Satisfaction classes and automorphisms of models of PA

Roman Kossak

1 Introduction

In recent years we have learned a lot about countable recursively saturated models of PA and their automorphisms. We know that there are continuum many nonisomorphic automorphism groups of such models, and we know that each of them contains a copy of the automorphism group of the order preserving permutations of the rationals. We can classify the closed normal subgroups of a given automorphism group, and we have a great deal of information about open maximal subgroups. We know much about the model theory of the arithmetically saturated countable models of PA; in particular we know that the automorphism group of a countable arithmetically saturated model of PA has the small index property. But still many questions remain open: Can we classify all normal and all maximal subgroups of the automorphism group a given model? Do non arithmetically saturated models have the small index property? Is the automorphism group of a countable recursively saturated model decidable?

Many other results and questions could be mentioned here. However, the purpose of this paper is not to give a complete survey; this has been done recently by Kotlarski in [17]. Instead, I will concentrate on a specific feature of countable recursively saturated models of PA — inductive satisfaction classes and their use.

My goal is twofold. Often satisfaction classes allow one to give easy answers to questions that otherwise seem difficult. A list of examples is presented in section 4. I hope those who work in the model theory of recursively saturated models of PA will find this list useful. The other goal is to propose the following problem. Much of the model theory of countable recursively saturated models of PA is based on specific techniques (resplendency arguments, special 'back-and-forth' constructions), but a significant number of results can be obtained as corollaries of classical results concerning models of PA* applied to structures of the form (M, S), where M is a recursively

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saturated model of PA and S is an inductive satisfaction class for M. A short survey such results constitutes sections 2 and 3. The advantage this approach is the availability of all the techniques developed for models of PA*, in particular the powerful machinery of definable types of Gaifman [1]. (PA* is Peano Arithmetic formulated in a language extending the language of PA.) The problem, I propose, is to determine what part of the model theory of countable recursively saturated models of PA can be obtained in this way. Of course, the answer might depend on the definition of model theory, and, perhaps, the biggest challenge is to give a precise formulation of the problem.

My presentation of the material will be a bit informal, as it is intended for an easy reading by the reader not necessarily interested in technical details. I assume that the reader is familiar with the notions of recursive saturation and of the standard system of a model of PA. I will also use the notion of arithmetic saturation. A model M of PA is *arithmetically saturated* if it is recursively saturated and its standard system satisfies the arithmetic comprehension schema. Definitions of all other concepts discussed here can be found in [3].

I will only discuss those problems concerning recursively saturated models of PA that can be formulated without direct references to satisfaction classes. There are many interesting results on the theory of satisfaction classes, most of them due to Kotlarski, Ratajczyk, and Smith. The interested reader should consult [17] and [20].

2 Recursive saturation and satisfaction classes

The notion of a satisfaction class requires an arithmetization of the language of arithmetic. The specifics of the arithmetization are not important, as long as it is can be formalized within PA. In what follows we will identify formulas and sentences of PA with their Gödel numbers.

Definition 2.1 *A set $S \subset M \models PA$ is an inductive satisfaction class for M if*

i) $\mathrm{Th}(M, a)_{a \in M} \subset S$;
ii) S satisfies Tarski's inductive definition of a satisfaction relation;
iii) $(M, S) \models PA^*$.

Tarski's argument shows that no inductive satisfaction class can be defined in a model of PA. The same argument can be applied to prove that if S extends $\mathrm{Th}(M, a)_{a \in M}$, and S does not contain sentences false in M, then S is undefinable in M. Definability here, and elswhere in the paper, means definability with parameters.

Here is the basic fact relating inductive satisfaction classes to recursive saturation.

Proposition 2.2 *Let M be a model of PA. i) If M has an inductive satisfaction class, then M is recursively saturated. ii) If M is countable and recursively saturated, then M has an inductive satisfaction class.*

The easy proof consists of an overspill argument in *i*, and a resplendency argument in *ii*.

It should be mentioned that the second part of proposition 2.2 is false for uncountable models. There are uncountable (even ω_1-like ones) recursively saturated models of PA without satisfaction classes. A rather classless model of Kaufmann [2] has this property. Direct constructions of such models can be found in [12] and [14].

