Towards Recursive Model Theory*

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Abstract. We argue that the models that are relevant to computer science are recursive and that Recursive Model Theory deserves being studied systematically, with at least the same vigor as Finite Model Theory has been. We study the status of some fundamental theorems from the classical model theory in this context and establish failure of several of them, including (generalized) Completeness, Compactness, Beth's Definability, Craig's Interpolation, and Lyndon's Lemma.

1 Introduction

Classical Model Theory deals with *all models.* If, for whatever reason, the class of models is restricted, this may potentially change model-theoretical laws that we take for granted. Take, for instance, the central for logic notion of *truth.* There may be sentences that are uniformly true in all the models of a certain class, but refutable in models not in the considered class. Hence, restricting the class of models may expand the class of true sentences. Conversely, the class of satisfiable sentences may shrink. If this actually happens, the equivalence between Model Theory and Proof Theory implied by Gödel's Completeness Theorem discontinues to hold, although in a specific situation a remedy can possibly be found by changing the axiomatization.

Recently, the Model Theory of finite models (those with finite universes) has been intensively investigated. The main motivation for Finite Model Theory has been the fact that, in several computer science applications, notably in databases, the models are often finite, and many issues in the theory of databases can be studied in the context of Finite Model Theory. Surprisingly or not, Finite Model Theory looks very much different from its classical counterpart.

The author is generally interested in Logic in Computer Science, and while finite models often are relevant to Computer Science, without question, not all the models that show up in CS applications are finite. Even in databases which have long been the Finite Model Theory refuge, infinite models not only show up, but actually move towards the central stage. In other CS playgrounds, say, in verification, finite models have never had any noticeable fraction of the market.

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Having said this, it is on the other hand the author's conviction that considering arbitrary models is irrelevant, as far as Computer Science is concerned. By way of example, suppose we knew that a constraint was going to be universally satisfied for all the databases over countable domains. Would it be a sufficient evidence to altogether eliminate this constraint from the list of constraints? My answer is "yes", because all the databases that a computer can handle are countable. Moreover, they are *recursive,* or *constructive.*

Of course, computer scientists do play with non-recursive structures. For example, a series of papers has appeared recently that deal with and establish interesting results about constraint databases over *o*-minimal and quasi *o*-minimal domains—a diverse family that includes several recursive structures, but at the same time includes such monstrous structures as the reals—see [PVV95, BDLW96, ST96]). The strength of these results is in providing uniform techniques for a wide class of domains, recursive and non-recursive alike, however, at the end it should not be forgotten that reals can not be represented inside a computer. In my view, "databases over the reals" can be used as a figure of speech, but should not be confused with real databases that are organized over strings, integers, and other *recursive* domains.

Generally, recursive models have been studied for decades. But the question, *"what Logic would look like if we had recursive models only?"*, has never been, to the best of the author's knowledge, systematically addressed before. In this paper, we consider several central results of Classical Model Theory:

- Gödel's Completeness Theorem
- Compactness Theorem
- Craig's Interpolation Theorem
- Beth's Definability Theorem
- Lyndon's Lemma on Monotonicity

from this viewpoint, and show that they all fail over recursive models—the situation remarkably similar to that with finite models. This indicates that Recursive Model Theory deserves being studied systematically, with at least the same vigor as the theory of finite models has been.

I wish to point the reader's attention to another paper in Recursive Model Theory [HH96] where Harel and Hirst (independently of this paper) show failure of Compactness and Completeness, and also prove several other results, including a very nice asymptotic law for recursive models.

Finally, a few words about the techniques used in this paper. We start off by pulling together several results about nonstandard models of Peano Arithmetic that jointly imply finite axiomatizability of the standard model $\mathbb{N}$ of arithmetic in the class of recursive models. This result is used as instrumental in refuting several of the aforementioned classical results in the context of recursive models. Of them the only one that requires a technically involved argument is Lyndon's Lemma. Generally, we further develop the technique introduced in [Sto95] for refuting Lyndon's Lemma over finite models. This requires substantial changes, but at the end, the proof turns out to be even a bit simpler.

