad \exists b \in A\Big((\mathfrak{A}, \overline{a}\tfrac{b}{j}) \in \alpha\Big) \longleftrightarrow \exists b' \in A'\Big((\mathfrak{A}', \overline{a}'\tfrac{b'}{j}) \in \alpha\Big).\end{aligned}$$

The limit $\approx_i \xrightarrow{i \to \infty} \approx$, where the equivalence relations $\approx_i$ now stand for equivalence with respect to the L^k-game, becomes equality of $L^k_{\infty\omega}$-types over finite structures. The $\approx_i$ also correspond to indistinguishability in the bounded quantifier rank fragments $L^k_{\infty\omega;i}$ of $L^k_{\infty\omega}$. $L^k_{\infty\omega;i}$-equivalence is the same as $L^k_{\omega\omega;i}$-equivalence by Corollary 1.40. Thus,

$$\equiv^{L^k_{\omega\omega}}, \equiv^{L^k_{\infty\omega}}, \text{ and } \approx$$

coincide, where $\approx$ now is equivalence in the L^k-game. This is precisely the statement of Corollary 2.3.

It is therefore justified to write $\equiv^{L^k}$ for both, equivalence in $L^k_{\infty\omega}$ or $L^k_{\omega\omega}$. Accordingly we identify $L^k_{\infty\omega}$-types and $L^k_{\omega\omega}$-types over finite structures and address them as L^k-types.

Finally an algebraic characterization of $\equiv^{L^k}$ in the style of Remark 2.18 is obtained: $\equiv^{L^k}$ is the roughest equivalence relation $\approx$ on $\text{fin}[\tau; k]$ that is at least as fine as atomic equivalence and satisfies the following fixed-point equation:

$$
\begin{aligned}
&(\mathfrak{A}, \overline{a}) \approx (\mathfrak{A}', \overline{a}') \\
&\Longleftrightarrow \\
&\text{for all } j \in \{1, \ldots, k\} \text{ and all } \alpha \in \text{fin}[\tau; k] \,/\!\approx \\
&\exists b \in A\Big((\mathfrak{A}, \overline{a}\tfrac{b}{j}) \in \alpha\Big) \longleftrightarrow \exists b' \in A'\Big((\mathfrak{A}', \overline{a}'\tfrac{b'}{j}) \in \alpha\Big).
\end{aligned}
$$

2.2 Colour Refinement and the Stable Colouring

This section is an intermezzo on our way to obtain definable orderings with respect to C^k- and L^k-types. The basic technique in the underlying inductive processes is intimately related to a similar technique in combinatorial graph theory: the *colour refinement technique* and the *stable colouring*, often also considered under the name of *vertex classification*. We review these notions in some detail and consider variants that are useful in the present development. In particular some definability properties of variants of the stable colouring can later directly be transferred to definability statements for the invariants.

We use the terminology of *pre-orderings* as reviewed in Section 1.7. In particular compare Definition 1.62. We reserve variants of the symbol $\preccurlyeq$ to denote pre-orderings; $\prec$ then denotes the associated strict pre-ordering and $\sim$ the induced equivalence relation. Recall that the quotient $\preccurlyeq/\!\sim$ is a linear ordering in the sense of $\leqslant$, $\prec/\!\sim$ the corresponding linear ordering in the sense of $<$. Intuitively $\sim$ describes the *discriminating power* of $\preccurlyeq$. Recall that $\prec$ and $\preccurlyeq$ are quantifier free interdefinable and that $\sim$ is quantifier free definable form either.

2.2.1 The Standard Case: Colourings of Finite Graphs

Let (V, E) be a finite graph. A colouring of (V, E) with finitely many colours $0, \ldots, r-1$ is a function $c: V \to r$, where $r = \{0, \ldots, r-1\}$ as usual. We regard this set of colours as ordered in the natural way. To make the order in the colours explicit, the colouring may be formalized as a pre-ordering on V: $v_1 \preccurlyeq v_2$ if $c(v_1) \leqslant c(v_2)$. The associated $\sim$ is the relation of having the same colour. A particular refinement of c is induced by the following mapping:

$$
c': v \longmapsto \Big(c(v), \big|\{w | Evw \wedge c(w) = 0\}\big|, \ldots, \big|\{w | Evw \wedge c(w) = r-1\}\big|\Big).
$$

Let $\sim'$ be the relation of having the same new colour. Obviously $v_1 \sim' v_2$ if and only if v_1 and v_2 have the same colour under c and the same numbers of direct neighbours in any of the c-colours. We note the similarity of this refinement process with that encountered in the refinement for equivalence of positions in the C^k-game as expressed in Lemma 2.13.

The new colours can be ordered lexicographically so that one may also regard c' as a mapping into some initial subset $r' = \{0, \ldots, r'-1\}$ of natural numbers. With our conventions for lexicographic orderings (see Section 1.7.3) the colours under c' get ordered with dominating c-colour.

The new c' is the *colour refinement* of c. Let $\preccurlyeq'$, $\prec'$ and $\sim'$ be the characteristic descriptions of c' in terms of pre-orderings. The colouring c' is a refinement of c in the sense that $\sim'$ is a refinement of $\sim$ and that for $\prec$ and $\prec'$ we have: $\prec \subseteq \prec'$. The discriminating power of the colouring is possibly enhanced in the passage from c to c', but the new ordering of colours is compatible with the former one.

