3. NUMERATIONS OF R.E. SETS

Any set numerated in T is r.e. The question arises if the converse of this is true, in other words, if every r.e. set can be numerated in T. If T is Σ_1–sound, then, of course, the answer is "yes" (Corollary 1.4). If T is not Σ_1–sound, the answer is still "yes" although this is not so obvious. This is the first and most important result of this chapter. We also prove some refinements of this result.

Beginning in this chapter we omit most references to the Lemmas, Facts, and Corollaries of Chapter 1. To avoid too much repetition, proofs are sometimes left to the reader.

§1. Numerations of r.e. sets. Let X be any r.e. set. Our first task is to show that X can be numerated in T even if T is not Σ_1–sound. We have already solved a similar problem in generalizing Gödel's incompleteness theorem to non Σ_1–sound theories (Theorem 2.2). A similar construction will suffice for our present problem.

Theorem 1. Let X be any r.e. set. There is then a Σ_1 (Π_1) formula $\xi(x)$ which numerates X in T.

Proof. There is a primitive recursive relation R(k,m) such that $X = \{k: \exists mR(k,m)\}$. Let $\rho(x,y)$ be a PR binumeration of R(k,m). Let $\xi(x)$ be such that

(1) $Q\vdash \xi(k) \leftrightarrow \exists y(\rho(k,y) \wedge \forall z{\leq}y\neg Prf_T(\xi(k),z))$.

Then $\xi(x)$ is Σ_1. We are going to show that $\xi(x)$ numerates X in T.

Suppose first $k\in X$. There is then an m such that

(2) $Q\vdash \rho(k,m)$.

Now, for *reductio ad absurdum*, suppose $T\nvdash \xi(k)$. Then $Q\vdash \neg Prf_T(\xi(k),p)$ for every p. It follows that

(3) $Q\vdash \forall z{\leq}m\neg Prf_T(\xi(k),z)$.

Combining (2) and (3) we get

$$Q\vdash \exists y(\rho(k,y) \wedge \forall z{\leq}y\neg Prf_T(\xi(k),z)).$$

But then, by (1), $Q\vdash \xi(k)$ and so $T\vdash \xi(k)$ and we have reached the desired contradiction. Thus, $T\vdash \xi(k)$.

Next suppose $T\vdash \xi(k)$. Let p be a proof of $\xi(k)$ in T. Then $Q\vdash Prf_T(\xi(k),p)$ and so

(4) $Q\vdash \forall z{\leq}y\neg Prf_T(\xi(k),z) \rightarrow y < p$.

Suppose $k\notin X$. Then $Q\vdash \neg\rho(k,m)$ for every m. It follows that

(5) $Q\vdash \neg\exists y{<}p\rho(k,y)$.

Combining (4) and (5) we get

$$Q\vdash \neg\exists y(\rho(k,y) \wedge \forall z{\leq}y\neg Prf_T(\xi(k),z)),$$

whence, by (1), $Q\vdash \neg\xi(k)$ and so $T\vdash \neg\xi(k)$, impossible. Thus, $k\in X$ and we have shown that $\xi(x)$ numerates X in T.

Next let $\xi(x)$ be such that

$$Q \vdash \xi(k) \leftrightarrow \forall y(\mathrm{Prf}_T(\xi(k),y) \rightarrow \exists z \leq y \rho(k,z)).$$

Then $\xi(x)$ is Π_1. We leave the proof that $\xi(x)$ numerates X in T to the reader. ■

Let us say that $\xi(x)$ *correctly numerates* X *in* T if $\xi(x)$ numerates X in T and for every k, $T \vdash \xi(k)$ iff $\xi(k)$ is true. We can partially improve Theorem 1 as follows.

Theorem 1′. The Σ_1 formula $\xi(x)$ defined in the proof of Theorem 1 numerates X correctly in T.

Proof. If $\xi(k)$ is true, then $T \vdash \xi(k)$, since $\xi(x)$ is Σ_1. Conversely, suppose $T \vdash \xi(k)$. Let p be the least proof of $\xi(k)$ in T. Then (4) holds. Suppose there is no $m < p$ such that $R(k,m)$. Then (5) follows and so, as before, we get $T \vdash \neg\xi(k)$, which is impossible. Thus, there is an $m < p$ such that $R(k,m)$. But then $\rho(k,m)$ is true. Also, p being minimal, $\forall z \leq m \neg \mathrm{Prf}_T(\xi(k),z)$ is true. It follows that

$$\exists y(\rho(k,y) \wedge \forall z \leq y \neg \mathrm{Prf}_T(\xi(k),z))$$

is true and so, by (1), $\xi(k)$ is true, as desired. ■

Note that if X is numerated correctly in T by a Π_1 formula, then X is recursive.

