Lecture Notes in Logic 10

Per Lindström

# Aspects of Incompleteness

Springer

# Editorial Policy

§1. Submissions are invited in the following categories:

i) Research monographs
ii) Lecture and seminar notes
iii) Reports of meetings
iv) Texts which are out of print.

Those considering a project which might be suitable for the series are strongly advised to contact the publisher or the series editors at an early stage.

§2. Categories i) and ii). These categories will be emphasized by Lecture Notes in Logic and are normally reserved for works written by one or two authors. The goal is to report new developments – quickly, informally, and in a way that will make them accessible to non-specialists. In the evaluation of submissions timeliness of the work is an important criterion. Texts should be well-rounded and reasonably self-contained. In most cases the work will contain results of others as well as those of the authors. In each case the author(s) should provide sufficient motivation, examples, and applications. In this respect, articles intended for a journal and Ph.D. theses will usually be deemed unsuitable for the Lecture Notes series. Proposals for volumes in this category should be submitted (preferably in duplicate) either to one of the series editors or to Springer-Verlag, Heidelberg, and will be refereed. A provisional judgment on the acceptability of a project can be based on partial information about the work: a detailed outline describing the contents of each chapter, the estimated length, a bibliography, and one or two sample chapters – or a first draft. A final decision whether to accept will rest on an evaluation of the completed work which should include

- at least 100 pages of text;
- a table of contents;
- an informative introduction perhaps with some historical remarks which should be accessible to readers unfamiliar with the topic treated;
- a subject index.

§3. Category iii). Reports of meetings will be considered for publication provided that they are both of exceptional interest and devoted to a single topic. In exceptional cases some other multi-authored volumes may be considered in this category. One (or more) expert participants will act as the scientific editor(s) of the volume. They select the papers which are suitable for inclusion and have them individually refereed as for a journal. Papers not closely related to the central topic are to be excluded. Organizers should contact Lecture Notes in Logic at the planning stage.

§4. Category iv). This category provides an avenue whereby books which have gone out of print but which are still in demand can be made available to the new generations of logicians.

§5. Format. Works in English are preferred. They should be submitted in camera-ready form according to Springer-Verlag's specifications. Technical instructions and/or TeX macros will be sent on request.

# Lecture Notes in Logic 10

Editors:
S. Buss, San Diego
A. Lachlan, Burnaby
T. Slaman, Chicago
A. Urquhart, Toronto
H. Woodin, Berkeley

Springer
*Berlin*
*Heidelberg*
*New York*
*Barcelona*
*Budapest*
*Hong Kong*
*London*
*Milan*
*Paris*
*Santa Clara*
*Singapore*
*Tokyo*

Per Lindström

# Aspects of Incompleteness

Springer

Author

Per Lindström
Department of Philosophy
University of Gothenburg
S-41298 Gothenburg, Sweden
e-mail: per.lindstrom@phil.gu.se

Library of Congress Cataloging-in-Publication Data

Lindström, Per, 1936-
Aspects of incompleteness / Per Lindström.
p. cm. -- (Lecture notes in logic ; 10)
Includes bibliographical references and index.
ISBN 3-540-63213-1 (softcover : alk. paper)
1. Incompleteness theorems. 2. Recursion theory. I. Title.
II. Series.
QA9.65.L56 1997
511.3--dc21 97-25726
CIP

Mathematics Subject Classification (1991): 03F30, 03F40

ISBN 3-540-63213-1 Springer-Verlag Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting, reproduction on microfilms or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1997
Printed in Germany

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typesetting: Camera-ready by the authors
SPIN: 10633635 46/3142-543210 - Printed on acid-free paper

To Eva
for all she's done for me

# PREFACE

By Gödel's incompleteness theorem there is no complete axiomatization of mathematics, not even of first order arithmetic. This leads naturally to the project of investigating the family of, inevitably incomplete, arithmetical theories. In this book we present some of the results obtained in pursuing this aim. A brief summary of the contents of the book is given in the introduction.