Since (V, E) is finite, repeated colour refinement must terminate in a stationary colouring after at most $|V|$ steps. In the standard graph theoretic setting this limit process is applied to the trivial monochromatic colouring $c_0 : V \rightarrow \{0\}$. Note that this trivial colouring corresponds to the pre-ordering $\preccurlyeq_0 = V \times V$ (with associated strict pre-ordering $\prec_0 = \emptyset$). The limit colouring obtained in this way is called the ***stable colouring*** of the graph. At the level of the associated strict pre-orderings the stable colouring is the ***least fixed point*** of the monotone operator corresponding to the single colour refinement step sending $\prec$ to $\prec'$:

$$\prec = \bigcup_i \prec_i \qquad \text{where} \quad \prec_0 = \emptyset \qquad \text{and} \quad \prec_{i+1} = (\prec_i)'.$$

The first successor level $\prec_1$ is just the pre-ordering according to the degree of vertices. Note that the description of the refinement process is monotone increasing in terms of $\prec$ and monotone decreasing in terms of $\sim$ and $\preccurlyeq$.

2.2.2 Definability of the Stable Colouring

A slight generalization of the setting in which the colour refinement technique is applicable concerns k-graphs with any given initial pre-ordering on the set of vertices. We use the term ***k-graph*** to denote structures with k binary relations $E_1, \ldots, E_k$ instead of the single edge relation in the standard case. Also we here need not require these relations to be irreflexive or symmetric. An additional arbitrary pre-ordering $\preccurlyeq_0$ serves as an initial stage for the colour refinement. In terms of colourings we now pass from a colouring $c: V \rightarrow r$ to a refinement c' obtained from a lexicographic ordering of the new colours

$$c' : v \longmapsto \Big(c(v), \big(\nu_j^s(v)\big)_{1 \leqslant j \leqslant k, 0 \leqslant s < r} \Big),$$

where $\nu_j^s(v) = \big|\{w | E_j vw \wedge c(w) = s\}\big|$.

Recall once more our conventions for the lexicographic ordering: a new colour $\overline{m} = (m, (m_{js}))$ is regarded as a tuple with first component m and consecutive components m_{js} listed according to the lexicographic ordering on the index pairs (j,s). For $\overline{m} = (m, (m_{js}))$ and $\overline{m}' = (m', (m'_{js}))$ we get that $\overline{m} < \overline{m}'$ if $m < m'$ or if $m = m'$ and $m_{js} < m'_{js}$ for the least (j,s) such that $m_{js} \neq m'_{js}$.

For the description in terms of the associated pre-orderings $\preccurlyeq$ and $\preccurlyeq'$ with corresponding strict $\prec$ and $\prec'$ and equivalences $\sim$ and $\sim'$ this becomes:

$$\begin{aligned} v_1 \prec' v_2 \quad & \text{iff} \\ & v_1 \prec v_2 \quad \text{or} \\ & v_1 \sim v_2 \text{ and } \big(\nu_j^s(v_1)\big) <_{\text{lex}} \big(\nu_j^s(v_2)\big), \\ & \text{where } \nu_j^s(v) = \big|\{w | E_j vw \wedge c(w) = s\}\big|. \end{aligned} \tag{2.1}$$

The structural similarity of this refinement process with that in Proposition 2.15 is most apparent for the associated equivalences $\sim$ and $\sim'$:

$$\begin{aligned} v_1 \sim' v_2 \quad & \text{iff} \\ & v_1 \sim v_2 \text{ and for all } j \in \{1,\dots,k\} \text{ and all } \alpha \in V/\sim \\ & \Big|\{u \in V \mid E_j v_1 u \wedge u \in \alpha\}\Big| = \Big|\{u \in V \mid E_j v_2 u \wedge u \in \alpha\}\Big|. \end{aligned} \tag{2.2}$$

Definition 2.20. *The* stable colouring *of a pre-ordered finite k-graph is the limit pre-ordering $\preccurlyeq$ obtained through application of the above refinement operation with the given $\preccurlyeq_0$ as the initial stage:*

$$\preccurlyeq \text{ is the limit } \preccurlyeq_i \xrightarrow{i \to \infty} \preccurlyeq \qquad \text{where inductively } \preccurlyeq_{i+1} := (\preccurlyeq_i)'.$$

We regard the $\preccurlyeq_i$ and $\preccurlyeq$ as global relations on finite pre-ordered k-graphs.

The standard version of the stable colouring of graphs is comprised as a special case for $k = 1$ and for trivial initial pre-ordering $\preccurlyeq_0 = V \times V$. In this form the following result is due to Immerman and Lander, see Theorem 2.23 below.

Lemma 2.21. *The stable colouring $\preccurlyeq$ of finite pre-ordered k-graphs is definable in $C^2_{\infty\omega}$.*