The following strengthening of Theorem 1 is occasionally useful.

Lemma 1. Suppose X and Y are r.e. and Y is monoconsistent with Q. There is then a Σ_1 (Π_1) formula $\xi(x)$ such that for every k, if $k \in X$, then $Q \vdash \xi(k)$ and if $k \notin X$, then $\xi(k) \notin Y$.

The proof is again left to the reader. Lemma 1 also follows from Lemma 2 (a), below.

We now ask if there are (Σ_1) formulas $\xi(x)$ which not only numerate X in T but also satisfy additional conditions in terms of provability or nonprovability of (propositional combinations of) sentences of the form $\xi(k)$ with $k \notin X$. The following result is a first step in that direction.

Theorem 2. Let X_0 and X_1 be disjoint r.e. sets. There is then a Σ_1 formula $\xi(x)$ such that $\xi(x)$ numerates X_0 in T and $\neg\xi(x)$ numerates X_1 in T.

Proof. Let $R_i(k,m)$ be a primitive recursive relation such that $X_i = \{k: \exists m R_i(k,m)\}$, $i = 0, 1$. Let $\rho_i(x,y)$ be a PR binumeration of $R_i(k,m)$. Let $\xi(x)$ be such that

(1) $Q \vdash \xi(k) \leftrightarrow \exists y\big((\rho_0(k,y) \vee \mathrm{Prf}_T(\neg\xi(k),y)) \wedge \forall z \leq y(\neg\rho_1(k,z) \wedge \neg \mathrm{Prf}_T(\xi(k),z))\big)$.

We show that $\neg\xi(x)$ numerates X_1 in T; the proof that $\xi(x)$ numerates X_0 in T is similar and is left to the reader.

Suppose first $k \in X_1$ and, for *reductio ad absurdum*, $T \nvdash \neg\xi(k)$. Then there is an m such that $T \vdash \rho_1(k,m)$ and $T \vdash \neg \mathrm{Prf}_T(\neg\xi(k),p)$ for all p. Also $k \notin X_0$ and so $T \vdash \neg\rho_0(k,n)$ for all n. It follows that

$$T \vdash \forall z \leq y \neg\rho_1(k,z) \rightarrow y < m,$$

$T\vdash \neg\exists y<m((\rho_0(k,y) \vee Prf_T(\neg\xi(k),y))$.

But then, by (1), $T\vdash \neg\xi(k)$, contrary to assumption. Thus, $T\vdash \neg\xi(k)$.

Next suppose $T\vdash \neg\xi(k)$ and let p be such that $T\vdash Prf_T(\neg\xi(k),p)$. We also have and $T\nvdash \xi(k)$ and so $T\vdash \neg Prf_T(\xi(k),m)$ for all m. Suppose now $k\notin X_1$. Then $T\vdash \neg\rho_1(k,m)$ for all m. It follows that

$T\vdash Prf_T(\neg\xi(k),p) \wedge \forall z\leq p(\neg\rho_1(k,y) \wedge \neg Prf_T(\xi(k),z))$.

But then, by (1), $T\vdash \xi(k)$, impossible. Thus, $k\in X_1$. ■

One aspect of the above question is to ask to what extent results on numerations of r.e. sets can be combined with results on independent formulas. For example, does there exist a (Σ_1) formula $\xi(x)$ which numerates X in T and is *independent on* X^c (= N − X) *over* T in the sense that the only propositional combinations of sentences $\xi(k)$, with $k\in X^c$, provable in T are the tautologies? We now show that the answer is affirmative.

To prove this we need part (a) of the following lemma; Lemma 2 (b) will be needed later, in the proof of Theorem 7.13.

Lemma 2. Suppose X and Y are r.e. and Y is monoconsistent with Q.

(a) There is then a Σ_1 (Π_1) formula $\xi(x)$ numerating X in Q and such that

(*) for every finite subset X' of X^c, $\bigvee\{\xi(k)\colon k\in X'\}\notin Y$.

(b) Suppose $PA\dashv T$. There are then formulas $\xi(x)$ and $\xi'(x)$ such that $\xi(x)$ is Π_1, $\xi'(x)$ is Σ_1, $PA\vdash \xi'(x) \to \xi(x)$, $\xi'(x)$ numerates X in Q, and (*) holds.

