IP Current Developments in Mathematics, 2005

Modularity of 2-dimensional Galois representations

Mark Kisin

Introduction

Our aim is to explain some recent results on modularity of 2-dimensional potentially Barsotti-Tate Galois representations. That such representations should arise from modular forms is a special case of a remarkable conjecture of Fontaine and Mazur [**FM**]. One of its concrete consequences is that if $A/\mathbb{Q}$ is an abelian variety of GL_2-type, then A is a subquotient of a product of Jacobians of modular curves.

The first breakthrough in the direction of this conjecture was the work of Wiles and Taylor-Wiles [**Wi**], [**TW**], which established that (under mild hypothesis) the conjecture holds for 2-dimensional p-adic representations ρ which are Barsotti-Tate at p provided that the associated mod p representation $\bar{\rho}$ is modular and irreducible. These results were extended by a number of authors [**Di 1**], [**CDT**], [**BCDT**], and a lifting theorem of this type for fairly general potentially Barsotti-Tate representations was proved in [**Ki 1**]. For ordinary representations with $\bar{\rho}$ reducible, the conjecture was proved by Skinner-Wiles [**SW**].

The condition that $\bar{\rho}$ was modular could be verified in certain special cases. The results mentioned in the previous paragraph were then sufficient to deduce the conjecture of Shimura-Taniyama-Weil that any elliptic curve over $\mathbb{Q}$ is modular. The case of semi-stable elliptic curves was established by Wiles [**Wi**] and the general case by Breuil-Conrad-Diamond-Taylor [**BCDT**].

However, Serre [**Se 1**] had conjectured that any two-dimensional mod p representation with odd determinant was modular. A few years ago Taylor established a weaker form of this conjecture [**Ta 1**], [**Ta 2**], which asserts that for some totally real field F in which p is unramified, $\bar{\rho}|_F$ arises from a Hilbert modular form. Combining this with the kind

The author was partially supported by NSF grant DMS-0400666 and a Sloan Research Fellowship.

of modularity lifting theorem mentioned above, he was able to show that certain Barsotti-Tate representations - or more generally crystalline representation of small weight - could be put into compatible systems of λ-adic representations (cf. also [**Die**]).

In a spectacular development, Khare-Wintenberger [**KW 1**] and Khare [**Kh 1**] were able to build on these results and prove Serre's conjecture for representations of level 1. More recently [**KW 2**], [**KW 3**] they have extended their methods to prove the conjecture for odd level, and to reduce the case of even level to a 2-adic modularity lifting theorem which was finally proved in [**Ki 5**]. One of the consequences of Serre's conjecture - observed by Serre [**Se 1**, 4.7] and Ribet [**Ri 2**] - is the modularity of abelian varieties of GL_2-type.

To state the results we are going to explain, fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ for each finite prime p. If E is a number field, we will refer to an embedding $\lambda : E \hookrightarrow \bar{\mathbb{Q}}_p$ as a finite prime $\lambda|p$. We write E_λ for the closure of $\lambda(E)$. This is an abuse of terminology, since a prime of E (in the usual sense) may correspond to several different embeddings, however this convention will prove to be very useful. For a finite set of primes S of $\mathbb{Q}$ we denote by $G_{\mathbb{Q},S}$ the Galois group of the maximal subfield of $\bar{\mathbb{Q}}$ unramified outside S.

The following theorem was proved by Eichler and Shimura for $k = 2$, Deligne [**De**] for $k \geq 2$, and Deligne-Serre when $k = 1$ [**DS**].

THEOREM. *Let $k \geq 1$, $N \geq 1$, and $f \in S_k(\Gamma_1(N), \mathbb{C})$ a cuspidal eigenform on $\Gamma_1(N)$, normalized so that f has Fourier expansion $f = \sum_{i=1}^{\infty} a_n q^n$, with $a_1 = 1$. Then*

(1) *The field $E_f := \mathbb{Q}(a_n)_{n\geq 1} \subset \mathbb{C}$ is a number field.*
(2) *For any finite prime $\lambda|p$ of E_f, there exists a continuous representation*

$$\rho_{f,\lambda} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(E_{f,\lambda})$$

such that $\mathrm{tr}(\rho_{f,\lambda}(\mathrm{Frob}_v)) = a_v$ for any rational prime $v \nmid pN$. Here S is the set of primes dividing pN together with ∞, and $\mathrm{Frob}_v \in G_{\mathbb{Q},S}$ denotes an arithmetic Frobenius.