Proof. Let $\prec_i$ and $\prec$ stand for the associated strict pre-orderings, $\sim_i$ and $\sim$ for the induced equivalences. It is sufficient to show that each level $\prec_i$ in the fixed-point process that generates $\prec$ is definable by some $C^2_{\infty\omega}$-formula $\varphi_i(x,y)$. Then the limit of the sequence $\prec_0 \subseteq \prec_1 \subseteq \cdots$ is defined by

$$\varphi(x,y) := \bigvee_{i\in\omega} \varphi_i(x,y).$$

i) Suppose that φ_i defines $\prec_i$. Then for each $s \geqslant 0$ there is a formula $\psi_{i,s}(x)$ of $C^2_{\infty\omega}$ in a single free variable which defines the s-th equivalence class with respect to $\sim_i$ in the sense of the ordering $\prec_i$. We first generate auxiliary $\chi_{i,s}(x)$ that define the union of the classes up to s: $\chi_{i,0}(x) := \neg\exists y(\varphi_i(y,x))$

defines the $\prec_i$-least $\sim_i$-class. As usual, $\varphi_i(y,x)$ is the result of exchanging all occurrences of x and y in $\varphi_i(x,y)$. Inductively let $\chi_{i,s+1}(x) := \forall y(\varphi_i(y,x) \rightarrow \chi_{i,s}(y))$. Finally $\psi_{i,s}(x) := \chi_{i,s}(x) \wedge \neg\chi_{i,s-1}(x)$ is as desired.
ii) Definability of the $\prec_i$ is established by an induction with respect to i. $\varphi_0(x,y) := x \preccurlyeq_0 y \wedge \neg\, y \preccurlyeq_0 x$ defines $\prec_0$ as the strict variant of the given $\preccurlyeq_0$. Recall from the definitions that

$$\begin{aligned} x \prec_{i+1} y \quad & \text{iff} \\ & x \prec_i y \quad \text{or} \\ & x \sim_i y \text{ and } \big(\nu_j^s(x)\big) <_{\text{lex}} \big(\nu_j^s(y)\big). \end{aligned} \tag{2.3}$$

$\nu_j^s(x) = \big|\{z|E_jxz \wedge \psi_{i,s}(z)\}\big|$ is the number of E_j-neighbours to x that are in the s-th class with respect to $\sim_i$.

The crucial lexicographic comparison $\big(\nu_j^s(x)\big) <_{\text{lex}} \big(\nu_j^s(y)\big)$ can be expressed as follows:

$$\bigvee_{(j,s)} \Big(\bigwedge_{(j',s')<(j,s)} \nu_{j'}^{s'}(x) = \nu_{j'}^{s'}(y) \;\; \wedge \;\; \nu_j^s(x) < \nu_j^s(y) \Big).$$

Since $\nu_j^s(x) = \big|\{y \mid E_jxy \wedge \psi_{i,s}(y)\}\big|$ it only remains to dissolve the cardinality equalities and inequalities in the last formula into infinite disjunctions according to the following pattern:

$$\Big|\{u|\chi(x,u)\}\Big| < \Big|\{u|\chi(y,u)\}\Big| \iff \bigvee_{m<n} \Big(\exists^{=m} y\, \chi(x,y) \wedge \exists^{=n} x\, \chi(y,x)\Big).$$

□

Beside infinitary definability in only two variables with counting we also get definability in an extension of fixed-point logic just sufficiently expressive to permit cardinality comparison. Recall the definition of the Rescher quantifier from Definition 1.53.

Lemma 2.22. *The stable colouring $\preccurlyeq$ of finite pre-ordered k-graphs is globally definable in* FP(Q_{R}), *fixed-point logic with the Rescher quantifier. In particular it is computable in* PTIME.

Proof. Note that equation 2.3 for the inductive refinement is directly adequate for the definition of $\prec$ as an inductive fixed point. Only, in standard fixed-point processes we initialize the fixed-point variable to $\emptyset$, whereas here we want to substitute the given $\prec_0$ for the initial stage. This is possible with the following standard trick. To obtain the inductive fixed-point for the operator given by $\chi(X,\overline{x})$ but with initialization to an X_0 defined by some $\varphi_0(\overline{x})$ one may use the usual inductive fixed-point over $\chi'(X,\overline{x}) = \big(\neg\exists\overline{x}X\overline{x} \wedge \varphi_0(\overline{x})\big) \vee \big(\exists\overline{x}X\overline{x} \wedge \chi(X,\overline{x})\big)$.

It therefore suffices to show that the lexicographic comparison in equation 2.3 is definable with the Rescher quantifier. $\big(\nu_j^s(x)\big) <_{\text{lex}} \big(\nu_j^s(y)\big)$ can now be reformulated as follows:

$$
\begin{aligned}
&\big(\nu_j^s(x)\big) <_{\text{lex}} \big(\nu_j^s(y)\big) \\
\Leftrightarrow\quad & \exists (j,s)\Big[\forall (j',s')\big((j',s') < (j,s) \rightarrow \nu_{j'}^{s'}(x) = \nu_{j'}^{s'}(y)\big) \;\wedge\; \nu_j^s(x) < \nu_j^s(y)\Big] \\
\Leftrightarrow\quad & \bigvee_{j=1}^{k} \exists s \left[\begin{array}{c} \bigwedge_{j'<j} \forall s'\big(\nu_{j'}^{s'}(x) = \nu_{j'}^{s'}(y)\big) \;\wedge\; \forall s'{<}s\,\big(\nu_j^{s'}(x) = \nu_j^{s'}(y)\big) \\ \wedge\; \nu_j^s(x) < \nu_j^s(y) \end{array}\right].
\end{aligned}
$$

The quantifications over s and s' can be replaced by quantifications over elements z and z' that represent the s-th and s'-th classes with respect to $\sim_i$. If for instance z is in the s-th $\sim_i$-class then $\nu_j^s(x) = \big|\{u \mid E_j x \wedge u \sim_i z\}\big|$. It follows that the cardinality equalities and comparisons in the above formulae can be expressed with applications of Q_{R}. Thus $\big(\nu_j^s(x)\big) <_{\text{lex}} \big(\nu_j^s(y)\big)$ is in first-order logic with the Rescher quantifier in terms of $\prec_i$.