Proof. We may assume that $Th(Q) \subseteq Y$. (If necessary, replace Y by $Th(Q) \cup Y$; this set is still monoconsistent with Q.) Let R(k,m) and R*(k,m) be primitive recursive relations such that $X = \{k\colon \exists mR(k,m)\}$ and $Y = \{k\colon \exists mR^*(k,m)\}$ and let $\rho(x,y)$ be a PR binumeration of R(k,m). Let $S(\eta,m)$ be the following primitive recursive relation:

there are $r \leq m$ and $k_0,\dots,k_r \leq m$ such that $R^*(\bigvee\{\eta(k_s)\colon s \leq r\},m)$ and $\forall s\leq r\forall p\leq m\neg R(k_s,p)$.

Let $\sigma(x,y)$ be a PR binumeration of $S(\eta,m)$.

(a) We construct a Σ_1 formula as desired. Let $\xi(x)$ be such that

(1) $Q\vdash \xi(x) \leftrightarrow \exists z(\rho(x,z) \wedge \forall u\leq z\neg\sigma(\xi,u))$.

We now show that

(2) $\neg S(\xi,m)$ for every m.

Suppose $S(\xi,m)$ is true. Then $Q\vdash \sigma(\xi,m)$. Hence, by (1), for every k,

(3) $Q\vdash \xi(k) \to \exists z\leq m\rho(k,z)$.

Moreover, there are $r \leq m$ and $k_0,\dots,k_r \leq m$ such that $\bigvee\{\xi(k_s)\colon s \leq r\}\in Y$ and $\forall s\leq r\forall p\leq m\neg R(k_s,p)$. It follows that $\forall s\leq r\forall p\leq m Q\vdash \neg\rho(k_s,p)$. But then, by (3), $Q\vdash \neg\bigvee\{\xi(k_s)\colon s \leq r\}$, contradicting the fact that Y is monoconsistent with Q. This proves (2).

We may assume that if $R^*(\bigvee\{\eta(k_s)\colon s \leq r\},m)$, then $r \leq m$ and $k_s \leq m$ for $s \leq r$. But then (*) follows at once from (2). Finally, the fact that $\xi(x)$ numerates X in T follows from (*), since $Th(Q) \subseteq Y$.

Next let $\xi(x)$ be such that

(4) $Q\vdash \xi(x) \leftrightarrow \forall u(\sigma(\xi,u) \rightarrow \exists z{\le}u\rho(x,z))$.

Then $\xi(x)$ is Π_1 and has the desired properties; details are left to the reader. ♦

(b) Let $\xi(x)$ be the formula defined by (4) and let $\xi'(x) :=$

$\exists z(\rho(x,z) \wedge \forall u{\le}z\neg\sigma(\xi,u))$.

The verification that $\xi(x)$ and $\xi'(x)$ are as claimed should now be easy. ■

Lemma 2 (b) can also be obtained as an easy consequence of Theorem 5, below.

Theorem 3. Let X be any r.e. set. There is then a Π_1 (Σ_1) formula $\eta(x)$ which numerates X in T and is independent on X^c over T.

Proof. By Theorem 2.9, there is a Π_1 (Σ_1) formula $\mu(x)$ which is independent (on N) over T. Let

$$Y = \bigcup\{Th(T + \{\mu^{f(k)}(k)\colon k \le n \ \&\ f\in 2^{n+1}\})\colon n\in N\}.$$

Then Y is r.e. and monoconsistent with Q. Let $\xi(x)$ be the Π_1 (Σ_1) formula given by Lemma 2 (a). Let $\eta(x) := \xi(x) \vee \mu(x)$.

If $k\in X$, then $Q\vdash \eta(k)$. To see that $\eta(x)$ is independent on X^c, suppose, for example, that $k_0, k_1, k_2, k_3\in X^c$ are distinct and that

$$T\vdash \eta(k_0) \vee \eta(k_1) \vee \neg\eta(k_2) \vee \neg\eta(k_3).$$

Then

$$T + \neg\mu(k_0) + \neg\mu(k_1) + \mu(k_2) + \mu(k_3)\vdash \xi(k_0) \vee \xi(k_1),$$

contrary to the choice of $\xi(x)$. ■

In Chapter 4 Theorem 3 will be used to construct not irredundantly axiomatizable theories (Theorem 4.7).