If S is an inductive satisfaction class for a model M, then, by overspill S decides all Σ_e sentences, with parameters, in the sense of M, for some nonstandard e. Also, either S decides all sentences in the sense of M, in this case we say that S is *full*, or there is the largest e such that S decides all Σ_e sentences of M, in such a case we will say that S is a Σ_e-inductive satisfaction class for M.

To illustrate how inductive satisfaction classes are used, I will now prove four well-known results, whose first proofs did not use satisfaction classes explicitely. These results form a basis for the model theory of recursively saturated models of PA. The original proofs are not very difficult, so the point here is not that the proofs involving satisfaction classes offer significant simplification (although the new proofs are shorter). What is interesting about the proofs I want to present, is that no reference to recursive saturation is made directly, and that they only use standard results concerning models of PA*.

One of the results, which turns out to be applicable in many situations, and which the reader might not be familiar with, is the following version of Gaifman's theorem on cofinal extensions, due to Kotlarski [16] and Schmerl [18].

Lemma 2.1 *If (M, X) is a model of PA* and N is a cofinal extension of N, then there exists (exactly one) $Y \subset N$ such that $(M, X) \prec (N, Y)$.*

Model theory of recursively saturated models of arithmetic started with Smoryński's paper [21]. One of the important results of that paper was the following analog of the MacDowell-Specker theorem.

Theorem 2.3 *Every countable recursively saturated model of PA has a proper countable recursively saturated elementary end extension.*

Proof: Let M be a countable recursively saturated model of PA. Let S be an inductive satisfaction class for M. By the MacDowell-Specker theorem (M, S) has a countable elementary end extension (N, T). Since T is an inductive satisfaction class for N, N is recursively saturated and the result follows. □

The next result that contributed much to further developments is known as the Smoryński-Stavi theorem [22].

Theorem 2.4 *Every cofinal extension of a recursively saturated model is recursively saturated.*

Proof: We will consider the countable case first. Let M be a countable recursively saturated model of PA, and let N be its cofinal extension. Let S be an inductive satisfaction class for M. By lemma 2.1 there is $T \subset N$ such that $(M,S) \prec (N,T)$, hence N is recursively saturated.

If M is not countable then M is a union of a tower of elementary submodels each of which has a cofinal countable recursively saturated cofinal submodel, and the result follows, by the previous argument. □

The two theorem quoted above seem to indicate that the model theory of recursively saturated models of PA does not differ much from the model theory of PA. The next proposition, first noted in [5], shows that it is not so.

Recall that if $M \prec N$, then a subset X of M is *coded* in N if $X = M \cap Y$ for some set Y which is definable in N. We say that N is a *conservative extension* of M if the only subsets of M coded in N are the definable subsets of M, or, equivalently, the type of every element of $N \setminus M$ over M is definable. If N is a conservative extension of M, then N must be an end extension. Also, by a suitable version of the MacDowell-Specker theorem, every model of PA has a conservative elementary end extension.

Proposition 2.5 *If N is a recursively saturated model of PA and $M \prec N$, then N is not a conservative extension of M.*

Proof: W.l.o.g. we can assume that N has a cofinal countable elementary submodel; hence, by proposition 2.2 and lemma 2.1, N has an inductive satisfaction class S. Let $T = M \cap S$. Since $(M,T) \models$PA*, it is easy to show that T is coded in M. T might not be an inductive satisfaction class for M (and indeed, it is not an inductive satisfaction class for M if M is not recursively saturated), but still by Tarski's argument T is undefinable in M and the result follows. □

According to proposition 2.5, recursively saturated elementary end extension of a model M always codes a nondefinable subset of M, but still we have a weak version of conservativeness for recursively saturated elementary end extensions: if M is countable, then every nondefinable subset of M can be omitted in a recursively saturated elementary end extension of M. This lemma was proved first by Kaufmann [2] using different methods, another short proof using satisfaction classes can be found in [18].

Lemma 2.2 *Let X be a nondefinable subset of a countable recursively saturated model M of PA, then there is a countable recursively saturated elementary end extension N of M in which X is not coded.*

Proof: Let us first suppose that (M, X) is not recursively saturated. Let S be an inductive satisfaction class for M such that (M, S) is recursively saturated. Such satisfaction classes exist by chronic resplendency of countable recursively saturated models (see [3]). Let (N, K) be a conservative elementary end extension of (M, S). Then, if Y is a subset of M coded in N, then Y must be definable in (M, S); hence (M, Y) is recursively saturated. Thus X is not coded in N.