2 Preliminaries

2.1 Recursive models

A model is called *recursive* iff it is isomorphic to a model over the natural numbers (or an initial segment thereof) with all the signature functions and predicates defined by recursive functions.

For example, every finite model is recursive. There are, however, infinite recursive models, some of them well known: the standard model $\mathbb{N}$ of Peano Arithmetic, or rational numbers with equality, $+$ and $\times$. Recursive models should not, however, be confused with models with decidable theories, as these two notions are independent. Indeed, the theory of real numbers with $+, \times, 0, 1$ is decidable, but the model is not recursive by virtue of its uncountability; on the other hand, the aforementioned standard model of arithmetic, while recursive, is undecidable.

2.2 Tennenbaum's Theorem and generalizations

The goal of this subsection is to pull together several results jointly implying finite axiomatizability of $\mathbb{N}$, in the class of recursive models. As is well known, Tennenbaum [Ten59] showed that the standard model of arithmetic is the only recursive model of Peano Arithmetic PA. However, PA is not finitely axiomatizable. Second however, McAloon [McA82] extended the result by showing that $\mathbb{N}$ continues to be the only recursive model of $I\Delta_0$, and, by implication, of $I\Sigma_n$, for any n. IX denotes a restricted version of PA with the induction axiom applied to X-formulas only.

Now, although the question of finite axiomatizability of $I\Delta_0$ continues to be open, it is known that $I\Sigma_n$, for $n \geqslant 1$, is finitely axiomatizable—see, e.g., Exercise 10.4 in [Kay91]. Hence:

Theorem 1. *The standard model $\mathbb{N}$ of Peano Arithmetic is finitely axiomatizable in the class of recursive models.*

Fix a finite axiomatization $\mathcal{A}$ of $\mathbb{N}$—for example, a finite axiomatization of $I\Sigma_1$.

2.3 Monotone formulas

The classical result by Roger Lyndon [Lyn59] is that any first order formula, monotone in a certain predicate, is equivalent to some formula of the same signature that is positive in the predicate. Apart from the fact that the lemma is used in proofs of Lyndon Interpolation Theorem, it reflects a very interesting logical phenomenon and is hence extremely interesting by itself.

By definition, a formula is *positive* in a predicate symbol iff, syntactically, every occurrence of this predicate symbol in the formula is under an even number of negations.

Monotonicity of a formula in a predicate is, on the other hand, a semantical property. Namely, a formula $\varphi(P, \mathbf{x})$ with free variables $\mathbf{x}$ is *monotone* in a predicate symbol P iff for every model M, tuple $\mathbf{a}$, and every extension $Q \supseteq P^M$,

$$M \models \varphi(P^M, \mathbf{a}) \text{ implies } M \models \varphi(Q, \mathbf{a}).$$

The "every model" refers to every model of the class we consider. The standard definition means "every possible model", but now we are interested in recursive models only.

All the same, a formula positive in a predicate symbol is necessarily monotone in this symbol. In the other direction, over all models Lyndon's Lemma says that every monotone formula is equivalent to a positive formula. To show that Lyndon's Lemma fails over recursive models we will need to construct a formula monotone in a predicate symbol *over all recursive models,* but not equivalent, over *recursive models,* to any positive formula.

3 Classical Theorems in Recursive Models

The goal of this section is to look at several important classical results about first order logic and to clarify their status in the recursive model theory.

To begin with, several classical theorems are syntactical by nature, and their status does not depend on a particular class of models being considered. A good example is the Prenex Normal Form.

Several other results, however, directly relate to the class of models, and it makes a perfect sense to inquire as to their status. Surprisingly or not, we establish here failure of several such results.

Theorem 2 No Gödel's Completeness. *In recursive models, first order truth is not recursively axiomatizable.*

PROOF: By Theorem 1, $\mathbb{N}$ is finitely axiomatizable. If the truth over recursive models were recursively axiomatizable, then so would be arithmetical truth. But, by Gödel's Incompleteness Theorem, it is not. *Q.E.D.*

Theorem 3 No Compactness. *Compactness Theorem fails over recursive models.*

In other words, there exists a set of first order sentences whose every finite subset has a recursive model, while the set itself does not have a recursive model.