§2. Types of independence. By a *type* (*of independence*) we understand a consistent r.e. set F of propositional formulas P in the propositional variables p_n, $n\in N$, closed under tautological consequence. Let $<\varphi_k\colon k<\omega>$ be a sequence of sentences. Let $P(<\varphi_k\colon k<\omega>)$ be obtained from P by replacing p_k by φ_k for each k. If $\xi(x)$ is a formula, let $P(\xi) = P(<\xi(k)\colon k<\omega>)$. $<\varphi_k\colon k<\omega>$ is of *type* F *over* T if

$$F = \{P\colon T\vdash P(<\varphi_k\colon k<\omega>)\}.$$

$\xi(x)$ is of *type* F *over* T if this is true of $<\xi(k) : k<\omega>$.

Theorem 4. For each type F, there is a primitive recursive sequence $<\varphi_k\colon k<\omega>$ of B_1 sentences of type F over T.

Proof. In what follows p_k^i is p_k, if i = 0, and $\neg p_k$, if i = 1. Let s be a sequence of 0's and 1's, $s = <i_0,\dots,i_k>$. Then P^s is $p_0^{i_0} \wedge\dots\wedge p_k^{i_k}$. Assuming that $\varphi_0,\dots,\varphi_k$ have been defined, we define φ^s in a similar manner.

We now define $\varphi_0, \varphi_1, \varphi_2,\dots$ It will be clear that the sentences φ_k are B_1 and that the sequence $<\varphi_k: k<\omega>$ is primitive recursive. In addition to this it is sufficient to guarantee that for every k and every $s = <i_0,...,i_k>$,

(1) $F + P^s$ is consistent iff $T + \varphi^s$ is consistent.

Without loss of generality we may assume that $p_0 \in F$. Let $\varphi_0 := 0 = 0$. Then (1) holds for $k = 0$. Suppose (1) holds for $k = n$. Let

$$X_0^s = \{m: (P^s \to p_m) \in F\}, \qquad X_1^s = \{m: (P^s \to \neg p_m) \in F\}.$$

Next let $\xi_s(x)$ be a Σ_1 formula defined as in the proof of Theorem 2 with X_0 and X_1 replaced by X_0^s and X_1^s and T replaced by $T + \varphi^s$. Then

(2) if $F + P^s$ is consistent, then X_0^s and X_1^s are disjoint,

(3) if $T + \varphi^s$ is consistent and X_0^s and X_1^s are disjoint, then

$$X_0^s = \{m: T + \varphi^s \vdash \xi_s(m)\}, \qquad X_1^s = \{m: T + \varphi^s \vdash \neg\xi_s(m)\}.$$

Let $s_0,...,s_q$ be all sequences of 0's and 1's of length n+1. Finally, set

$$\varphi_{n+1} := (\varphi^{s_0} \wedge \xi_{s_0}(n+1)) \vee ... \vee (\varphi^{s_q} \wedge \xi_{s_q}(n+1)).$$

Then

(4) $T + \varphi^s \vdash \varphi_{n+1} \leftrightarrow \xi_s(n+1)$.

To complete the induction, we now have to show that

(5) $F + P^s \wedge p_{n+1}$ is consistent iff $T + \varphi^s \wedge \varphi_{n+1}$ is consistent,

(6) $F + P^s \wedge \neg p_{n+1}$ is consistent iff $T + \varphi^s \wedge \neg\varphi_{n+1}$ is consistent.

To prove (5), suppose first $F + P^s \wedge p_{n+1}$ is consistent. Then $n+1 \notin X_1^s$. Moreover, $F + P^s$ is consistent and so, by (2), X_0^s and X_1^s are disjoint and, by the inductive assumption, $T + \varphi^s$ is consistent. It follows, by (3), that $T + \varphi^s \nvdash \neg\xi_s(n+1)$ and so, by (4), $T + \varphi^s \wedge \varphi_{n+1}$ is consistent.

Next suppose $T + \varphi^s \wedge \varphi_{n+1}$ is consistent. Then $F + P^s$ is consistent. Hence, by (2), (3), (4), $n+1 \notin X_1^s$ and so $F + P^s \wedge p_{n+1}$ is consistent.

This proves (5). The proof of (6) is similar. ■

From Theorem 4 and Fact 10 (b) we get:

Corollary 1. Suppose PA⊣ T. Then for each type F, there is a Δ_2 formula of type F over T

Suppose T is Σ_1–sound, $\xi(x)$ is Σ_1, and $\xi(x)$ is of type F over T. Then F is *positively prime* (*p.p.*) in the sense that for all propositional variables $p_{n_0},...,p_{n_k}$, if $p_{n_0} \vee ... \vee p_{n_k} \in F$, there is an $i \leq k$ such that $p_{n_i} \in F$. (A formula P is *p.p.* if the set of tautological consequences of P is p.p.) We now prove that, for extensions of PA, the converse of this is true.

