Noninterpretability of Infinite Linear Orders

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Abstract. We prove that no infinite linear order can be interpreted without parameters in any structure of the form $B \times B$, and describe applications of this result in recursion theory.

1 Squares

We are interested structures **A** which *fail* to satisfy the following property.

(ILOP) It is possible to interpret an infinite linear order in **A** without parameters.

The abbreviation stands for Infinite Linear Order Property. Interpretations of a structure **C** in a structure **A** are studied in [Ho 93]. In brief, an interpretation is a way to encode **C** into **A** using a finite collection of first–order formulas in the language of **A** as a decoding key. Elements of **C** are represented by tuples of a fixed length m of elements of B, modulo some definable equivalence relation. For the special case of an interpretation of a reflexive linear order Q in **A**, it is enough to consider a decoding key consisting of a single formula $\varphi(\overline{x}, \overline{y})$, also written as $\overline{x} \leq_Q \overline{y}$. Here $\overline{x}, \overline{y}$ are tuples of variables of length m, and, in **A**, the formula defines a linear pre–ordering on the domain $\{\overline{x} : \overline{x} \leq_Q \overline{x}\}$. For instance, $(\mathbb{Q}, \leq)$ can be interpreted in the ring $\mathbb{Z}$ using the formula

$$(z, w) \leq_Q (z', w') \equiv \psi_{\leq}(1, w) \wedge \psi_{\leq}(1, w') \wedge \psi_{\leq}(z \cdot w', z' \cdot w),$$

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where $\psi_{\leq}(x, y)$ expresses in the language of rings that $x \leq y$ (which is possible using Lagranges Theorem: an integer is nonnegative if and only if it is the sum of 4 squared integers).

We write $\overline{x} <_Q \overline{y}$ for $\overline{x} \leq_Q \overline{y} \wedge \overline{y} \not\leq_Q \overline{x}$ and $\equiv_Q$ for the equivalence relation on m-tuples

$$\{\overline{x}, \overline{y} : \overline{x} \leq_Q \overline{y} \leq_Q \overline{x}\}.$$

The fact that (ILOP) holds via a fixed decoding key can be expressed by an infinite collection of first-order sentences saying for each n that in the interpreted linear order there is a chain of length n. Therefore, (ILOP) only depends on the theory of $\mathbf{A}$.

Recall that we are interested in structures where (ILOP) fails. A weaker property than (ILOP) is the strict order property ([Ho 93], p. 317) of a structure $\mathbf{A}$: some formula $\varphi(\overline{x}, \overline{y})$ defines a *partial* order with arbitrarily long finite chains in $\mathbf{A}$. Even the weaker strict order property implies that the theory of $\mathbf{A}$ is unstable. However, the converse implication fails to hold: The random graph for instance has an unstable theory, but does not satisfy the strict order property. The random partial order (i.e. the Fraissé limit in the sense of [Ho 94] of the class of finite partial orders) or the random distributive lattice obviously have the strict order property, but still fail to satisfy (ILOP).

In asking whether (ILOP) holds via some given decoding key, it is safe to assume that

$$\{\overline{x} : \overline{x} \leq_Q \overline{x}\} = A^m$$

because else one can form a new equivalence class $\{\overline{x} : \overline{x} \not\leq_Q \overline{x}\}$ and put it at the beginning of an extended definable linear order, modifying $\leq_Q$ in the appropriate way.

The product $\mathbf{C} \times \mathbf{D}$ of structures $\mathbf{C}$, $\mathbf{D}$ for the same language is defined in the expected way ([Ho 93]). We some times call structures of the form $\mathbf{C} \times \mathbf{C}$ *squares.*

Theorem 1. *It is not possible to interpret an infinite linear order Q in a structure of the form $\mathbf{A} \times \mathbf{A}$ without parameters.*

Proof. Let $\mathbf{B} = \mathbf{A} \times \mathbf{A}$. The proof is by contradiction. Suppose that an interpretation of an infinite linear order in $\mathbf{B}$ is given so that $\{\overline{x} : \overline{x} \leq_Q \overline{x}\} = B^m$. Since a decoding key uses only finitely many symbols, we can assume that the language of $\mathbf{A}$ has a finite signature. To make the argument clearer, we will first assume that $m = 1$. We make use of the following fact. Suppose $\leq_Q$ is a definable linear order on a structure $\mathbf{C}$ such that the domain $\{x : x \leq_Q x\}$ equals C. If an automorphism π of $\mathbf{C}$ exchanges u and v, then $u \equiv_Q v$. This is because

$$u \leq_Q v \Rightarrow \pi(u) \leq_Q \pi(v) \Rightarrow v \leq_Q u. \tag{1}$$

Clearly $\mathbf{B}$ can be interpreted in $\mathbf{A}$. Thus "$\leq_Q$" can be viewed both as a definable binary relation on $\mathbf{B}$ and as a definable 4–ary relation on $\mathbf{A}$. Moreover the statement "$\leq_Q$ defines an infinite linear order" is a statement about $\mathrm{Th}(\mathbf{A})$.