If $E/\mathbb{Q}_p$ is a finite extension (always assumed contained in $\bar{\mathbb{Q}}_p$) with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$, and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(E)$ is a continuous representation, then ρ is called modular if there exists a modular eigenform f, and a prime $\lambda|p$ of E_f, such that $\rho \sim \rho_{f,\lambda}$. (That is, ρ and $\rho_{f,\lambda}$ become isomorphic after an extension of scalars.)

Given such a ρ, we will denote by $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ the representation obtained by choosing a Galois stable $\mathcal{O}$-lattice and reducing modulo the radical of $\mathcal{O}$. Although $\bar{\rho}$ depends on the choice of lattice, its semi-simplification does not. We will say $\bar{\rho}$ is modular if there exists f and λ such that the semi-simplifications of $\bar{\rho}$ and $\bar{\rho}_{f,\lambda}$ are equivalent. If $\det \bar{\rho}$ sends complex conjugation to -1, we say that $\bar{\rho}$ is odd.

THEOREM 0.1. *Suppose that*

(1) $\bar{\rho}$ *is modular and* $\bar{\rho}|_{\mathbb{Q}(\zeta_p)}$ *is absolutely irreducible and has insoluble image if* $p = 2$.
(2) ρ *is potentially Barsotti-Tate at* p, *and* $\det \rho = \psi\chi$, *where* χ *denotes the* p*-adic cyclotomic character, and* ψ *has finite order.*

Then ρ *is modular.*

This result was proved in [**Ki 1**] and [**Ki 5**] when $p = 2$.. Recall that the second condition means that there exists a finite extension $K/\mathbb{Q}_p$ such that $\rho|_{G_K}$ arises from the Tate module of a p-divisible group, where $G_K = \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$.

As already remarked, thanks to the work of Khare-Wintenberger, completed in [**Ki 5**], we have the following result conjectured by Serre:

THEOREM 0.2. *Suppose that* $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ *is odd and absolutely irreducible. Then* $\bar{\rho}$ *is modular.*

COROLLARY 0.3. *Let* $A/\mathbb{Q}$ *be an abelian variety of dimension* g, *and suppose that there exists a number field* F *with* $[F : \mathbb{Q}] = g$, *and an embedding* $F \hookrightarrow \mathrm{End}_{\mathbb{Q}} A \otimes_{\mathbb{Z}} \mathbb{Q}$. *(i.e* A *is of* GL_2*-type). Then* A *is a quotient of* $\mathrm{Jac}(X_1(N))^m$ *for some* $N, m \geq 1$. *Moreover the* L*-function* $L(A, s)$ *is entire and satisfies a functional equation with respect to* $s \mapsto 2 - s$.

That the theorem implies the corollary was established by Ribet [**Ri 2**, Thm. 4.4] following an argument of Serre [**Se 1**, Thm. 5], who considered the case when F is totally real. Khare [**Kh 1**] observed that one can adapt Serre's argument to show that Theorem (0.2) implies the odd two dimensional Artin conjecture:

COROLLARY 0.4. *Let* $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{C})$ *be a continuous, irreducible, odd representation. Then* ρ *arises from a weight* 1 *cusp form on* $\Gamma_1(N)$ *for some* $N \geq 1$. *In particular, the Artin* L*-function* $L(\rho, s)$ *is entire.*

We remark that Artin's conjecture for odd, two dimensional representations was previously known in many cases, thanks to the work of Langlands [**La**], Tunnel [**Tu**], Buzzard–Dickinson–Shepherd-Barron–Taylor [**BDST**], and Taylor [**Ta 4**].

Finally, one may re-inject Theorem (0.2) into Theorem (0.1) (which is of course *used* in the proof of Serre's conjecture) to obtain

COROLLARY 0.5. *Let* $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathcal{O})$ *be an continuous, representation such that*

(1) ρ *is potentially Barsotti-Tate at* p, *and* $\det \rho$ *is equal to the cyclotomic character times an even character of finite order.*
(2) $\bar{\rho}|_{\mathbb{Q}(\zeta_p)}$ *is absolutely irreducible and has insoluble image if* $p = 2$.