Theorem 5. Suppose PA⊣ T. Then for each p.p. type F, there is a Σ_1 formula of type F over T.

The proof of Theorem 5 is different from the other proofs in this book. We shall have to rely on the reader's ability to formalize (fairly simple) intuitive arguments in PA (or willingness to believe that these arguments can be so formalized). It will be essential to distinguish between the claims (i): for every k, PA proves: ...k... and (ii): PA proves: for every k, ...k... Here (ii) is the stronger claim; it may very well be the case that (i) is true and (ii) is false.

We are going to define a certain primitive recursive function f(k,m,n); the details of the definition will be crucial. The (inductive) definition of the function f(k,m,n) is given in the metalanguage and the task of formalizing this definition is left to the reader.

The numbers 0, 1 will be thought of as the truth–values *falsity* and *truth*, respectively. A function $t \in 2^N$ can then be regarded as a truth–value assignement: t assigns truth to p_i iff t(i) = 1. We always assume that t(i) = 0 for all but finitely many i. Thus, t is essentially a finite object and can be coded by, and treated as, a natural number. t[P] is the truth–value assigned by t to the formula P; for example, $t[p_i]$ = t(i).

By induction on the length of P, it is easy to show that for every P,

(1) PA proves: if for every i such that p_i occurs in P, $\xi(i)$ iff t(i) = 1, then $P(\xi)$ iff t[P] = 1.

Suppose $g, h \in 2^N$. Then g precedes h in the *lexiographic ordering* if $g \neq h$ and $g(k) < h(k)$, where k is the least number such that $g(k) \neq h(k)$. We shall also use the following partial ordering of 2^N: $g \preceq h$ iff $g(k) \leq h(k)$ for all k.

Lemma 3. (a) Suppose F is p.p. Then there is a primitive recursive function Q(s) such that (i) F is tautologically equivalent to $\{Q(s): s \in N\}$, (ii) for every s, Q(s) is p.p., (iii) PA proves: for all s, s′, if s < s′, then $Q(s') \rightarrow Q(s)$ is a tautology (we may assume that Q(0) is a tautology), (iv) PA proves: for all i, s, if p_i occurs in Q(s), then $i \leq s$.

(b) If P is p.p. and consistent, then there is a $\preceq$–least t such that t[P] = 1.

(c) Let t^s be the $\preceq$–least t such that t[Q(s)] = 1. For every s, $t^s \preceq t^{s+1}$.

Proof. (a) There is a primitive recursive function $Q_0(s)$ such that $F = \{Q_0(s): s \in N\}$. Let $Q_1(s,n)$ be the conjunction of the set of tautological consequences of $\bigwedge\{Q_0(s'): s' \leq s\}$ which contain no propositional variables other than p_i for $i \leq n$. Next let r(s) = $\max\{n \leq s: Q_1(s,n)$ is p.p.$\}$. Finally, let $Q(s) = Q_1(s,r(s))$. Then Q(s) is primitive recursive and (i) – (iv) are satisfied. ♦

(b) Let $t_0,\ldots,t_n$ be all assignments t such that t[P] = 1 and t(i) = 0 for every p_i not in P. Let $p_{i_0},\ldots,p_{i_m}$ be all propositional variables p_i such that $P \rightarrow p_i$ is a tautology. Let t′(i) = 1 iff $i \in \{i_0,\ldots,i_m\}$. Then $t' \preceq t_k$ for $k \leq n$. Suppose t′[P] = 0. Then for every $k \leq n$, there is a $j_k \notin \{i_0,\ldots,i_m\}$ such that $t_k(j_k) = 1$. But then $P \rightarrow p_{j_0} \vee \ldots \vee p_{j_n}$ is a tautology and so the same is true of $P \rightarrow p_{j_k}$ for some $k \leq n$, a contradiction. Thus, t′[P] = 1. ♦

(c) This is clear, since, by (a) (iii), $t^{s+1}[Q(s)] = 1$. ■

Proof of Theorem 5. Let f(s,m,i) be the primitive recursive function defined below; m will always be assumed to be a formula $\eta(x)$, the value of f(s,m,i), when m is not a formula, is irrelevant and we may set f(s,m,i) = 0. Now let $\xi(x)$ be such that

(2) $PA \vdash \xi(x) \leftrightarrow \exists z(f(z,\xi,x) = 1)$

and let $h(s,i) = f(s,\xi,i)$. Also, let h_s be such that for all i,

$$h_s(i) = h(s,i).$$

$h_s(i)$ may be thought of as the truth–value assigned to p_i at stage s. It will be clear from the definition of f that for fixed η and i, $f(s,\eta,i)$ is nondecreasing in s. Thus, informally, $\xi(i)$ is true iff the truth–value eventually assigned to p_i is 1.