Suppose now that (M, X) is recursively saturated. Consider the theory Σ saying that S is an inductive satisfaction class and X is undefinable in (M, S). Each finite fragment of Σ is modeled by (M, Tr_n), for n large enough, where Tr_n is the universal Σ_n relation (n is standard here). Hence M can be expanded to a model of Σ. If (M, S) is such an expansion, then we again take (N, T) to be a conservative extension of (M, S), and the result follows. □

3 Minimal satisfaction classes and s-ultrapowers

In the last proof of the previous section we have, tacitly, used definable ultrapowers of structures of the form (N, S). A *definable ultrapower* of a structure $\mathcal{A}$ is an ultrapower of $\mathcal{A}$ built using the set of definable functions of $\mathcal{A}$ and an ultrafilter on the definable subsets of $\mathcal{A}$. For brevity, we will call a recursively saturated model N an *s-ultrapower* if there is a recursively saturated model M and an inductive satisfaction class S for M such that (N, T) is a definable ultrapower of (M, S), for some $T \subset N$.

It is shown in [6] that every countable recursively saturated model of PA can be expanded by adding an inductive satisfaction class in continuum many elementary inequivalent ways. Also, for every such expansion (M, S) there are continuum many nonisomorphic expansions (M, T) which are all elementarily equivalent to (M, S). Using this variety of satisfaction classes it is not difficult to show that for every countable recursively saturated model M, there are continuum many nonisomorphic pairs of the form (M, K) where $K \prec_{end} M$ and M is an s-ultrapower of K. Thus, the question to consider is: Suppose M and N are countable recursively saturated models of PA and $M \prec_{end} N$, is N an s-ultrapower of M? The answer to such a general question is negative. For a model M and $a \in M$, let $M[a]$ denote the largest elementary submodel of M not containing a (if there is such a model). If M is a recursively saturated model of PA, then, for every $a \in M$, M is not a s-ultrapower of $M[a]$, an outline of the argument proving this is given in the discussion of question 4.1 in section 4. We also have the following proposition.

Proposition 3.1 *If M is countable recursively saturated but not arithmetically saturated, model of PA then there is a recursively saturated $K \prec_{end} M$ such that none of the inductive satisfaction classes of K is coded in M.*

For a proof see proposition 4.6 of [7]. Clearly, if K is as in the above proposition, then M is not an s-ultrapower of K.

Problem 3.2 *For a given countable recursively saturated model M of PA, classify all recursively saturated $K \prec_{end} M$ such that M is an s-ultrapower of K.*

An important class of s-ultrapowers is obtained by using minimal satisfaction classes.

For a structure $\mathcal{A}$, DEF$(\mathcal{A})$ will denote the family of sets definable in $\mathcal{A}$ (with parameters), and Def$(\mathcal{A})$ will be the set of points definable without parameters in $\mathcal{A}$.

Definition 3.3 *An inductive satisfaction class S for a model M is minimal, if* $\mathrm{Def}(M, S) = M$.

Theorem 3.4 *Every countable recursively saturated model of PA has a minimal inductive satisfaction class. Moreover, if M has a Σ_e inductive satisfaction class, for some $e \in M$, then M has continuum many pairwise elementarily inequivalent structures of the form (M, S), where S is a minimal Σ_e inductive satisfaction class for M.*

Theorem 3.4 was proved in a slightly weaker form in [6], the present formulation is taken from [14]. Here is the outline of the proof. Start with the theory $T_0 = \mathrm{Th}(M)$ plus the set of sentences saying that S is a Σ_e inductive satisfaction class. T_0 is coded in the standard system of M. The crucial point now is that one can find a complete consistent theory T extending T_0, and such that the family of sets of natural numbers represented in T is exactly the standard system of M. If (M_0, S_0) is the minimal model of T, then S_0 is a minimal inductive satisfaction class for M_0. M_0 is elementarily equivalent to M, both models are recursively saturated, and they have the same standard systems, hence M_0 is isomorphic to M, and the result follows.

To obtain the second part of the theorem it is enough to notice that the completion T in the above proof can be obtained in continuum many different ways.

If S is a minimal inductive satisfaction class for a model M, then the srtucture (M, S) is rigid (i.e. has no nontrivial automorphisms). But we can prove more.