Then ρ *arises from a holomorphic modular eigenform of weight* 2.

In the first section of this article we explain the refined form of Serre's conjecture, and why it implies (0.3) and (0.4) above. In fact we extend this result to motives over $\mathbb{Q}$ of rank 2. This extension is already suggested by Serre's article, where the case of a smooth projective variety with 2-dimensional cohomology is considered. In the second section we explain some of the ideas behind the proof of Theorem (0.1). In the third we sketch the argument for Taylor's result that $\bar{\rho}|_F$ is modular for some totally real field F. Finally we explain some of the ideas which go into the work of Khare-Wintenberger on Serre's conjecture. In particular, we explain Khare's argument in the level one case.

Acknowledgment. I would like to thank A. Beilinson, G. Böckle, K. Buzzard, M. Emerton, C. Khare and the referee for useful remarks.

1. Serre's conjecture and its consequences

1.1. The strong Serre conjecture. For a subfield $F \subset \bar{\mathbb{Q}}$ (resp. $F \subset \bar{\mathbb{Q}}_p$) we will denote by G_F the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ (resp. $\mathrm{Gal}(\bar{\mathbb{Q}}_p/F)$). If S is a finite set of places of F, we denote by $G_{F,S}$ the Galois group of the maximal extension of F in $\bar{Q}$ unramified outside S.

In this section we recall the precise form of Serre's conjecture which predicts not only that an odd representation $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ arises from a modular form, but also the minimal weight and level of the form which gives rise to it. We then recall Serre's argument deducing the modularity GL_2-type abelian varieties (or more generally of two dimensional odd motives), as well as Khare's modification of this argument, which allows one to deduce the Artin conjecture for odd, two dimensional representations. These applications use the refined form of Serre's conjecture, which will also be needed at various points later in this article.

1.1.1. Let $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$ denote the inertia subgroup. We denote by

$$\omega_i : I_{\mathbb{Q}_p} \to \mathbb{F}_{p^i}^{\times}; \quad \sigma \mapsto \sigma(\sqrt[p^i-1]{p})/\sqrt[p^i-1]{p} \pmod{p}$$

the fundamental character of level i. We will write ω for ω_1, which is the mod p reduction of the p-adic cyclotomic character, and we again denote by $\omega : G_{\mathbb{Q},S} \to \mathbb{F}^{\times}$ the mod p cyclotomic character.

Suppose we are given a representation $\bar{\rho}_p : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$. Then $\bar{\rho}_p|_{I_{\mathbb{Q}_p}}$ is either of the form $\left(\begin{smallmatrix} \omega^i & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \omega^j$ with $i, j \in \mathbb{Z}$ or $\begin{pmatrix} \omega_2^i & 0 \\ 0 & \omega_2^{pi} \end{pmatrix} \otimes \omega^j$ for some integers $i, j \in \mathbb{Z}$, and $p+1 \nmid i$.

When $\bar{\rho}_{I_{\mathbb{Q}_p}}$ is semi-simple (i.e., tamely ramified) we may choose $i, j \geq 0$ and such that $j \in [0, p-2]$ and $i + j \in [1, p-1]$. When $\bar{\rho}_{I_{\mathbb{Q}_p}}$ is wildly ramified, $i, j \in [0, p-2]$ are uniquely determined. We set $k(\bar{\rho}) = 1 + i + (p+1)j$ unless $\bar{\rho}_p|_{I_{\mathbb{Q}_p}} \sim \left(\begin{smallmatrix} \omega & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \omega^j$ with $*$ très ramifiée, which means that the cocycle $*$ has the form $\sigma \mapsto \sigma(\sqrt[p]{u})/\sqrt[p]{u}$ for some $u \in \mathbb{Q}_p^\times$ such that $p \nmid v_p(u)$. In this exceptional case we set $k(\bar{\rho}) = (p+1)(j+1)$.