Our goal is to define f in such a way that the following two claims can be established; Q(s) is as in Lemma 3 (a).

Claim 1. For every s, $PA \vdash (Q(s))(\xi)$.

Claim 2. For every P, if $T \vdash P(\xi)$, then $P \in F$.

By Lemma 3 (a) (i), Theorem 5 follows from Claims 1 and 2.

Cases 1.1 and 2 of the definition of f are designed to ensure the validity of Claim 2: If $T \vdash P(\xi)$ and, for a suitable s, $Q(s) \rightarrow P$ is not a tautology, Case 1.1 applies at Stage s+1 and so $h_{s+1}[P] = 0$. Also Case 2 applies at all later stages and so $h_{s'} = h_{s+1}$, whence $h_{s'}[P] = 0$, for all $s' > s$. This is provable in PA. It follows, by (1), that $PA \vdash \neg P(\xi)$, contradicting the assumption that $T \vdash P(\xi)$.

We now define $f(s,\eta,i)$ and at the same time an auxiliary function $g(s,\eta)$ as follows:

Stage 0. $f(0,\eta,i) = g(0,\eta) = 0$.

Stage s+1. *Case 1*. $g(s,\eta) = 0$.

Case 1.1. $s = \langle P,m \rangle$, m is a proof of $P(\eta)$ in T, and there is a t such that

(3) $t[Q(s)] = 1$,

(4) $t[P] = 0$,

(5) $f(s,\eta,i) \leq t(i)$ for $i < s$.

Let t′ be the lexicographically least such t. Set $g(s+1,\eta) = 1$ and

$$f(s+1,\eta,i) = t'(i).$$

Case 1.2. Not Case 1.1 and there is a t such that (5) holds and

(6) $t[Q(s+1)] = 1$.

Let t′ be the lexicographically least such t. Set $g(s+1,\eta) = 0$ and

$$f(s+1,\eta,i) = t'(i).$$

Case 1.3. Otherwise. Set $g(s+1,\eta) = 0$ and

$$f(s+1,\eta,i) = f(s,\eta,i).$$

Case 2. $g(s,\eta) > 0$. Set $g(s+1,\eta) = 1$ and

$$f(s+1,\eta,i) = f(s,\eta,i).$$

Inspection of the above definition in conjunction with Lemma 3 (a) (iv) shows that $h_s(i) = 0$ whenever $i > s$; in fact, this can easily be proved in PA, in other words:

(7) PA proves: for all i and s, if $i > s$, then $h_s(i) = 0$.

Furthermore,

(8) if $s < s'$, then $h_s \trianglelefteq h_{s'}$.

For $s' = s+1$, this can be seen by inspection; the full result follows by induction.

Using (7), the proof of (8) can be formalized in PA and so we have

(9) PA proves: for all s, s′, if $s < s'$, then $h_s \trianglelefteq h_{s'}$.

Next we show that

(10) for every s, $g(s,\xi) = 0$; in other words, if $\eta := \xi$, Case 1.1 never applies.

Suppose not and let s′ be the least number such that $g(s',\xi) = 1$. Then Case 1.1 applies at Stage s′. Thus, $s'-1 = \langle P,m\rangle$, m is a proof of $P(\xi)$ in T, whence

(11) $T\vdash P(\xi)$,

and $h_{s'}[P] = 0$. Let $t' = h_{s'}$. For $\eta := \xi$ Case 2 now applies at every $s > s'$ and so $h_s = t'$ for every $s \geq s'$. By (8), $h_s \preceq t'$ for $s < s'$. It follows that $t'(i) = 1$ iff there is an s such that $h(s,i) = 1$.

Using (9), this argument can be formalized in PA and so

$$PA\vdash \exists z(h(z,x) = 1) \leftrightarrow t'(x) = 1.$$

But then, by (2), $PA\vdash \xi(x) \leftrightarrow t'(x) = 1$ and so, by (1), $PA\vdash P(\xi) \leftrightarrow t'[P] = 1$. But $PA\vdash t'[P] = 0$. It follows that $PA\vdash \neg P(\xi)$, contradicting (11). This proves (10).