PROOF: Recall that $\mathcal{A}$ is the finite axiomatization from Theorem 1. Consider the arithmetical signature *plus* an additional constant symbol ∞. Let ε_k, for $k \in \{0, 1, 2, \dots\}$, be the following sentence of this signature:

$$\mathcal{A} \wedge \infty > k.$$

The $\infty > k$ is an obvious abbreviation. Then the set

$$\Xi = \{\varepsilon_k\}_{k=0}^{\infty}$$

does not have a recursive model. However, every finite subset of Ξ is satisfied in the standard model when ∞ is interpreted as a large enough number (greater than all the ks in the sentences). *Q.E.D.*

Let us now formulate Craig's Interpolation Theorem and Beth's Definability Theorem that we are going to refute next:

Theorem 4 Craig's Interpolation Theorem. *Let φ, ψ be two sentences, such that $\varphi \models \psi$. There exists a sentence δ, such that:*

1. $\varphi \models \delta$
2. $\delta \models \psi$, *and*
3. *any relation, or function symbol, occurring in δ (except possibly for equality) does also occur in both φ, and ψ*

This sentence δ is often called an interpolant of φ, ψ.

Consider a signature Ω, and an extra predicate symbol $R \notin \Omega$. Let $\varphi(R)$ be a sentence over the signature $\Omega \bigcup \{R\}$. We say that $\varphi(R)$ *defines the predicate R implicitly* iff for any model $\mathcal{M}$ of the signature Ω there exists a unique interpretation $R^{\mathcal{M}}$ of the predicate R, such that the sentence $\varphi(R)$ is valid in $\mathcal{M}$ under this interpretation of R.

Theorem 5 Beth's Definability Theorem. *Let Ω be a signature, $R \notin \Omega$ be a new predicate symbol of an arity k, and let a sentence $\varphi(R)$ over the signature $\Omega \bigcup \{R\}$ define the predicate R implicitly. There exists a formula $\psi(x_1, x_2, \ldots, x_k)$ over the signature Ω such that*

$$\varphi(R) \models \forall x_1 \forall x_2 \ldots \forall x_k \left(R(x_1, x_2, \ldots, x_k) \longleftrightarrow \psi(x_1, x_2, \ldots, x_k) \right).$$

In other words, over all models implicit definability does not increase the power of first order logic in defining predicates. It will also be useful to have in mind the following "modern" proof of Beth's Definability Theorem: essentially, it demonstrates the well known fact that for any reasonable class of models, and any reasonable logic, the interpolation property implies the definability property.
PROOF OF THEOREM 5: Since $\varphi(R)$ defines R implicitly, by definition, we have (for the signature $\Omega \bigcup \{R, R'\}$, where R' is yet another new predicate symbol of the same arity k):

$$\varphi(R) \wedge \varphi(R') \models \forall x_1 \forall x_2 \ldots \forall x_k (R(x_1, x_2, \ldots, x_k) \longrightarrow R'(x_1, x_2, \ldots, x_k)).$$

Let us enrich the signature with the additional constants (0-ary function symbols) $c_1, c_2, \ldots, c_k$. Now,

$$\varphi(R) \wedge \varphi(R') \models R(c_1, c_2, \ldots, c_k) \longrightarrow R'(c_1, c_2, \ldots, c_k), \text{ or, in other words,}$$

$$\varphi(R) \wedge R(c_1, c_2, \ldots, c_k) \models \varphi(R') \longrightarrow R'(c_1, c_2, \ldots, c_k).$$

By Craig's Interpolation Theorem, there exists an interpolant $\psi(c_1, c_2, \ldots, c_k)$ over the signature $\Omega \bigcup \{c_1, c_2, \ldots, c_k\}$ (but *without* R, R'), such that:

1. $\varphi(R) \wedge R(c_1, c_2, \ldots, c_k) \models \psi(c_1, c_2, \ldots, c_k)$, and
2. $\psi(c_1, c_2, \ldots, c_k) \models \varphi(R') \longrightarrow R'(c_1, c_2, \ldots, c_k)$