For a continuous representation $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$, we set $k(\bar{\rho}) = k(\bar{\rho}|_{G_{\mathbb{Q}_p}})$ and we set

$$N(\bar{\rho}) = \prod_{l \neq p} \mathrm{cond}(\bar{\rho}|_{G_{\mathbb{Q}_l}})$$

where l runs over the finite primes of S not equal to p, and $\mathrm{cond}(\bar{\rho}|_{G_{\mathbb{Q}_l}})$ denotes the Artin conductor of $\bar{\rho}|_{G_{\mathbb{Q}_l}}$. If V denotes the underlying $\mathbb{F}$-vector space of $\bar{\rho}$, then this is an integral power of l with

$$v_l(\mathrm{cond}(\bar{\rho}|_{G_{\mathbb{Q}_l}})) = \sum_{i=0}^{\infty} \frac{1}{(G_0 : G_i)} \dim V/V^{G_i},$$

where the $G_i \subset G_{\mathbb{Q}_l}$ are the ramification subgroups.

Serre made the following:

Conjecture 1.1.2. *Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be odd and absolutely irreducible. Then $\bar{\rho} \sim \bar{\rho}_{f,\lambda}$ where f is an eigenform of weight $k(\bar{\rho})$ and level $N(\bar{\rho})$.*

1.1.3. *Remarks.*

(1) We have stated the conjecture in a way which differs slightly from the formulation in [**Se 1**]. First, Serre specified a character for the form f in terms of $\bar{\rho}$. As observed by Serre [**Se 2**, p. 197], this form of the conjecture is correct when $p \geq 5$ (in which case it is easily seen to be equivalent to the form we have given here using Carayol's lemma [**Ca 2**, Prop. 5]), but wrong for $p = 2, 3$. In the latter case this can be rectified by using Katz modular forms (cf. [**Ed**]). Secondly, when $p = 2$ and $\bar{\rho}_p|_{I_{\mathbb{Q}_p}} \sim \left(\begin{smallmatrix} \omega & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \omega^j$ with $*$ très ramifiée, Serre set $k(\bar{\rho}) = 4$ rather than 3. The reason for this is that Serre's choice of character is even, so $k(\bar{\rho}) = 3$ is impossible if one insists on it. Without this choice, $k(\bar{\rho}) = 3$ seems to be a more natural convention.

(2) When $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is a sum of two unramified characters - so that $k(\bar{\rho}) = p$ - Serre predicted that $\bar{\rho}$ also arises from a weight 1 Katz modular form. Here one is really forced to use Katz modular forms, since in general there is an obstruction to lifting a form from characteristic p to characteristic 0. However, if we fix N, then this obstruction is contained in the p-torsion of a finite $\mathbb{Z}[1/N]$-module (more precisely of the cohomology of a line bundle on a modular curve over $\mathbb{Z}[1/N]$-cf. [**Ka**, 1.6, 1.7]), and hence vanishes for almost all p.

Serre's conjecture on weight 1 forms was proved by Gross [**Gr**] assuming that $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ has non scalar semi-simplification, and by Coleman-Voloch [**CV**] assuming $p > 2$. The remaining case with $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ having scalar semi-simplification and $p = 2$ still seems to be open.

(3) One can ask for the weights of all the modular forms giving rise to $\bar{\rho}$. By using the mod p representation theory of $\mathrm{GL}_2(\mathbb{F}_p)$, and considering $\bar{\rho}$ together with all its twists, one can reduce this to the question of whether there is a modular form of weight $k \in [1, p+1]$ (and prime to p level) giving rise to $\bar{\rho}$.

For example, if $\bar{\rho}|_{I_{\mathbb{Q}_p}} \sim \begin{pmatrix} \omega_2^i & 0 \\ 0 & \omega_2^{pi} \end{pmatrix}$ for some $i \in [1, p-1]$, then $k(\bar{\rho}) = 1 + i$. However, given that $\bar{\rho}$ is modular, a twist of it also arises from a form of weight $p+2-i$, because $\bar{\rho}|_{I_{\mathbb{Q}_p}} \sim \begin{pmatrix} \omega_2^{p+1-i} & 0 \\ 0 & \omega_2^{p(p+1-i)} \end{pmatrix} \otimes \omega^{i-1}$. (For $i = 1$ this is not formally implied by (1.1.2), but is still true).