We claim that the following 6-type in the language of $\mathbf{A}$ is consistent with $\mathrm{Th}(\mathbf{A})$:

(a) $(x_0, y_0) <_Q (x_1, y_1) <_Q (x_2, y_2)$
(b) The collection of first order-statements expressing that, for each i, j, $0 \leq i < j \leq 2$, (x_i, y_i, x_j, y_j) realize the same 4-type.

The statements in (b) are of the form

$$\varphi(x_0, y_0, x_1, y_1) \Leftrightarrow \varphi(x_0, y_0, x_2, y_2) \Leftrightarrow \varphi(x_1, y_1, x_2, y_2),$$

for any formula φ with four free variables. They also imply that the same 2-type is realized by any pair (x_i, y_i), $0 \leq i \leq 2$.

To prove the consistency of the 6-type with $\mathrm{Th}(\mathbf{A})$, we apply Ramseys Theorem as in the construction of Ehrenfeucht–Mostowski models. Since Q is infinite, we can suppose that there is an infinite ascending chain

$$(a_0, b_0) <_Q (a_1, b_1) <_Q \dots$$

in Q (if there is an infinite descending chain, one can argue similarly). Given finitely many formulas φ_k ($k = 0, \dots, n-1$) in 4 free variables, we assign one of 2^n possible colours to an unordered pair $\{(a_i, b_i), (a_j, b_j)\}$ ($i < j$) depending on which formulas $\varphi_k(a_i, b_i, a_j, b_j)$ hold. Then any homogeneous (in the sense of Ramsey) set of cardinality 3 shows the consistency of $\mathrm{Th}(\mathbf{A})$,

$$(x_0, y_0) <_Q (x_1, y_1) <_Q (x_2, y_2)$$

and the statements in (b) determined by a formula φ_k ($k = 0, \dots, n-1$). Thus the 6-type is consistent with $\mathrm{Th}(\mathbf{A})$.

Now, by standard results from model theory we can choose a countable ω-homogeneous model $\widetilde{\mathbf{A}}$ of $\mathrm{Th}(\mathbf{A})$ such that elements

$$(a_0, b_0) <_Q (a_1, b_1) <_Q (a_2, b_2)$$

realize the 6-type above. We will derive a contradiction. We need the following fact: If two pairs (a, b) and (a', b') realize the same 2-type in $\widetilde{\mathbf{A}}$, then

$$(a, b') \equiv_Q (a', b). \tag{2}$$

For let α be an automorphism of $\widetilde{\mathbf{A}}$ mapping a to a' and b to b'. Let S be the automorphism of $\widetilde{\mathbf{B}} := \widetilde{\mathbf{A}} \times \widetilde{\mathbf{A}}$ mapping a pair (x, y) to (y, x). Then, the automorphism of $\widetilde{\mathbf{B}}$

$$S \circ (\alpha, \alpha^{-1}),$$

exchanges (a, b') and (b, a').

Applying fact (1), we can conclude that $(a, b') \equiv_Q (b, a')$, and applying the same fact to S alone, $(a', b) \equiv_Q (b, a')$. So, by the transitivity of $\equiv_Q$,

$$(a, b') \equiv_Q (a', b).$$

We apply (2) two times to obtain the desired contradiction. First, a_0, b_1 and a_0, b_2 realize the same 2–type in $\tilde{\mathbf{A}}$ since they are the projections to the first and fourth components of the 4–types of a_0, b_0, a_1, b_1 and of a_0, b_0, a_2, b_2, respectively. Thus $(a_0, b_1) \equiv_Q (a_0, b_2)$. Also, a_0, b_2 and a_1, b_2 realize the same 2–type, so $(a_0, b_2) \equiv_Q (a_1, b_2)$. Now, by transitivity,

$$(a_0, b_1) \equiv_Q (a_1, b_2).$$

However, since $\tilde{\mathbf{A}}$ is ω–homogeneous, again by our definition of the 6–type, we can choose an automorphism β of $\tilde{\mathbf{A}}$ such that $\beta(b_i) = b_{i+1}$ $(i = 0, 1)$. Then, since $(\mathrm{Id}_{\tilde{\mathbf{A}}}, \beta)$ is an automorphism of $\tilde{\mathbf{B}}$, $(a_0, b_0) <_Q (a_1, b_1)$ implies that

$$(a_0, b_1) <_Q (a_1, b_2),$$

a contradiction.