Now, the sentence 1 refers to the models of the signature $\Omega \bigcup \{R, c_1, c_2, \ldots, c_k\}$, while 2 does to ones of $\Omega \bigcup \{R', c_1, c_2, \ldots, c_k\}$. As any model of the former signature is also a model of the latter, if we interpret R' as R, we finally have:

- $\varphi(R) \wedge R(c_1, c_2, \ldots, c_k) \models \psi(c_1, c_2, \ldots, c_k)$, and
- $\psi(c_1, c_2, \ldots, c_k) \models \varphi(R) \longrightarrow R(c_1, c_2, \ldots, c_k)$

in the signature $\Omega \bigcup \{R, c_1, c_2, \ldots, c_k)$, or

- $\varphi(R) \models R(c_1, c_2, \ldots, c_k) \longrightarrow \psi(c_1, c_2, \ldots, c_k)$, and
- $\varphi(R) \models \psi(c_1, c_2, \ldots, c_k) \longrightarrow R(c_1, c_2, \ldots, c_k)$

Combining these two assertions, we have:

$$\varphi(R) \models R(c_1, c_2, \ldots, c_k) \longleftrightarrow \psi(c_1, c_2, \ldots, c_k), \text{ or, further,}$$

$$\varphi(R) \models \forall x_1 \forall x_2 \ldots \forall x_k (R(x_1, x_2, \ldots, x_k) \longleftrightarrow \psi(x_1, x_2, \ldots, x_k)),$$

for the signature $\Omega \bigcup \{R\}$. *Q.E.D.*

The situation changes dramatically over all recursive models:

Theorem 6 No Beth's Definability. *Beth's Definability Theorem fails for recursive models.*

Specifically, there exist a signature Ω, and a sentence $\varphi(P)$ in the signature $\Omega \bigcup \{P\}$, where P is a unary predicate symbol not in Ω, such that:

1. *in every recursive model M of the signature Ω, $\varphi(P)$ is satisfied for one and only one unary predicate P_M substituted for P*
2. *there exists a recursive model M of the signature Ω such that for any formula $\psi(x)$ in the signature Ω with one free variable x:*

$$M \not\models \varphi(\{a \in |M| : \psi(m)\})$$

PROOF: The set of all true arithmetical sentences is implicitly definable in arithmetic (see, e.g., [Rog67]). Let Ω be the arithmetical signature $+, \times, <, 0, 1$, and let $\mathcal{T}(P)$ implicitly defines the set of numbers of true arithmetical sentences. This set, however, is not *explicitly* definable in arithmetic [Rog67].

To finish the proof, define the required $\varphi(P)$ as follows:

$$(\mathcal{A} \wedge \mathcal{T}(P)) \vee (\neg\mathcal{A} \wedge (\forall x)(P(x) \longrightarrow \textbf{false})),$$

where, again, $\mathcal{A}$ is the axiomatization from Theorem 1.

Observe that $\varphi(P)$ defines the set of numbers of true arithmetical sentences in the standard model, and an empty set otherwise. *Q.E.D.*

Now, let us refute Craig's Interpolation Theorem.

Theorem 7 No Craig's Interpolation. *Craig's Interpolation fails over recursive models.*

In other words, there exist two sentences φ, ψ such that $\varphi \models \psi$ over all recursive models, but no interpolant sentence δ exists such that:

1. *$\varphi \models \delta$ over all recursive models*
2. *$\delta \models \psi$ over all recursive models, and*
3. *any relation, or function symbol, occurring in δ (except possibly for equality) does also occur in both φ, and ψ*

PROOF: A closer look at our proof of Theorem 5 reveals that, if Craig's Interpolation Theorem (Theorem 4) held over recursive models, we would be able to prove Beth's Definability Theorem over recursive models as well. But we already know that the latter fails for recursive models. *Q.E.D.*

4 Failure of Lyndon's Lemma

Finally, we are going to refute Lyndon's Lemma. Our method of refutation builds on the technique developed in [Sto95] for finite models. There, the author introduced Positive Pebble Games and proved that they capture "preservation" under positive formulas. Independently, similar games (together with one direction of proof of the Theorem 9) were introduced in [McC95].[3] Let us recall the definitions from [Sto95].