Similarly, if $\bar{\rho}|_{I_{\mathbb{Q}_p}} \sim \begin{pmatrix} \omega^i & 0 \\ 0 & 1 \end{pmatrix}$, so $k(\bar{\rho}) = i+1$, then $\bar{\rho}|_{I_{\mathbb{Q}_p}} \sim \begin{pmatrix} \omega^{p-1-i} & 0 \\ 0 & 1 \end{pmatrix} \otimes \omega^i$, which suggests that $\bar{\rho} \otimes \omega^{-i}$ is modular of weight $p-i$. More precisely, if $i \neq p-1$ then this is predicted by (1.1.2) and was proved in above papers of Gross and Coleman-Voloch. If $i = p$ then this is the situation already mentioned in (2). Serre called the weight $i+1$ form giving rise to $\bar{\rho}$ and the weight $p+1-i$ form giving rise to $\bar{\rho} \otimes \omega^{-i}$, companion forms.

THEOREM 1.1.4. *Suppose that $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ arises from some modular eigenform. Then it arises from a form of weight $k(\bar{\rho})$ and level $N(\bar{\rho})$.*

PROOF. This is the consequence of the work of several people, chiefly Ribet [**Ri 1**], [**Ri 3**] for results regarding the level $N(\bar{\rho})$ and Gross, Coleman-Voloch and Edixhoven [**Ed**] for the weight $k(\bar{\rho})$. For $p > 2$, the general statement was completed by Diamond [**Di 3**].

When $p = 2$ this was proved by Buzzard [**Bu**, Prop. 2.4, Thm. 3.2] provided that $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is not scalar, and by Wiese [**Wie**] when $\bar{\rho}$ is dihedral (cf. also [**Se 1**, Prop. 10]). The general non-dihedral case when $k(\bar{\rho}) = 2$ is contained in the work of Khare-Wintenberger [**KW 2**, Thm. 1.2]. However, this uses the techniques introduced by Khare-Wintenberger in a serious way. $\square$

1.2. Modularity of Abelian varieties of GL_2-type. Recall [**Ri 2**] that an abelian variety over $\mathbb{Q}$ of GL_2-type is an abelian variety $A/\mathbb{Q}$ with an embedding $E \hookrightarrow \mathrm{End}_{\mathbb{Q}} A \otimes_{\mathbb{Z}} \mathbb{Q}$, where E is a number field of degree $g = \dim A$. (Here and below, by such an embedding we will always mean a map of rings with unit, although we do not repeat this condition explicitly below).

If $\lambda | p$ is a prime of E, let $V_{A,\lambda} = T_p(A) \otimes_{E \otimes_{\mathbb{Q}} \mathbb{Q}_p} E_\lambda$, where $T_p(A)$ denotes the p-adic Tate module of A. For $v \nmid p$ a prime where A has good reduction the characteristic polynomial $P_v(A, T) := \det(1 - T\mathrm{Frob}_v | V_\lambda)$ has coefficients in the ring of integers $\mathcal{O}_E$ of E, and is independent of λ.

The following result was proved by Serre [**Se 1**, 4.7] for E totally real and by Ribet [**Ri 1**, §4] in general.

THEOREM 1.2.1. *Assume that* (1.1.2) *holds. Let* $A/\mathbb{Q}$ *be of* GL_2 *type, and* E *a number field of degree* $\dim A$, *which admits an embedding* $E \hookrightarrow \mathrm{End}_{\mathbb{Q}} A \otimes_{\mathbb{Z}} \mathbb{Q}$. *Then for any prime* $\lambda|p$ *of* E, V_λ *is modular. That is* $V_\lambda \sim \rho_{f,\lambda'}$ *for some cusp eigenform* f *of weight* 2, *and a prime* $\lambda'|p$ *of* E_f.