The limit $\prec$, and with it $\preccurlyeq$, therefore are definable in $\text{FP}(Q_{\text{R}})$. □

2.2.3 $C^2_{\infty\omega}$ and the Stable Colouring

For this section we return to the standard case of the stable colouring, with just one edge relation E and initialization to the trivial pre-ordering. Lemma 2.21 was first stated by Immerman and Lander [IL90] in this form:

Theorem 2.23 (Immerman, Lander). *The stable colouring of graphs is $C^2_{\infty\omega}$-definable in the finite: there is a $C^2_{\infty\omega}$-formula $\eta(x,y)$ defining on all finite graphs the pre-ordering associated with the stable colouring.*

The stable colouring receives special attention in graph theory since on generic graphs it provides canonization up to isomorphism. On *almost all* finite graphs the pre-ordering associated with the stable colouring is a linear ordering. This result is due to Babai, Erdös and Selkow [BES80]. The 'almost all' is to say that the proportion of graphs of size n satisfying the statement tends to 1 as n goes to infinity. In [BK80] this result was further used to provide a graph normalization algorithm that operates in average linear time.

Theorem 2.24 (Babai, Erdös, Selkow). *For almost all finite graphs the stable colouring gives different colours to any two distinct vertices. In other words, almost all finite graphs are in fact linearly ordered (in the sense of $\leqslant$) by the pre-ordering $\preccurlyeq$ associated with the stable colouring. It follows that almost all finite graphs are characterized up to isomorphism by their $C^2_{\infty\omega}$-theories, hence also by their $C^2_{\omega\omega}$-theories.*

Immerman and Lander proved that not only is the stable colouring C^2-definable, but it exactly classifies vertices up to C^2-equivalence:

Theorem 2.25 (Immerman, Lander). *The equivalence relation $\sim$ associated with the stable colouring of finite graphs is equality of C^2-types of singletons. The associated pre-ordering $\preccurlyeq$ therefore is a pre-ordering with respect to C^2-types of single vertices.*

Sketch of Proof. Let $\mathfrak{G} = (V, E)$ be a graph. It suffices to show that $u \sim u'$ for $u, u' \in V$ implies that player **II** has a strategy in the infinite game on $(\mathfrak{G}, uu; \mathfrak{G}, u'u')$. Then $\sim$ is at least as fine as equality of C^2-types. It cannot be strictly finer because each $\sim$-class is $C^2_{\infty\omega}$-definable as we have seen in the proof of Lemma 2.21. We show that player **II** can maintain the following condition on game positions $(\mathfrak{G}, uv)$ and $(\mathfrak{G}, u'v')$:

$$(*) \quad u \sim u' \text{ and } v \sim v' \quad \text{and} \quad \mathrm{atp}_{\mathfrak{G}}(u, v) = \mathrm{atp}_{\mathfrak{G}}(u', v').$$

Let this condition be satisfied in the current stage $(\mathfrak{G}, uv; \mathfrak{G}, u'v')$. Assume without loss of generality that player **I** chooses to play in the second component, $j = 2$, and proposes $B \subseteq V$ as a subset over the first copy of $\mathfrak{G}$. Let the colour classes in $V/\sim$ be enumerated as $\alpha_1, \ldots, \alpha_l$. Split B into colour classes $B_i = B \cap \alpha_i$. Since $u \sim u'$ and since $\sim = \sim'$ is stationary with respect to a further colour refinement step, we have for all α_i:

$$\left|\{w \mid Euw \wedge w \in \alpha_i\}\right| = \left|\{w' \mid Eu'w' \wedge w' \in \alpha_i\}\right|.$$

It follows that also $\left|\{w \mid \neg Euw \wedge w \in \alpha_i\}\right| = \left|\{w' \mid \neg Eu'w' \wedge w' \in \alpha_i\}\right|$. Therefore **II** can choose subsets $B'_i \subseteq \alpha_i$ such that u' has exactly as many E-neighbours and non-neighbours in B'_i as u has in B_i. Let **II** put $B' = \bigcup_i B'_i$. If **I** now chooses for instance a neighbour of u' in B'_i, then **II** can answer with a neighbour of u from B_i. Thus $(*)$ is realized in the resulting stage again. □

2.2.4 A Variant Without Counting

There is also an inductively definable pre-ordering adapted to capture the refinement that corresponds to the moves in the ordinary pebble game for L^k. Its definition does not require cardinality comparison so that it turns out to be FP-definable. In fact, the rôle of cardinality comparisons in the colour refinement is taken by the boolean distinction whether or not there are neighbours (no matter how many) of respective kinds. Consider some colouring $c: V \to r$ on a k-graph. For the refinement step pass to a new colouring

$$c': v \longmapsto \Big(c(v), \big(d^s_j(v)\big)_{1 \leqslant j \leqslant k, 0 \leqslant s < r}\Big),$$

$$\text{where} \quad d^s_j(v) := \begin{cases} 0 & \text{if } \neg\exists w\big(E_j vw \wedge c(w) = s\big) \\ 1 & \text{if } \exists w\big(E_j vw \wedge c(w) = s\big). \end{cases}$$

Note that the entries in all but the first component are boolean values. These take the place of cardinalities in the colour refinement. The new colours are ordered lexicographically just as in the colour refinement. The corresponding refinement in the associated strict pre-orderings can easily be described in a form analogous to condition 2.1 on page 69.