4.1 Positive preservation and pebble games

Definition 8 positive n-preservation. Let A, B be two models of a certain relational signature $\Omega = \Delta \bigcup \Theta$, Δ and Θ be disjoint, and let $\mathbf{a} = a_1, a_2, \ldots, a_k$, $\mathbf{b} = b_1, b_2, \ldots, b_k$ be elements of $|A|$, $|B|$, respectively.

We say that $(A, \mathbf{a})$ is *Θ-positively n-preservable by* $(B, \mathbf{b})$, iff any first order Θ-positive formula $\varphi(x_1, x_2, \ldots, x_k)$ of quantifier depth n with k free variables $x_1, x_2, \ldots, x_k$ that is true in A under $x_1, x_2, \ldots, x_k := a_1, a_2, \ldots, a_k$, is also true in B under $x_1, x_2, \ldots, x_k := b_1, b_2, \ldots, b_k$.

If $k = 0$, we will simply say that A is Θ-positively n-preservable by B.

Let $\Omega = \Delta \bigcup \Theta$ be a relational signature, Δ and Θ be disjoint, and let A, B be two models of the signature. Let $\mathbf{a} = a_1, a_2, \ldots, a_k$, $\mathbf{b} = b_1, b_2, \ldots, b_k$ be elements of $|A|$, $|B|$, respectively. Consider the following game of two players on the pair $\langle (A, \mathbf{a}), (B, \mathbf{b}) \rangle$, called n-positive pebble game, for $n \geq 0$.

Each player initially has n pebbles, numbered $1, 2, \ldots, n$. In the first step of the game, the first player, whom we will call *Spoiler*, chooses a model among A, B, and places his pebble with number 1 onto some element of the model.

[3] An anonymous referee has suggested that the idea of "positive pebble games" might have been known to certain model theorists in some form.

Then the second player, whom we will call *Duplicator*, takes the other model, and places his pebble with number 1 onto some element of this model.

After $(r-1)$-th step of the game, for $1 < r \leq n$, each player retains the pebbles numbered $r, r+1, \ldots, n$, while the pebbles numbered $1, 2, \ldots, r-1$ are somehow placed onto elements of the models A, B. Then in the r-th step of the game, again, the first player chooses a model among A, B, and places his pebble r onto some element of the model. Then the second player takes the other model, and places his pebble r onto some element of this model.

The n-positive pebble game ends after its n-th step. After the game ends, each of the models A, B has pebbles numbered $1, 2, \ldots, n$ placed somehow onto its elements. Let $a_{k+1}, a_{k+2}, \ldots, a_{k+n}$ be the elements of A covered with, respectively, pebbles numbered $1, 2, \ldots, n$, and let $b_{k+1}, b_{k+2}, \ldots, b_{k+n}$ be the elements of B covered with, respectively, pebbles numbered $1, 2, \ldots, n$.

Let A' be the substructure of A generated with $a_1, a_2, \ldots, a_{k+n}$, and let B' be the substructure of B generated with $b_1, b_2, \ldots, b_{k+n}$.

By definition, the second player, Duplicator, wins the game, if the mapping $a_i \mapsto b_i$, for $i = 1, 2, \ldots, k+n$, is correctly defined and this mapping is:

1. an isomorphism of Δ-reducts of A' and B';
2. a homomorphism of Θ-reduct of A' onto Θ-reduct of B'.

Otherwise the first player, Spoiler, wins the game.

Theorem 9. *Let A, B be two models of a finite relational signature $\Omega = \Delta \bigcup \Theta$ (with or without constants), Δ and Θ be disjoint, and let $\mathbf{a} = a_1, a_2, \ldots, a_k$, $\mathbf{b} = b_1, b_2, \ldots, b_k$ be elements of $|A|$, $|B|$, respectively.*

$(A, \mathbf{a})$ is Θ-positively n-preservable by $(B, \mathbf{b})$, iff Duplicator has a winning strategy in the n-positive pebble game on the pair $\langle (A, \mathbf{a}), (B, \mathbf{b}) \rangle$.