PROOF. Let S be the set of primes of bad reduction of A together with the infinite prime and p. Let $c \in G_{\mathbb{Q},S}$ be a complex conjugation. Since c is a continuous involution of $A(\mathbb{C})$, it induces an involution of the rational Betti cohomology $H^1_B(A,\mathbb{Q})$, and the induced involution of $H^1_B(A,\mathbb{C})$ exchanges the $H^{1,0}$ and $H^{0,1}$ pieces of the Hodge decomposition. Hence c cannot act as a scalar on the 2-dimensional E-vector space $H^1(A,\mathbb{Q})$, so that it has eigenvalues -1 and $+1$, and $\det(c) = -1$. It follows that V_λ, which is dual to the étale cohomology $H^1(A,\mathbb{Q}_p)$, is an odd representation.

Now let $\lambda|p$ be any prime of E. Write $\rho_{A,\lambda}$ for the representation of $G_{\mathbb{Q},S}$ on V_λ. Suppose that A has good reduction at p. Then $V_{A,\lambda}$ arises from a p-divisible group over $\mathbb{Z}_p$, and hence $\bar\rho_{A,\lambda}$ arises from a finite flat group scheme. A result of Raynaud [**Ra**, Thm. 3.4.3] therefore implies that $\bar\rho_{A,\lambda}|_{I_{\mathbb{Q}_p}}$ has the form $\begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_2^p \end{pmatrix}$ or $\left(\begin{smallmatrix} \omega & * \\ 0 & 1 \end{smallmatrix}\right)$ with $*$ peu ramifiée. Hence $k(\bar\rho_{A,\lambda}) = 2$. Moreover, if N denotes the conductor of A, then $N(\bar\rho_{A,\lambda})$ divides N. Hence, if $\bar\rho_{A,\lambda}$ is absolutely irreducible, then $\bar\rho_{A,\lambda} \sim \bar\rho_{f_\lambda,\lambda'}$ for some eigenform $f_\lambda \in S_2(\Gamma_1(N),\mathbb{C})$, and some prime $\lambda'|p$ of E_{f_λ}.

Now suppose that $\bar\rho_{A,\lambda}$ is absolutely irreducible for infinitely many λ. Since the space $S_2(\Gamma_1(N),\mathbb{C})$ is finite dimensional, there are only finitely many possibilities for the form f_λ. Hence there is an infinite set I of primes of E such that $\bar\rho_{A,\lambda}$ is absolutely irreducible, and $f = f_\lambda$ is independent of $\lambda \in I$. Let $\tilde E_f \subset \mathbb{C}$ be a finite extension of E_f which contains all embeddings $E \hookrightarrow \mathbb{C}$, and for each $\lambda \in I$, fix an extension $\tilde E_f \hookrightarrow \bar{\mathbb{Q}}_p$ of λ'. We again denote this extension by λ'. Then $\lambda = \lambda' \circ i_\lambda : E \hookrightarrow \bar{\mathbb{Q}}_p$ for some embedding $i_\lambda : E \to \tilde E_f$. After replacing I by an infinite subset, we may assume that all the i_λ are equal to a fixed embedding i.

Then for any $\lambda \in I$, and any $v \nmid Np$, we have

$$i(\mathrm{tr}_{E_\lambda}(\mathrm{Frob}_v|V_{A,\lambda})) = \mathrm{tr}_{E_{f,\lambda'}}(\mathrm{Frob}_v|\rho_{f,\lambda'}) = a_v(f) \pmod{\lambda'}.$$

Note that the left hand side depends only on v and not on λ. Since this holds for infinitely many primes λ' of $\tilde E_f$, we conclude that $i(\mathrm{tr}_{E\lambda}(\mathrm{Frob}_v|V_{A,\lambda})) = a_v(f)$ for all $v \nmid Np$, and that $E_f \subset i(E)$. (More precisely, we see that $a_v(f) \in E$ for $v \nmid N$. One can show that this implies that also $a_v(f) \in E$ for $v|N$.) Hence, for any prime $\lambda|p$, $\rho_{A,\lambda} \sim \rho_{f,\lambda'}$ where $\lambda' = \lambda \circ i^{-1} : E_f \to \bar{\mathbb{Q}}_p$. Note that $\rho_{A,\lambda}$ is absolutely irreducible, since we have assumed this for infinitely many $\bar\rho_{A,\lambda}$.