We now show that for all s,

(12) $h_s = t^s$,

where t^s is as in Lemma 3 (c). Since Q(0) is a tautology, this holds for s = 0. Suppose (12) holds for s. Then, by Lemma 3 (c), $h_s \preceq t^{s+1}$. Since $t^{s+1}[Q(s+1)] = 1$, either Case 1.1, Case 1.2 or Case 2 applies at s+1. By (8), Cases 1.1 and 2 don't and so Case 1.2 does. Also, the lexicographically least t′ mentioned in Case 1.2 with $\eta := \xi$ is t^{s+1}. It follows that $h_{s+1} = t^{s+1}$. This proves (12).

From (7) and (12) it follows that

(13) for every s, PA proves: $h_s = t^s$.

Next we show that

(14) for every s, PA proves: for every $s' \geq s$, $h_{s'}[Q(s)] = 1$.

Argue in PA: "For $s' = s$ we have $h_{s'} = t^s$, by (13), and so $h_{s'}[Q(s)] = 1$. Suppose $s' \geq s$ and the statement holds for s′. If Case 1.3 or Case 2 applies at s′+1, then $h_{s'+1} = h_{s'}$ and so, by the inductive assumption, $h_{s'+1}[Q(s)] = 1$. If Case 1.1 or Case 1.2 applies at s′+1, then $h_{s'+1}[Q(s')] = 1$ or $h_{s'+1}[Q(s'+1)] = 1$ and so, by Lemma 3 (a) (iii), $h_{s'+1}[Q(s)] = 1$. Now the desired conclusion follows by induction." (Since this argument takes place in PA, Cases 1.1 and 2 cannot be ruled out.) This proves (14).

Proof of Claim 1. Fix s. Argue in PA: "By (2) and (9), there is an $s' \geq s$ such that for every $i \leq s$, $h_{s'}(i) = 1$ iff $\xi(i)$. By (14), $h_{s'}[Q(s)] = 1$. By Lemma 3 (a) (iv), no p_i with $i > s$ occurs in Q(s). Thus, by (1), $(Q(s))(\xi)$." ♦

Proof of Claim 2. Let m be a proof of $P(\xi)$ in T. Let $s = \langle P,m\rangle$. By Lemma 3 (a) (i), it is sufficient to show that $Q(s) \rightarrow P$ is a tautology. Suppose not. Let t be such that $t[Q(s)] = 1$ and $t[P] = 0$. Then $t^s \preceq t$. By (13), $h_s = t^s$ and so $h_s \preceq t$. But then Case 1.1 applies at s+1 and so $g(s+1,\xi) = 1$, contrary to (10). Thus, $Q(s) \rightarrow P$ is a tautology. Finally, $Q(s)\in F$ and so $P\in F$. ♦

This concludes the proof of Theorem 5. ■

For $PA \dashv T$, Theorems 1, 2, 3 are, of course, special cases of Theorem 5.

Exercises for Chapter 3.

1. Suppose $Q \dashv T_0 \dashv T_1$. Show that for every r.e. set, there is a Σ_1 formula which numerates X in both T_0 and T_1.

2. We write $S \dashv_p T$ to mean that S is a proper subtheory of T.

(a) Suppose $Q \dashv T_0 \dashv_p T_1$. Let X_0 and X_1 be r.e. sets such that $X_0 \subseteq X_1$. Show that there is a formula $\xi(x)$ numerating X_i in T_i, $i = 0, 1$. [Hint: Let θ be such that $T_0 \nvdash \theta$ and $T_1 \vdash \theta$. There exist a formula $\xi_1(x)$ numerating X_1 in T_0 and in T_1 and a formula $\xi_0(x)$ numerating X_0 in $T_0 + \neg\theta$. Let $\xi(x) := \xi_1(x) \wedge (\theta \vee \xi_0(x))$.]

(b) Suppose $Q \dashv T_0 \dashv_p \dots \dashv_p T_n$. Let X_i, $i \leq n$, be r.e. sets such that $X_i \subseteq X_{i+1}$ for $i < n$. Show that there is a formula $\xi(x)$ numerating X_i in T_i for $i \leq n$.

(c) Suppose $Q \dashv T_0 \dashv T_1$ and suppose there is a formula $\sigma(x)$ which numerates Th(S) in S for every S such that $T_0 \dashv S \dashv T_1$. Show that $T_1 \dashv T_0$. [Hint: Suppose $T_1 \vdash \theta$ and let φ be such that $Q \vdash \varphi \leftrightarrow \neg\sigma(\varphi \vee \theta)$. Show that $T_0 \vdash \neg\varphi$.]