4.2 Rotated grids

The goal of this subsection is to define the class of recursive models that is going to be instrumental in refuting Lyndon's Lemma. These structures will be called *Rotated Grids of Finite Height,* or simply *Rotated Grids.*

First, let's attempt an informal explanation. Imagine a grid on points (i, j) on the integer surface, with two unary functions—North and West—connecting (i, j) with $(i, j+1)$ and $(i-1, j)$, respectively. In the grid structure, there is going to be a binary relation H which is the union of the graphs of the two functions.

Now, rotate this grid 45 degrees clockwise, draw two parallel horizontal lines, and cut off everything above the upper, and below the lower line. This constitutes an informal definition of a rotated grid.

Additionally, the second binary relation, $<$, is defined in a rotated grid as the transitive closure of H. If $<$ is a proper subset of the transitive closure of H, the result is called a rotated *under*grid.

On the side, in the rotated (under)grids we will have the standard model of arithmetic (with $+, \times, 0, 1$ and $\lessdot$—distinguished from $<$), whose elements—natural numbers—are distinguished from the pairs by a unary predicate N.

These two parts of the models are going to be related by a binary relation $\equiv$ that connects all the points of a horizontal level in an (under)grid to the "number" of this level. There are finitely many levels.

Formally, the non-arithmetical part of an (under)grid of k levels is defined as the collection of elements:

$$\{(i,j) \mid i = 1, \ldots, k;\ j \in \mathbb{Z}\},$$

with the binary predicate H defined as follows:

$$((i,j),(l,m)) \in H \text{ iff } l = i+1 \wedge (j = m \vee j+1 = m).$$

Notice that, for technical convenience, the first coordinate of a tuple numbers the level—this is different from the informal explanation attempted above, although the structure continues to be precisely the same.

The relation $\equiv$ connecting the grid part with $\mathbb{N}$ is defined as follows:

$$(i,j) \equiv i.$$

The intention of this relation $\equiv$ is to make the finiteness of the height of grids axiomatizable.

Finally, a rotated undergrid of height k is called *canonical* iff adding the pair $((0,0),(k,0))$ to its relation H makes it a rotated grid.

Theorem 10. *For any n, a rotated grid of height 2^{n+2} is positively n-preservable by its canonical undergrid.*

PROOF: Due to Theorem 9, it suffices to demonstrate a winning strategy for the second player in the positive n-game played in the rotated grid and its canonical undergrid.

First, in the arithmetical part, the games is played by replicating all the moves, while in the non-arithmetical part (in the grids), the moves will at least preserve the levels. Because of the definition of the relation $\equiv$, specifics of the pebble positions within the arithmetical part become irrelevant to winning the game, and we can henceforth concentrate on the grid parts only.

On the grids, Duplicator starts from drawing an imaginary line cutting the two grids into two parts. This line is drawn through the points $(0,-2^{n+1})$ and $(2^{n+2},2^{n+1})$. The part of the grid on or at the left from the line will be called left, and the other part will be called right.

Generally, any move of Spoiler in the right part of either of the grids is answered by replicating. In the left part, when Spoiler pebbles an element (i,j) in the grid, Duplicator answers by pebbling $(i,j+1)$ in the undergrid. Conversely, when Spoiler pebbles (i,j) in the undergrid, duplicator answers $(i,j-1)$ in the grid. At the step k of the game, Duplicator moves the imaginary line 2^{n+1-k} steps left or right, as follows:

- if the last move was in the left part, move the line to the right
- if the last move was in the right part, move the line to the left

("left" and "right" here refer to changing the second coordinate).

It is easy to see by induction, that the distance from the imaginary line to the closest pebble is, after the kth step, at least 2^{n-k}. Hence, between right and left pebbles, the relation H is impossible.

Observe, that although the line moves throughout the game, a pebble once attributed to the left (right) part, continues to be in this part. Therefore, separately, the positions of the left (right) pebbles in the two structures are isomorphic. To show that Duplicator wins the game, it then suffices to show that, if in the grid, ℓ, r are two pebbled elements from the left and right parts, respectively, and $r < \ell$,[4] then in the undergrid, the corresponding pebbled elements r', ℓ' are in the same relation $r' < \ell'$.