Proposition 3.5 *If S is a minimal inductive satisfaction class for a model M, then the structure $(M, \mathrm{DEF}(M, S), \in)$ is rigid.*

The key to the proof of 3.5 is the following lemma:

Lemma 3.1 *If S is a Σ_e inductive satisfaction class for a model M, then there exists a Σ_a inductive satisfaction class that is definable in (M, S) iff $a < e + n$ for some standard n.*

Proofs of 3.5 and 3.1 are not difficult, they can be found in [14].

If K and M are countable recursively saturated models of PA, and M is an elementary end extension of K, then it is important to know which automorphisms of K can be extended to M.

Let $\mathrm{Coded}_M(K)$ denote the family of those subsets of K which are coded in M.If $f \in \mathrm{Aut}(K)$ extends to M, then, f must be an automorphism of $(K, \mathrm{Coded}_M(K), \in)$. In most cases the converse is also true [10]:

Lemma 3.2 *Let M be a countable recursively saturated model of PA, then for all but countably many recursively saturated $K \prec_{end} M$, an automorphism f of K can be extended to an automorphism of M iff f is an automorphism of $(K, \mathrm{Coded}_M(K), \in)$.*

See [11] for the discussion of the special cases in which lemma 3.2 fails.

The interesting problem is: What are the groups of automorphisms of structures of the form $(K, \mathrm{Coded}_M(K), \in)$?. It turns out that they can be almost anything, and the key to this result is an application of minimal satisfaction classes. First an extreme example:

Theorem 3.6 *If M is an s-ultrapower of a recursively saturated model K, built with a minimal inductive satisfaction class for K, then no nontrivial automorphism of K can be extended to M.*

Theorem 3.6 is an easy consequence of 3.5. The proof is given in [14].

Let M be a given countable recursively saturated model of PA, and let G be the automorphism group of M. For $X \subset M$ let $G_{\{X\}}$ be the setwise stabilizer of X and let $G_{(X)}$ be the pointwise stabilizer of X. According to lemma 3.2, for most elementary initial segments $K \prec_{end} M$ the group $G_{\{K\}}/G_{(K)}$ is isomorphic to $\mathrm{Aut}(K, \mathrm{Coded}_M(K), \in)$, in particular, theorem 3.6 can be reformulated as follows. There is $K \prec_{end} M$ such that the quotient group $G_{\{K\}}/G_{(K)} = \mathrm{Aut}(K, \mathrm{Coded}_M(K), \in)$ is trivial. Much more can be done in this direction. Using minimal satisfaction classes and a result of Gaifman [1] we have shown in [9] that:

Theorem 3.7 *For every countable recursively saturated model M and for every countable linearly ordered set $(I, <)$ there is $K \prec_{end} M$ such that M is an s-ultrapower of K, and $G_{\{K\}}/G_{(K)} \cong \mathrm{Aut}(I, <)$.*

Since the models K obtained in the proof of 3.7 satisfy the assumptions of lemma 3.2 (i.e. they are not among the special cases), theorem 3.7 also provides information about the groups $\mathrm{Aut}(K, \mathrm{Coded}_M(K), \in)$.

One of the consequences of theorem 3.7 is that the group G is not divisible; hence it is not elementarily equivalent to the group of order preserving permutations of the rationals.

Recently Schmerl [19] has shown that in the formulation of theorem 3.7 "*linearly ordered set* $(I, <)$" can be replaced by "*linearly ordered structure* $(I, <, \ldots)$." Schmerl's proof uses s-ultrapowers.

4 Examples and counterexamples

Throughout this section, let M be a countable recursively saturated model of PA, and let G =Aut(M).

Question 4.1 *Can every recursively saturated $K \prec M$ be represented as* Def(M, S) *for some inductive satisfaction class S for M?*

Answer: In general the answer is negative. If K =Def(M, S) for some inductive satisfaction class S, then $(K,T) \prec (M, S)$, where $T = S \cap K$ is an inductive satisfaction class for K coded in M. Thus, the answer is provided by proposition 3.1. Also, since in (M, S) one can define functions majorizing all definable functions of M, if $(K,T) \prec (M, S)$, and $K \prec_{end} M$, then for every $a \in M \setminus K$ there is $b \in M \setminus K$ such that $t(b) < a$, for every function t definable in M without parameters. Hence models $M[a]$ cannot be represented as Def(M, S).