Gödel's main idea was to translate metamathematical statements, concerning formulas, formal theories, provability in such theories etc., into arithmetic – the arithmetization of metamathematics. Combining this with the technique of self–reference, he was able to construct a (formal) sentence which is undecidable, neither provable nor disprovable, in a given theory T of arithmetic, thereby demonstrating, in an entirely novel way, that T is incomplete.

The ideas developed by Gödel proved fruitful far beyond their original application. Later essentially new ideas were added. Most important are the basic concepts (and results) of recursion theory – these are needed for a completely general formulation (and proof) of Gödel's theorem – notable are also the so called partial truth–definitions, the idea of formalizing the proof of the completeness of first order logic in arithmetic, and the general concept of interpretability.

In this book we shall be concerned almost exclusively with properties that are common to all sufficiently strong, axiomatizable theories. Thus, for example, investigations of particular theories such as, for example, Peano Arithmetic and its fragments, requiring special, model–theoretic or proof–theoretic, methods, fall outside the scope of the book. But even so, the choice of material in some respects reflects the author's preferences; this is particularly true of Chapter 7.

Credits for results, proofs, and exercises, remarks on alternative proofs, related results, etc. are given in the notes following each chapter. Results (exercises) not attributed to anyone are either easy, folklore, or due to the author.

The reader is assumed to be acquainted with (first order) predicate logic including Henkin's completeness proof. We also presuppose knowledge of the elements of recursion theory. Finally, the reader may find our presentation difficult to follow unless (s)he has previously seen a more detailed development of formal arithmetic, an explicitly defined arithmetization of metamathematics, and a full proof of Gödel's theorem.

I am grateful to Daniel Vallström and Rineke Verbrugge; they have very kindly and carefully read substantial portions of this book, pointed out numerous (minor) errors and suggested many improvements. The responsibility for the blemishes that remain is, of course, mine.

*May 1997.* P.L.

## CONTENTS


CHAPTER 0. Introduction . . . . . 1

CHAPTER 1. Preliminaries . . . . . 5
Exercises . . . . . 20
Notes . . . . . 22

CHAPTER 2. Incompleteness . . . . . 23
§1. Incompleteness . . . . . 23
§2. Consistency statements . . . . . 25
§3. Independent formulas . . . . . 31
§4. The length of proofs . . . . . 33
Exercises . . . . . 35
Notes . . . . . 40

CHAPTER 3. Numerations of r.e. sets . . . . . 42
§1. Numerations of r.e. sets . . . . . 42
§2. Types of independence . . . . . 45
Exercises . . . . . 50
Notes . . . . . 51

CHAPTER 4. Axiomatizations . . . . . 52
§1. Finite and bounded axiomatizability; reflection principles . . . . . 52
§2. Irredundant axiomatizability . . . . . 57
Exercises . . . . . 59
Notes . . . . . 61

CHAPTER 5. Partial conservativity . . . . . 62
Exercises . . . . . 70
Notes . . . . . 73

CHAPTER 6. Interpretability . . . . . 75
§1. Interpretability . . . . . 75
§2. Faithful interpretability . . . . . 84
Exercises . . . . . 88
Notes . . . . . 91

CHAPTER 7. Degrees of interpretabilty . . . . . 94
§1. Algebraic properties . . . . . 94
§2. A classification of degrees . . . . . 102

§3. $\Sigma_1$ and $\Pi_1$ degrees . . . . . 104
Exercises . . . . . 114
Notes . . . . . 117

CHAPTER 8. Generalizations . . . . . 119
§1. Incompleteness . . . . . 119
§2. Axiomatizations . . . . . 121
§3. Interpretability . . . . . 122
Notes . . . . . 124

REFERENCES . . . . . 125
INDEX . . . . . 130
NOTATION . . . . . 132