Indeed, let $\ell = (h_\ell, v_\ell), r = (h_r, v_r)$. $r < \ell$ implies $h_r < h_l$ and $v_r \leq v_\ell$. By definition, $\ell' = (h_\ell, v_\ell{+}1), r' = (h_r, v_r)$. Taking into account that $(h_\ell, v_r{+}h_\ell{-}h_r)$ is, by definition, in the right part, we can conclude that $v_\ell + 1 < v_r + h_\ell - h_r$. Hence, $r' < \ell'$. *Q.E.D.*

4.3 Monotone separation of rotated grids

A binary relation $<$ defined in a set U is called *cyclic,* iff there exist elements $e_1, e_2, \ldots, e_n \in U, n > 0$, such that $e_1 < e_2 < \cdots < e_n < e_1$. $<$ is called *acyclic* iff it is not cyclic.

An acyclic relation $<$ in a set U is called *downward infinite* iff there exist infinitely many elements $\{e_i\}_{i=-\infty}^{0}$, such that $\cdots < e_{-2} < e_{-1} < e_0$. An acyclic $<$ is called *downward finite* iff it is not downward infinite.

To begin with, let's axiomatize a class of models of the rotated grids' signature where H is an acyclic downward finite relation in $\overline{N}$ (not all such models though). This class will include rotated (under)grids. In what follows, $\mathcal{A}_N$ is the axiom system $\mathcal{A}$ from Theorem 1, specialized to the predicate N. Consider the following list of axioms:

$$\mathcal{A}_N \tag{1}$$

$$\forall x \notin N \exists! n \in N (x \equiv n) \tag{2}$$

$$\forall xy \notin N \forall nm \in N (H(x,y) \wedge x \equiv n \wedge y \equiv m \longrightarrow m = n + 1) \tag{3}$$

Obviously, in the recursive models of this axiom system, H is acyclic and downward finite. For the rest of the section, consider only recursive models of this axiom system.

The following is a variation of the technique used in [Sto95].

In the grid structure, a is called an *immediate predecessor* of b, iff $H(a,b)$, and a *predecessor* of b, iff a is an immediate predecessor of either b, or one of its predecessors. Similarly, we can define the notion of $<$*-predecessor*. The following formula $\beta(x)$ with one free variable x asserts that if $b \leq x$, a is an immediate predecessor of b, and $c \leq b$, then $c < x$ and $c < b$:

$$\neg N(x) \wedge \forall yzu \notin N (H(z,y) \wedge u \leq z \wedge y \leq x \longrightarrow z < x \wedge z < y).$$

[4] Notice that, by the definition of the imaginary line, $\ell < r$ is impossible.

The following formula $\alpha(x)$ asserts the property β for x and for all its $<$-predecessors:

$$\forall y \notin N(y \leq x \longrightarrow \beta(y)).$$

In the above formulas, $u \leq v$ abbreviates $u = v \vee u < v$. Note that it is different from the arithmetical comparison $\leqslant$ defined in N. An important property implied by α is observed in the following:

Lemma 11. *Assume both the relations H and $<$ are acyclic and downward finite.*

If, for an element e, $\alpha(e)$, then the relation $<$ restricted to the set of H-predecessors of e together with e itself, extends the transitive closure of H on this set.

PROOF: Clearly, $<$ extends H. By induction, assume that if $(a, b) \in TC(H)$ and the difference in the levels of a and b is smaller than k, then the lemma holds.

Take $(a, b) \in TC(H)$ with the level difference k. Take an element c such that $(a, c) \in TC(H), H(c, b)$. Since, by induction, $\beta(b)$ and $a < c$, we have $a < b$.

Q.E.D.

Informally, our axioms for $<$ are organized into two groups—positive and negative—whose disjunction constitutes the axiomatization. The **positive part** is very simple, it asserts that $<$ is transitive and contains the transitive closure of H in N:

$$\forall xy \notin N(H(x, y) \longrightarrow x < y) \tag{4}$$

$$\forall xyz \notin N(x < y \wedge y < z \longrightarrow x < z) \tag{5}$$

For example, the rotated grids are going to be accepted because of this positive part alone.