Starting from a pre-ordered k-graph and applying this refinement procedure inductively, a limit pre-ordering is obtained. Let us call this resulting pre-ordering the *Abiteboul-Vianu colouring* of the pre-ordered k-graph.

In complete analogy with the proofs of Lemmas 2.21 and 2.22 above, we find that the Abiteboul-Vianu colouring of pre-ordered k-graphs is globally $L^2_{\infty\omega}$-definable as well as FP-definable.

We shall see in the next sections that the Abiteboul-Vianu colouring serves to construct global pre-orderings with respect to L^k-types just as the stable colouring serves to construct similar pre-orderings with respect to C^k-types. We have seen in Theorem 2.25 a first indication in this direction: the standard stable colouring of graphs provides a global pre-ordering of C^2-types of singletons. It may similarly be shown that the Abiteboul-Vianu colouring is a pre-ordering of L^2-types of singletons.

2.3 Order in the Analysis of the Games

The desired ordering with respect to types is obtained through an ordered classification of positions in the corresponding game. Formally the ordering of the quotients $A^k/\equiv^{\mathcal{L}}$ gets interpreted over each structure $\mathfrak{A}$ through a pre-ordering on the k-th power of the universe. The associated equivalence relation will be equality of types. We have seen a special case of this idea in Theorem 2.25. In the following we present the introduction of the desired pre-orderings in two different approaches, each with its specific advantages.

(a) The first view is an *internal* one in the sense that the pre-ordering is defined as a global relation on the game positions over each individual $\mathfrak{A}$ without reference to positions over other structures. This development is a direct application of the stable colouring to some k-graph associated with each individual $\mathfrak{A}$. From Section 2.2 we infer definability properties of the resulting pre-ordering as a global relation on $\mathrm{fin}[\tau]$.

(b) The other, and indeed more comprehensive, view defines the desired pre-ordering as a pre-ordering on $\mathrm{fin}[\tau; k]$, i.e. as a relation that serves to compare game positions over different structures. In this sense it involves considerations that are *external* to the individual structures. This is in good agreement, however, with the game analysis in terms of the equivalence relations $\approx$. These also primarily are equivalences over $\mathrm{fin}[\tau; k]$. Only their restrictions to the special case that both positions are over the same structure are global relations over $\mathrm{fin}[\tau]$.

Both views are presented in the following. The externally defined pre-ordering agrees with the internally defined one in restriction to each individual structure so that both views contribute to the understanding of the pre-ordering as a global relation. In order not to overburden notation we shall not distinguish between the two notationally. Wherever it matters it will be clear from context which view is intended.

We explicitly treat the case with counting quantifiers first and indicate the analogous treatment for the L^k in the sequel.

2.3.1 The Internal View

We introduce the desired orderings on $\mathrm{Tp}^{C^k}(\mathfrak{A};k) = A^k/\equiv^{C^k}$ as the stable colouring of some k-graph associated with $\mathfrak{A}$.

Let us fix some linear ordering $\leqslant_0$ on the finite set $\mathrm{Atp}(\tau;k)$ of atomic τ-types in k variables. This induces an initial pre-ordering $\preccurlyeq_0$ on the k-th power of the universe of any $\mathfrak{A} \in \mathrm{fin}[\tau]$:

$$\overline{a} \preccurlyeq_0 \overline{a}' \quad \text{if} \quad \mathrm{atp}_{\mathfrak{A}}(\overline{a}) \leqslant_0 \mathrm{atp}_{\mathfrak{A}}(\overline{a}').$$

The associated equivalence relation $\sim_0$ is equality of atomic types, i.e. the above $\approx_0$. With any finite τ-structure $\mathfrak{A}$ we associate a k-graph that encodes the game positions over $\mathfrak{A}$ in the k-pebble game together with the fixed initial pre-ordering with respect to atomic types.

Definition 2.26. *With structures $\mathfrak{A} \in \mathrm{fin}[\tau]$ associate the following structures over universe A^k.*

(i) The game k-graph *of $\mathfrak{A}$, $\mathfrak{A}^{(k)}$. Its vocabulary $\tau^{(k)}$ consists of binary relations E_j, for $j = 1, \ldots, k$, and unary predicates P_θ for each atomic type $\theta \in \mathrm{Atp}(\tau;k)$. These are interpreted on A^k according to $E_j\overline{a}\,\overline{a}'$ if $\bigwedge_{i\neq j} a_i = a_i'$, and $P_\theta\overline{a}$ if $\mathrm{atp}_{\mathfrak{A}}(\overline{a}) = \theta$.*

$$\mathfrak{A}^{(k)} = \Big(A^k, \big(E_j\big)_{1\leqslant j\leqslant k}, \big(P_\theta\big)_{\theta\in\mathrm{Atp}(\tau;k)}\Big).$$

(ii) For the pre-ordered k-graph *of $\mathfrak{A}$, the identification of the individual atomic types is replaced be the pre-ordering $\preccurlyeq_0$ according to atomic types (as induced by $\leqslant_0$). The pre-ordered k-graph of $\mathfrak{A}$ is*

$$\Big(A^k, \big(E_j\big)_{1\leqslant j\leqslant k}, \preccurlyeq_0\Big).$$

The E_j encode in both structures the accessibility between positions over $\mathfrak{A}$ in a moves that are carried out over the j-th component. It is important to note that both the game k-graph and the pre-ordered k-graph of $\mathfrak{A}$ are *quantifier free interpreted* over the k-th power of $\mathfrak{A}$. Also, the pre-ordering $\preccurlyeq_0$ of the pre-ordered k-graphs is atomically definable over the game k-graphs.