However, if K is a cofinal submodel of M, then the answer is positive. In this case, let T be a minimal inductive satisfaction class for K. By lemma 2.1, there is $S \subset M$ such that $(K,T) \prec (M, S)$. Thus, Def$(M, S) = K$. □

Question 4.2 *Let f be an automorphism of M. Is there an inductive satisfaction class such that $fS = S$?*

Answer: If M is arithmetically saturated, then the answer is negative. By fix(f) we will denote the set of points fixed by f. We will consider fix(f) as a submodel of M. Clearly, fix$(f) \prec M$. If $fS = S$ for some inductive satisfaction class S for M, fix$(f) \cap S$, is an inductive satisfaction class for fix(f). Thus, if fix(f) is nonstandard it is recursively saturated. It is well-known that if M is arithmetically saturated then there are $f \in G$ such that fix(f) is nonstandard and not recursively saturated (cf.[15], [4]), and the result follows.

If M is not arithmetically saturated, then, for every $f \in G$, fix$(f) \cong M$, hence the previous argument cannot be applied. The problem is still open in this case. □

Question 4.3 *Suppose M is arithmetically saturated. Is every $K \prec M$ of the form* fix(f) *for some $f \in G$?*

Answer: The answer is negative. It is shown in [15] that if K =Def(M, S) for an inductive satisfaction class S such that (M, S) is recursively saturated, then K has continuum many elementary substructures which are not of the form fix(f). □

Question 4.3 is related to the following important open problem posed in [4].

Problem 4.4 *Characterize elementary submodels of M which are of the form* fix(f), *for $f \in G$.*

Question 4.5 *Suppose that M is a model of true arithmetic. Is there an $f \in G$ such that $f(a) > a$ for all nonstandard $a \in M$?*

Answer: In [15] a special 'back-and-forth' argument was designed to show that the answer to the question is positive iff M is arithmetically saturated. Here is a much shorter proof using satisfaction classes. The proof is due to Jim Schmerl.

Let M be a countable arithmetically saturated model of true arithmetic. Let $A_0, A_1, \ldots$ be an enumeration of the standard system M. By a result of Gaifman [1], we can find an elementary extension $\mathcal{N}$ of $(\mathbf{N}, A_0, A_1, \ldots)$, which is generated by a set of indiscernibles of the order type of the integers. Hence, $\mathcal{N}$ has an automorphism f such that $f(a) > a$ for all nonstandard a. Let N be the reduct of $\mathcal{N}$ to the language of PA. To finish the proof we have to make two observations. Since M is a recursively saturated model of true arithmetic, the list $A_0, A_1, \ldots$ contains the standard satisfaction class for $\mathbf{N}$, hence N is recursively saturated. Also, since $\mathcal{N}$ is generated by elements realizing a minimal (hence definable) type over $(\mathbf{N}, A_0, A_1, \ldots)$, the standard system of $\mathcal{N}$ (which is the same as the standard system of N) is $\{A_0, A_1, \ldots\}$. Here we are using the fact that $\{A_0, A_1, \ldots\}$ is closed under arithmetic comprehension. Thus N is isomorphic to M and the result follows. □

Inductive satisfaction classes can be also applied to problems concerning isomorphism types of structures of the form (M, K), where $K \prec_{end} M$. Two such applications are given in [8]. They both involve the following concept.

Definition 4.6 *For $K \prec_{end} M$, let $\mathcal{S}(K)$ be the set of those $e \in M$ for which there exists a Σ_e inductive satisfaction class for K, which is coded in M.*

The cut $\mathcal{S}(K)$ is definable in (M, K), hence it can be used, in a way similar to that in which the cofinality of a cut is used, to construct structures (M, K) of different isomorphism types. Arguments using the cofinality of a cut apply to non-semiregular cuts only; arguments using $\mathcal{S}(K)$ apply to a much larger family of cuts.

Problem 4.7 *Characterize cuts of the form $\mathcal{S}(K)$ for a given countable recursively saturated model of PA.*

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Author address

Department of Mathematics and Computer Science
BCC/City University of New York
University Ave. & W 181 Street
Bronx, NY 10453, USA
email: `rkobb@cunyvm.cuny.edu`