The **negative part** accepts models where the relation $<$ is "weird". One example is when the relation disagrees with the level numbers, but, more importantly, when the following situation occurs:

- all the immediate predecessors of a certain a satisfy α
- a certain $b < a$
- for no immediate predecessor c of a, $c \leq b$

In this situation, it is guaranteed that b is not a predecessor of a (in the sense of H), still it is smaller than a, hence, the model is not an undergrid! Here are the **negative axioms**:

$$\exists xy \notin N \exists nm \in N(x \equiv n \wedge y \equiv m \wedge m \leqslant n \wedge x < y) \tag{6}$$

$$\exists xy \notin N(x < y \wedge \forall u \notin N(H(u, y) \longrightarrow \alpha(u)) \wedge \neg \exists z \notin N(H(z, y) \wedge z \leq x)) \tag{7}$$

The formal proof consists of a few lemmas:

Lemma 12. *The conjunction of the negative axioms is monotone in $<$.*

PROOF: If Axiom 6 is satisfied once, it is going to be satisfied in any extension of $<$. So the interesting case is when Axiom 6 is not satisfied for $<$, but Axiom 7 is satisfied, say, b, a witness the axiom. Extend $<$ arbitrarily. Again, assume Axiom 6 is not satisfied in the extension, hence $<$ is downward finite.

Call an element e *sound* (with respect to the current definition of $<$) iff all its predecessors are sound, and for any element f, $f < e$ iff f is a predecessor of e. Observe that all the elements at the level 1 are sound in the extension, and that soundness of an element implies the property α.

Observe that by Lemma 11, $<$ extends the transitive closure of H on the set of H-predecessors of a (because α holds for all the immediate predecessors of a). The same remains true in the extension. Therefore, if b, a do not witness Axiom 7 anymore, among the predecessors of a there is going to be an element a' which is not sound, but its all predecessors are sound. Hence, there exists a b' such that $b' < a'$, but $b' \not< c$ for any immediate predecessor c of a'.

These $b' < a'$ will now witness Axiom 7 in the extension.

Clearly, our proof relies on the downward finiteness of H.

Q.E.D.

Lemma 13. *If $<$ strictly extends the transitive closure of H, one of the negative axioms is satisfied.*

PROOF: Again, Axiom 6 is the trivial case. If the extension, however, does not satisfy Axiom 6, there is going to be an element which is not sound with all its predecessors being sound, which finishes the proof as in the case of Lemma 12.

Q.E.D.

Finally, our axiomatization is going to be:

$$\text{Axiom } 1 \wedge \cdots \wedge \text{Axiom } 3 \wedge$$

$$(\text{Axiom } 4 \wedge \text{Axiom } 5 \vee \text{Axiom } 6 \wedge \text{Axiom } 7)$$

Clearly, this axiomatization accepts every rotated grid, but does not accept any rotated undergrid. Combining this with Lemmas 12, 13, and Theorem 10, we finally have:

Theorem 14. *There exist a signature Ω and a sentence μ in this signature such that:*

- *μ is monotone in a predicate symbol $P \in \Omega$ over the class of recursive models of the signature Ω*
- *mu is not equivalent, over the class of recursive models of the signature, to any formula π in the signature Ω positive in the predicate symbol P*

5 Conclusion

There are many natural problems related to Recursive Model Theory whose status needs to be resolved. One fascinating open problem that we want to specifically attract public attention to is, whether an analogue of Gurevich-Shelah Theorem holds for recursive models. Recall that, although Lyndon's Lemma fails over finite models, the fixpoints of positive and monotone formulas over finite models were shown to be the same (see [GS86]). Likewise, the fixpoints continue to be the same in the standard model of arithmetic (and in some even more powerful recursive structures), even though the status of Lyndon's Lemma there continues to be open (see [Dou87]).

The Beth definability theorem as used in this paper sometimes is called the "weak Beth definability theorem", while the "Beth definability theorem" is used to refer to the version of the theorem in which "one and only one" is replaced to "at most one". Clearly, this relaxed version is refuted by this paper as well.

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