We now adapt the argument to the case that $m = 2$. Then, by obvious notational changes the general case can be obtained. Note that $\leq_Q$ now is a 4–ary relation on $\mathbf{B}$, which can be viewed as an 8–ary relation on $\mathbf{A}$ via

$$(x_0, y_0, z_0, w_0) \leq_Q (x_1, y_1, z_1, w_1) \text{ in } \mathbf{A}$$
$$\Leftrightarrow ((x_0, y_0), (z_0, w_0)) \leq_Q ((x_1, y_1), (z_1, w_1)) \text{ in } \mathbf{B}$$

Fact (1) is used in a modified form: If an automorphism π of $\mathbf{C}$ (a structure like $\mathbf{B}$) exchanges u with v and p with q, then $(u, p) \equiv_Q (v, q)$.

The following 12–type in the language of $\mathbf{A}$ is consistent with $\mathrm{Th}(\mathbf{A})$:

(a') $(x_0, y_0, z_0, w_0) <_Q (x_1, y_1, z_1, w_1) <_Q (x_2, y_2, z_2, w_2)$

(b') The collection of first order–statements expressing that, for each i, j, $0 \leq i < j \leq 2$, $(x_i, y_i, z_i, w_i, x_j, y_j, z_j, w_j)$ realize the same 8–type.

The consistency of the 12–type with $\mathrm{Th}(\mathbf{A})$ is proved as before. Now let $\tilde{\mathbf{A}}$ be a countable ω–homogeneous model of $\mathrm{Th}(\mathbf{A})$ such that the elements

$$(a_0, b_0, c_0, d_0) <_Q (a_1, b_1, c_1, d_1) <_Q (a_2, b_2, c_2, d_2)$$

realize the 12–type. We derive a contradiction by applying the previous argument simultaneously to the first and the second half of the quadruples. If two pairs (a, b, c, d) and (a', b', c', d') realize the same 4–type in $\tilde{\mathbf{A}}$, then

$$(a, b', c, d') \equiv_Q (a', b, c', d).$$

For if α is an automorphism of $\tilde{\mathbf{A}}$ mapping (a, b, c, d) to (a', b', c', d'), and S is as before, then $S \circ (\alpha, \alpha^{-1})$ exchanges the pairs of elements of $\tilde{\mathbf{B}}$ $((a, b'), (c, d'))$ and $((b, a'), (d, c'))$, and S exchanges the last pair of pairs with $((a', b), (c', d))$.

Now, by similar arguments as before,

$$(a_0, b_1, c_0, d_1) \equiv_Q (a_0, b_2, c_0, d_2) \equiv_Q (a_1, b_2, c_1, d_2).$$

But we can also choose an automorphism β of $\tilde{\mathbf{A}}$ such that $\beta(b_i) = b_{i+1}$ and $\beta(d_i) = d_{i+1}(i = 0, 1)$. Then $(a_0, b_0, c_0, d_0) <_Q (a_1, b_1, c_1, d_1)$ implies that $(a_0, b_1, c_0, d_1) <_Q (a_1, b_2, c_1, d_2)$, contradiction. $\Diamond$

The statement of the theorem becomes false if one allows interpretations with parameters. For instance, if $B = \mathbb{N} \times \mathbb{N}$, consider $a = (1,0)$ as a parameter. The set of elements u in B which satisfy $a \cdot u = u$ is definable from a and equals the set of elements $\{(x,0) : x \in \mathbb{N}\}$, which is infinite and linearly ordered by $\leq^{\mathbf{B}}$. The same example shows that the statement is false in general for structures $\mathbf{C} \times \mathbf{D}$, even if $\mathbf{C}, \mathbf{D}$ are infinite: let $\mathbf{C} = (\mathbb{N}, \mathbf{0})$ and $\mathbf{D} = (\mathbb{N}, \mathbf{1})$.