Finally we have to check our assumption that infinitely many of the $\bar{\rho}_{A,\lambda}$ are absolutely irreducible. If this were not true then the semi-simplification of $\bar{\rho}_{A,\lambda}$ would be the mod p reduction of $\epsilon_1 + \epsilon_2\chi$, where $\chi : G_{\mathbb{Q},S} \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character, and ϵ_1, ϵ_2 are Dirichlet characters of conductor dividing N. Then a finiteness argument as above would show that the character of $\rho_{A,\lambda}$ itself was of this form. This is impossible, because for $v \nmid Np$, $\rho_{f,\lambda}(\mathrm{Frob}_v)$ has eigenvalues whose complex absolute values are $|v|^{1/2}$. $\square$

1.2.2. We could also have proved (1.2.1) by using a modularity lifting theorem. As in the argument above, all but finitely many of the $\bar{\rho}_{A,\lambda}$ are absolutely irreducible, and a similar argument shows that if all but finitely many of these representations are dihedral, then the $\rho_{A,\lambda}$ are themselves dihedral and arise from a CM form. Since the $\bar{\rho}_{A,\lambda}$ are modular by assumption, $\rho_{A,\lambda}$ is modular by (0.1).

COROLLARY 1.2.3. *Let $A/\mathbb{Q}$ be an abelian variety of* GL_2*-type. Then A is a quotient of* $\mathrm{Jac}(X_1(N))^m$ *for some $N, m \geq 1$.*

PROOF. This follows immediately from (1.2.1) by Faltings' isogeny theorem [**Fa**]. $\square$

1.2.4. Recall that for a finite prime v, and any abelian variety $A/\mathbb{Q}$, the local L-factor $L_v(A,T)$ is defined by

$$L_v(A,T) = \det{}_{\mathbb{Q}_p}(1 - \mathrm{Frob}_v^{-1}T | H^1(A, \mathbb{Q}_p)^{I_v})^{-1},$$

where I_v denotes the inertia subgroup at v. This is the inverse of a polynomial with rational coefficients, which depends only on v and not on $p \nmid v$. The complex L-function is defined by the Euler product

$$L(A,s) = \prod_{\ell} L_\ell(A, \ell^{-s})$$

where ℓ runs over the finite primes. This converges in the half plane Re $s > 3/2$.

If $\lambda | p$ is a prime of E, and $\lambda \nmid v$, we may also define

$$L_{v,E}(A,T) = \det{}_{E_\lambda}(1 - \mathrm{Frob}_v^{-1}T | H^1(A, \mathbb{Q}_p)^{I_v} \otimes_{E \otimes_{\mathbb{Q}} \mathbb{Q}_p} E_\lambda)^{-1}.$$

This is the reciprocal of a polynomial with coefficients in E, and depends only on v and not on λ. We have $L_v(A,T) = \prod_{\sigma : E \hookrightarrow \mathbb{C}} \sigma(L_{v,E}(A,T))$.

COROLLARY 1.2.5. *Assuming* (1.1.2), *let A be an abelian variety of* GL_2*-type. Then $L(A,s)$ has an analytic continuation to the whole complex plane, and satisfies a functional equation with respect to the symmetry $s \mapsto 2 - s$.*

More precisely, if $N(A)$ denotes the conductor, $g = \dim A$, and we set

$$\Lambda(A,s) = N(A)^{s/2}((2\pi)^{-s}\Gamma(s))^g L(A,s),$$

then we have

$$\Lambda(A, s) = \pm\Lambda(A, 2-s).$$

PROOF. Let $E \subset \mathrm{End}_{\mathbb{Q}} A \otimes_{\mathbb{Z}} \mathbb{Q}$ be a field of degree g, and for a prime $\lambda | p$ of E, let

$$f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_1(N), \mathbb{C})$$

be the normalised eigenform given by (1.2.1), so that $V_{A,\lambda} \sim \rho_{f,\lambda'}$ for a prime $\lambda' | p$ of E_f. More precisely, we fix inclusion $E_f \subset E \subset \mathbb{C}$ so that $\lambda|_{E_f} = \lambda'$. We may also assume that f is a newform, so that there is no $f' \in S_2(\Gamma_1(N'), \mathbb{C})$ with $N' < N$ and $a_v(f') = a_v(f)$ for $v \nmid N$.