From Section 2.2.2 we obtain a stable colouring $\preccurlyeq$ on the pre-ordered k-graphs.

Proposition 2.27. *The stable colouring of the pre-ordered k-graph of $\mathfrak{A}$ is a pre-ordering with respect to C^k-types: its associated equivalence relation is equality of C^k-types over A^k.*

Proof. Let the $\preccurlyeq_i$ be the stages in the generation of the stable colouring $\preccurlyeq$ on the associated k-graph. Let $\sim$ and the $\sim_i$ be the corresponding equivalence relations on A^k. The proposition is equivalent with the statement that $\sim$ coincides with $\approx$ over A^k. It suffices to show inductively that $\sim_i = \approx_i$ for all i, since we know that

$$\sim_i \xrightarrow{i\to\infty} \sim \quad \text{and} \quad \approx_i \xrightarrow{i\to\infty} \approx .$$

Agreement between $\sim_0$ and $\approx_0$ is clear from the definition.

Consider the refinement step in the generation of the stable colouring on the k-graph associated with $\mathfrak{A}$. Recall the inductive definition of the stages for the stable colouring, in particular the formula governing the refinement step for the associated equivalence relation from equation 2.2 on page 69:

$$\begin{aligned} \bar{a} \sim_{i+1} \bar{a}' \quad \text{if} \quad & \bar{a} \sim_i \bar{a}' \text{ and for all } j \in \{1,\dots,k\} \text{ and all } \alpha \in A^k/\sim_i \\ & \left|\{\bar{b} \mid E_j\bar{a}\bar{b} \wedge \bar{b} \in \alpha\}\right| = \left|\{\bar{b} \mid E_j\bar{a}'\bar{b} \wedge \bar{b} \in \alpha\}\right|. \end{aligned}$$

But obviously $\left|\{\bar{b} \in \alpha \mid E_j\bar{a}\bar{b} \wedge \bar{b} \in \alpha\}\right| = \left|\{b \in A \mid \bar{a}\frac{b}{j} \in \alpha\}\right|$ so that

$$\begin{aligned} \bar{a} \sim_{i+1} \bar{a}' \quad \text{if} \quad & \bar{a} \sim_i \bar{a}' \text{ and for all } j \in \{1,\dots,k\} \text{ and all } \alpha \in A^k/\sim_i \\ & \left|\{b \in A \mid \bar{a}\tfrac{b}{j} \in \alpha\}\right| = \left|\{b \in A \mid \bar{a}'\tfrac{b}{j} \in \alpha\}\right|. \end{aligned}$$

Comparing Proposition 2.15 for the inductive refinement step in the $\approx_i$ — and specializing to the case that both positions are over the same structure $\mathfrak{A}$ — it follows that $\sim_i = \approx_i$ implies $\sim_{i+1} = \approx_{i+1}$. This yields an inductive proof of the claim. □

Recall from Lemma 2.21 that the stable colouring of pre-ordered k-graphs is $C^2_{\infty\omega}$-definable. $\preccurlyeq$ is the stable colouring of a pre-ordered k-graph that itself is quantifier free interpreted over the k-th power of $\mathfrak{A}$. It follows with Lemma 1.50 that $\preccurlyeq$ is globally definable as a global relation over fin$[\tau]$ in $C^{2k}_{\infty\omega}[\tau]$.

By Lemma 2.22 $\preccurlyeq$ is definable in FP(Q_{R}). Thus we have the following.

Theorem 2.28. *For each k there is a global pre-ordering $\preccurlyeq$ over the k-th power of the universe of structures in* fin$[\tau]$, *such that*

- *(i) the associated equivalence relation is equality of C^k-types. Thus $\preccurlyeq$ is the quotient interpretation of a global linear ordering of the $A^k/\equiv^{C^k}$.*
- *(ii) as a global relation over* fin$[\tau]$, *$\preccurlyeq$ is definable in $C^{2k}_{\infty\omega}[\tau]$ as well as in* FP$(Q_{\mathrm{R}})[\tau]$, *fixed-point logic with the Rescher quantifier.*

2.3.2 The External View

Recall how the equivalence relation $\approx$ was introduced as a binary relation on $\mathrm{fin}[\tau; k]$. Together with its inductive stages $\approx_i$ it serves to analyze the equivalence of k-tuples over different structures. $\approx$ and the $\approx_i$ as global relations on structures in $\mathrm{fin}[\tau]$ merely are the restrictions of these externally defined relations. It is possible to treat $\preccurlyeq$ and its stages $\preccurlyeq_i$ under the same external view as pre-orderings not only on individual structures in $\mathrm{fin}[\tau]$, but on $\mathrm{fin}[\tau; k]$. In this view an inductive definition of the $\prec_i$ can be given as follows. We here choose the strict variants $\prec_i$ because their inductive definition is the formally more transparent one. $<_0$ is the strict variant of the fixed linear ordering $\leqslant_0$ on $\mathrm{Atp}(\tau; k)$.