(d) Suppose $Q \dashv T_0$, $Q \dashv T_1$, and T_0 and T_1 are incomparable (with respect to $\dashv$). Let X_0 and X_1 be any two r.e. sets. Show that there is a formula $\xi(x)$ which numerates X_i in T_i, $i = 0, 1$.

3. Suppose $Q \dashv T_0 \dashv_p T_1$. Show that there is a formula $\xi(x)$ such that for every recursive function f, the set

$$\{n: T_0 \vdash \xi(n) \text{ \& there is a proof } p \text{ of } \xi(n) \text{ in } T_1 \text{ such that } \xi(n) \text{ has no proof} \leq f(p) \text{ in } T_0\}$$

is infinite, in fact, nonrecursive (this improves Theorem 2.13). [Hint: Let X be an r.e. nonrecursive set and let $\xi(x)$ be a formula numerating X in T_0 and N in T_1.]

4. Let X_0 and X_1 be r.e. sets. Let $\xi_0(x)$ be a Σ_n formula numerating X_0 in T. Show that there is a Σ_n formula $\xi_1(x)$ numerating X_1 in T such that $\xi_0(x) \vee \xi_1(x)$ numerates $X_0 \cup X_1$ in T. (If n = 1 and T is Σ_1–sound, this is trivial.) [Hint: Let $\rho(x,y)$ be a PR formula such that $\exists y\rho(x,y)$ correctly numerates X_1 in T, let $\xi(x)$ be such that

$$Q \vdash \xi(k) \leftrightarrow \exists y(\rho(k,y) \wedge \forall z{\leq}y \neg Prf_T(\xi(k) \vee \xi_0(k),z)),$$

and let $\xi_1(x) := \xi(x) \vee (\exists y\rho(x,y) \wedge \xi_0(x))$.]

5. Suppose $PA \dashv T$. Let X be any r.e. set. Show that there is a Γ formula $\xi(x)$ numerating X in T and such that for every Γ formula $\eta(x)$, the theory $T + \{\xi(k) \leftrightarrow \eta(k): k \notin X\}$ is consistent. (This improves Theorem 3.)

6. (a) Suppose $PA \dashv T$ and T is not Σ_1–sound. Show that the sentences φ_k in Theorem 4 can be taken to be Δ_1^T. [Hint: Use Lemma 1.3 (vi).]

(b) Suppose $Q \dashv S$. Show that there are primitive recursive enumerations $\varphi_0, \varphi_1, \varphi_2, \dots$ and $\psi_0, \psi_1, \psi_2, \dots$ of *all* sentences such that the type of $<\varphi_n: n<\omega>$ over S is the same as the type of $<\psi_n: n<\omega>$ over T.

7. (a) Let $\rho_i(y)$, i = 0, 1, be PR formulas. Let φ be such that

$$Q\vdash \varphi \leftrightarrow \exists y((\mathrm{Prf}_T(\neg\varphi,y) \vee \rho_0(y)) \wedge \forall z{\le}y(\neg\mathrm{Prf}_T(\varphi,z) \wedge \neg\rho_1(z))).$$

Show that

$$T\vdash \varphi \text{ iff } \exists y(\rho_0(y) \wedge \forall z{\le}y\neg\rho_1(z)) \text{ is true,}$$

$$T\vdash \neg\varphi \text{ iff } \exists z(\rho_1(z) \wedge \forall y{<}z\neg\rho_0(y)) \text{ is true.}$$

(b) Obtain Rosser's theorem (Theorem 2.2), Theorem 2, and Exercises 2.21, 2.22, 5.2 (a) as special cases of (a).

Notes for Chapter 3.

Theorems 1 and 2 are essentially due to Ehrenfeucht and Feferman (1960) and Putnam and Smullyan (1960), respectively; the present proofs are due to Shepherdson (1960). Lemmas 1 and 2 are due to Lindström (1979), (1984a).

Theorem 4 follows from a result of Pour–El and Kripke (1967) restricted to theories in L_A (see Exercise 6 (b)); the proof is just an "effective" version of the proof that every denumerable Boolean algebra is embeddable in every denumerable atomless Boolean algebra. Theorem 5 is new; the proof is an adaption of a proof of Solovay (1985); the result solves Problem 32 of Friedman (1975); an interesting special case of Theorem 5 is proved in Montagna and Sorbi (1985).

Exercise 3 is due to di Paola (1975). Exercise 6 (b) is a result of Pour–El and Kripke (1967) restricted to theories in L_A. Exercise 7 (a) is the so called Shepherdson–Smoryński fixed point theorem (see Smoryński (1980) and Hájek and Pudlák (1993)); a more general result is proved in Smoryński (1981a).