2 Applications in recursion theory

In this section we give examples of structures arising naturally in recursion theory which are squares and therefore fail to satisfy (ILOP). First we discuss the relevance of (ILOP) in recursion theory. Examples of structures arising naturally are the lattice $\mathcal{E}$ of r.e. sets under inclusion and the structure $\mathcal{R}_{\mathcal{T}}$ of r.e. Turing degrees. The use of coding methods has been a very successful method of analyzing those structures. For all structures, coding methods were first used to prove the undecidability of the elementary theory. Stronger results were obtained by giving a (natural) many–one reduction of true arithmetic, i.e. Th($\mathbb{N}$), to the theory of the structure. Such a reduction shows that the theory is as complex as possible, provided that the structure itself is arithmetical, which is the case for $\mathcal{E}$, $\mathcal{R}_{\mathcal{T}}$ and many other structures occuring in recursion theory. However, the existence of such a reduction does not mean that an interpretation of the model $\mathbb{N}$ in the recursion theoretic structure can be given. Instead, to obtain the reduction, some coding of copies of $\mathbb{N}$ with parameters $\overline{p}$ is developed. As a next step, one finds a first–order correctness condition $\psi(\overline{p})$ on parameter lists which holds for some list, and always implies that the model coded using $\overline{p}$ is a standard model. Then, existentially quantifying over correct lists $\overline{p}$, one obtains an interpretation of true arithmetic.

All theories we consider in this section have the same computational complexity. If one looks at the structures themselves rather than just at their theories, finer distinctions emerge. In $\mathcal{R}_{\Updownarrow}$ (the structure of r.e. many-one degrees) and in $\mathcal{R}_{\mathcal{T}}$, a further development of the coding methods allows one to find an interpretation of $\mathbb{N}$ without parameters ([N95] and [NSSlta], respectively). Certainly this implies (ILOP). However, for $\mathcal{E}$, L. Harrington showed that even (ILOP) fails. Harrington used $\mathcal{E}$-specific methods, thereby paving the way for further noncoding and nondefinability results (see [HaNta]). Here we obtain Harringtons result in a purely model theoretic way, since $\mathcal{E}$ is a square. To see this, let $S \subseteq \omega$ be any recursive infinite and coinfinite set. Then the collection of subsets of S forms a lattice isomorphic to $\mathcal{E}$. Thus the map

$$X \mapsto (X \cap R, X \cap (\omega - R))$$

gives an isomorphism as required.

We now consider other structures from recursion theory where (ILOP) fails. The major subset relation is defined as follows: for $A, B \in \mathcal{E}$,

$$A \subset_m B \Leftrightarrow A \subset_\infty B \wedge (\forall W)[B \cup W = \omega \Rightarrow A \cup W =^* \omega].$$

Maass and Stob [MSt83] proved that the interval $[A, B]_{\mathcal{E}}$ has the same structure for any pair A, B such that $A \subset_m B$. This structure is denoted by $\mathcal{M}$. Thus, $\mathcal{M} = [A, B]$, where $A \subset_m B$. From the Maas–Stob result, it follows that $\mathcal{M}$ is a distributive lattice with strong homogeneity properties: all nontrivial closed intervals are isomorphic to the whole structure, and all nontrivial complemented elements are automorphic within $\mathcal{M}$. The undecidability of Th($\mathcal{M}$) is proved in [Nta1].

Theorem 2. *$\mathcal{M}$ fails to satisfy (ILOP).*

Proof. Choose $A \subset_m B$. Clearly, $\overline{B} \cup A$ is non–r.e. Then, by the Owings Splitting Theorem (see [So87]), there is a splitting $B = B_1 \cup B_2$ into disjoint sets such that $\overline{B_i} \cup A$ is non–r.e. for $i = 1, 2$. If $C_i = A \cup B_i$, then $A \subset_m C_i$ and hence, by the Maass–Stob result, $[A, C_i] \cong \mathcal{M}$. Now the map

$$X \mapsto (X \cap C_1, X \cap C_2)$$

$(X \in [A, B])$ gives an isomorphism $\mathcal{M} \cong \mathcal{M} \times \mathcal{M}$. So $\mathcal{M}$ fails to satisfy (ILOP). $\diamondsuit$

Note that the arguments above also work for the quotient lattices modulo finite differences $\mathcal{E}^*$ and $\mathcal{M}^*$.

Question. As mentioned in the introduction, the random graph, random partial order, and the random distributive lattice fail to satisfy (ILOP). On the other hand, $(\mathbb{Q}, \leq)$ certainly satisfies (ILOP). It would be interesting to find a criterion when a countable structure **A** such that Th(**A**) is ω–categorical and has quantifier elimination satisfies (ILOP) (see also [Ho93], p. 350). m

Acknoledgement. The question answered by Theorem 1.1 arose in discussions between L. Harrington and the second author.

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