$$
\begin{array}{ll}
(\mathfrak{A}, \overline{a}) \prec_0 (\mathfrak{A}', \overline{a}') & \text{if} \quad \mathrm{atp}_{\mathfrak{A}}(\overline{a}) <_0 \mathrm{atp}_{\mathfrak{A}'}(\overline{a}') \\
(\mathfrak{A}, \overline{a}) \prec_{i+1} (\mathfrak{A}', \overline{a}') & \text{if} \\
\quad (\mathfrak{A}, \overline{a}) \prec_i (\mathfrak{A}', \overline{a}') \quad \text{or} & \\
\quad (\mathfrak{A}, \overline{a}) \sim_i (\mathfrak{A}', \overline{a}') \text{ and } \big(\nu_j^\alpha(\mathfrak{A}, \overline{a})\big) <_{\mathrm{lex}} \big(\nu_j^\alpha(\mathfrak{A}', \overline{a}')\big) & \\
\quad \text{where} \quad \nu_j^\alpha(\mathfrak{A}, \overline{a}) = \Big|\big\{b \in A \mid (\mathfrak{A}, \overline{a}\tfrac{b}{j}) \in \alpha\big\}\Big|. &
\end{array}
\tag{2.4}
$$

The indices (j, α) range over $\{1, \ldots, k\} \times \mathrm{fin}[\tau; k] /{\approx_i}$. The ordering of the index sets in the lexicographic comparison is chosen with dominant first component j. Note that the tuples involved in this comparison each only have a finite number of non-zero entries. Only types that are realized over $\mathfrak{A}$ or $\mathfrak{A}'$ enter non-trivially. Comparison with the inductive generation of the $\approx_i$ in Proposition 2.15 shows that the equivalence relations $\sim_i$ associated with the $\prec_i$ defined in this manner are indeed the $\approx_i$. It follows that the limit $\prec$ of the $\prec_i$ is a strict pre-ordering with respect to C^k-types over $\mathrm{fin}[\tau; k]$.

Lemma 2.29. *The pre-orderings $\prec_i$, as inductively defined on* $\mathrm{fin}[\tau; k]$ *according to equations 2.4, and their limit $\prec$ coincide in restriction to each individual $\mathfrak{A} \in$* $\mathrm{fin}[\tau]$ *with those defined through the stable colouring of the k-graph associated with $\mathfrak{A}$.*

Sketch of Proof. One need only specialize equations 2.4 to a single structure $\mathfrak{A} = \mathfrak{A}'$. The obvious equality $\big|\{b \in A \mid (\mathfrak{A}, \overline{a}\frac{b}{j}) \in \alpha\}\big| = \big|\{\overline{b} \in A^k \mid E_j \overline{a}\,\overline{b} \wedge (\mathfrak{A}, \overline{b}) \in \alpha\}\big|$ shows the agreement of the lexicographic comparison in 2.4 with that of the colour refinement over the k-graph associated with $\mathfrak{A}$, cf. equation 2.1 on page 69. This proves equality for the inductive stages and implies equality in the limits as well. □

This external view of $\preccurlyeq$ and the $\preccurlyeq_i$ really goes beyond the view of these as global relations on individual structures: it immediately shows that two C^k-types that are both realized in two different structures get ordered the same way in both structures.

Corollary 2.30. *As global relations on* fin$[\tau]$, *the* $\preccurlyeq$ *provide a coherent ordering with respect to* C^k*-types across all structures in* fin$[\tau]$*:*
if $\mathrm{tp}^{C^k}_{\mathfrak{A}}(\overline{a}_1) = \mathrm{tp}^{C^k}_{\mathfrak{A}'}(\overline{a}'_1)$ *and* $\mathrm{tp}^{C^k}_{\mathfrak{A}}(\overline{a}_2) = \mathrm{tp}^{C^k}_{\mathfrak{A}'}(\overline{a}'_2)$, *then* $\overline{a}_1 \preccurlyeq^{\mathfrak{A}} \overline{a}_2$ *if and only if* $\overline{a}'_1 \preccurlyeq^{\mathfrak{A}'} \overline{a}'_2$.

This is immediate here from Lemma 2.29. The same coherence claim can also be proved directly on the basis of the global definition of the individual pre-orderings. Note, however, that it does not follow directly from the fact that the associated equivalence relation is equality of C^k-types. Even though it is clear that whenever $\mathfrak{A}$ and $\mathfrak{A}'$ share even a single C^k-type they must be C^k-equivalent, coherent ordering of the types might a priori seem to require C^{2k}-equivalence.

2.3.3 The Analogous Treatment for L^k

We sketch the introduction of a pre-ordering with respect to L^k-types. An inductive characterization of the relation $\equiv^{L^k}$ or equality of L^k-types has been obtained in the analysis of the L^k-game. Recall Proposition 2.19 for the inductive generation of equivalences $\approx_i$ appropriate for the L^k-game. Their limit $\approx$ over fin$[\tau; k]$ is $\equiv^{L^k}$.

The desired pre-ordering, for which we also write $\preccurlyeq$, can once more be defined as a global relation internal to each individual structure, or externally as a pre-ordering on fin$[\tau; k]$ whose restriction to individual structures is the same as the former. As global relations internal to each $\mathfrak{A}$ the pre-ordering $\preccurlyeq$ and its stages $\preccurlyeq_i$ are obtained as the limit and the stages of the Abiteboul-Vianu colouring applied to the pre-ordered k-graphs associated with $\mathfrak{A}$. This immediately gives the analogous definability results as in the case of the C^k, cf. Theorem 2.28.

Theorem 2.31. *For each k there is a global pre-ordering* $\preccurlyeq$ *over the k-th power of the universe of structures in* fin$[\tau]$, *such that its associated equivalence relation is equality of* L^k*-types. This pre-ordering is obtained as the Abiteboul-Vianu colouring of the pre-ordered k-graphs associated with structures in* fin$[\tau]$. *As a global relation over* fin$[\tau]$, $\preccurlyeq$ *is definable in* $L^{2k}_{\infty\omega}[\tau]$ *as well as in* FP$[\tau]$.

The more general external version of $\preccurlyeq$ over fin$[\tau; k]$ is obtained in an inductive definition analogous to equations 2.4:

$$\begin{array}{ll}
(\mathfrak{A},\overline{a}) \prec_0 (\mathfrak{A}',\overline{a}') & \text{if} \quad \mathrm{atp}_{\mathfrak{A}}(\overline{a}) <_0 \mathrm{atp}_{\mathfrak{A}'}(\overline{a}') \\
(\mathfrak{A},\overline{a}) \prec_{i+1} (\mathfrak{A}',\overline{a}') & \text{if} \\
\quad (\mathfrak{A},\overline{a}) \prec_i (\mathfrak{A}',\overline{a}') & \text{or} \\
\quad (\mathfrak{A},\overline{a}) \sim_i (\mathfrak{A}',\overline{a}') & \text{and } \big(d^{\alpha}_j(\mathfrak{A},\overline{a})\big) <_{\mathrm{lex}} \big(d^{\alpha}_j(\mathfrak{A}',\overline{a}')\big)
\end{array}$$

$$\text{where} \quad d^{\alpha}_j(\mathfrak{A},\overline{a}) := \begin{cases} 0 & \text{if } \neg\exists b (\mathfrak{A}, \overline{a}\tfrac{b}{j}) \in \alpha \\ 1 & \text{if } \exists b (\mathfrak{A}, \overline{a}\tfrac{b}{j}) \in \alpha\,. \end{cases}$$

The indices (j, α) range over $\{1, \ldots, k\} \times \text{fin}[\tau; k] / \approx_i$.

Recall that $\approx_i$ is the i-th stage in the generation of $\approx$ — where now $\approx$ is $\equiv^{L^k}$ on $\text{fin}[\tau; k]$. In order to verify that indeed $\approx_i$ also is the equivalence relation associated with $\preccurlyeq_i$ as defined here, compare Proposition 2.19. In analogy with Lemma 2.29 it is shown that in restriction to individual structures this externally defined pre-ordering coincides with the one obtained internally. In particular, as a global relation on $\text{fin}[\tau]$, $\preccurlyeq$ is a coherent pre-ordering with respect to L^k-types.

Lemma 2.32. *As a global relation on* $\text{fin}[\tau]$ *the pre-ordering* $\preccurlyeq$ *obtained as the Abiteboul-Vianu colouring of the k-graphs of structures in* $\text{fin}[\tau]$ *provides a coherent ordering with respect to L^k-types across all structures in* $\text{fin}[\tau]$*: its associated equivalence relation is equality of L^k-types, and if* $\text{tp}^{L^k}_{\mathfrak{A}}(\overline{a}_1) = \text{tp}^{L^k}_{\mathfrak{A}'}(\overline{a}'_1)$ *and* $\text{tp}^{L^k}_{\mathfrak{A}}(\overline{a}_2) = \text{tp}^{L^k}_{\mathfrak{A}'}(\overline{a}'_2)$*, then* $\overline{a}_1 \preccurlyeq^{\mathfrak{A}} \overline{a}_2$ *if and only if* $\overline{a}'_1 \preccurlyeq^{\mathfrak{A}'} \overline{a}'_2$.

Sources and attributions. As pointed out above, the Fraïssé style analysis for finite variable logics in terms of back-and-forth systems is due to Barwise [Bar77], the introduction of the corresponding pebble games and their analysis to Immerman [Imm82]. For some more background on the finite variable fragments of first-order logic see also [Poi82]. The games for finitely many variables and counting quantifiers were introduced by Immerman and Lander in [IL90]. Cai, Fürer and Immerman applied these games in the analysis of their construction of non-isomorphic but C^k-equivalent graphs in [CFI89]. In this construction counting is, however, easily eliminated. A systematic analysis of the C^k-game over graphs is presented in [CFI92] and was independently developed in [GO93, Ott96a]. Cai, Fürer and Immerman attribute the underlying graph theoretic technique connected with the stable colouring in higher dimension to Lehman and Weisfeiler. The approach in [GO93, Ott96a] grew out of the generalization of the Abiteboul-Vianu approach to the case with counting. It should be remarked that the notions of a k-ary stable colouring underlying [GO93, Ott96a] — which is the one used here as well — differs in some technical details from the one attributed to Lehman and Weisfeiler in the work of Cai, Fürer and Immerman. Our k-ary variant is tuned exactly to yield a classification of k-tuples with respect to C^k; the other one rather corresponds to the classification of k-tuples with respect to types in C^{k+1}. Both ways have their merits but the difference has to be kept in mind to avoid confusion when comparing the statements. We find our convention more suitable in connection with definability issues concerning the pre-orderings with respect to types and the invariants to be introduced in the next chapter.

The work of Abiteboul and Vianu [AV91] is the essential source for the introduction of the definable ordered quotients $A^k / \equiv^{L^k}$ that will form the backbone of the invariants. An excellent presentation of the related results for the L^k-game in logical terms is given by Dawar, Lindell and Weinstein in [Daw93, DLW95].