If $\sigma : E_f \hookrightarrow \mathbb{C}$ is an embedding, then $f^\sigma := \sum_{n=1} \sigma(a_n) q^n$ is again a normalized eigenform in $S_2(\Gamma_1(N), \mathbb{C})$. This follows from the fact that the space $S_2(\Gamma_1(N), \mathbb{C})$ is spanned by the Hecke stable $\mathbb{Q}$-subspace consisting of cusp forms with rational Fourier coefficients.

If $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ denotes the character of f, (extended to a function on $\mathbb{Z}$ by setting $\varepsilon(m) = 0$ if $(m, N) > 1$) then

$$L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_{\ell} (1 - a_\ell \ell^{-s} + \varepsilon(\ell) \ell^{1-2s})^{-1}$$

has analytic continuation and satisfies a functional equation with respect to the symmetry $s \mapsto 2 - s$. More precisely, if

$$\Lambda(f, s) = N^{s/2} (2\pi)^{-s} \Gamma(s) L(f, s),$$

then

$$\Lambda(f, s) = W(f) \Lambda(f^c, 2 - s),$$

where $W(f) \in E_f$ is a root of unity, and c denotes complex conjugation.

Since $V_{A,\lambda} \sim \rho_{f,\lambda'}$ we have

$$L(f, s) = \prod_{\ell} \det{}_{E_\lambda} (1 - \mathrm{Frob}_\ell \ell^{-s} | (V_{A,\lambda})_{I_\ell})^{-1} = \prod_{\ell} L_{\ell,E}(A, \ell^{-s}),$$

where we view $E \subset \mathbb{C}$ via the embedding chosen earlier. More precisely, we see that the Euler factors corresponding to primes $\ell \nmid N$ on the two sides agree. It follows from a result of Carayol [**Ca 1**], building on work of Deligne and Langlands, that the factors at $\ell \nmid N$ also agree. Finally we compute

$$L(A, s) = \prod_{\ell} L_\ell(A, \ell^{-s}) = \prod_{\ell} \prod_{\sigma : E \hookrightarrow \mathbb{C}} \sigma(L_{\ell,E}(A, \ell^{-s})) = \prod_{\sigma : E \hookrightarrow \mathbb{C}} L(f^\sigma, \mathbb{C}).$$

This shows that $L(A, s)$ is entire, and that $\Lambda(A, s) = \prod_\sigma \Lambda(f^\sigma, \mathbb{C})$. That $N(A) = N^g$ follows from the work of Carayol *loc. cit.* □

1.3. Artin's conjecture. Khare [**Kh 2**] observed that one could modify the above arguments to show that (1.1.2) implies Artin's conjecture for odd, two dimensional representations.

THEOREM 1.3.1. *Assume* (1.1.2), *and let* $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{C})$ *be an odd, irreducible Artin representation. Then the Artin L-function*

$$L(\rho, s) = \prod_{\ell} \det{}_{\mathbb{C}}(1 - \mathrm{Frob}_\ell \ell^{-s} | \rho^{I_\ell})^{-1}$$

is entire. More precisely, there exists an eigenform f *of weight* 1 *such that* $L(\rho, s) = L(f, s)$.

PROOF. This was observed by Khare [**Kh 2**]. Since ρ has finite image, after conjugation we may assume that ρ factors through $\mathrm{GL}_2(E)$ for some number field E. For $\lambda | p$ a prime of E, denote by $\bar{\rho}_\lambda$ the reduction of ρ modulo λ. It is not hard to see that $\bar{\rho}_\lambda$ is absolutely irreducible, except for finitely many λ.

Suppose that ρ is unramified at p, and $\lambda | p$. Then $k(\bar{\rho}_\lambda) = p$, and so $\bar{\rho}_\lambda = \bar{\rho}_{f,\lambda'}$ for some eigenform $f \in S_p(\Gamma_1(N), \mathbb{C})$ and $\lambda' | p$ a prime of E_f. Here N denotes the conductor of ρ. The result of Coleman-Voloch mentioned in (1.1.3)(2) shows that there exists an eigenform $g \in S_1(\Gamma_1(N), \mathbb{C})$ such that $\bar{\rho}_{g,\lambda''} = \bar{\rho}_{f,\lambda'} = \bar{\rho}_\lambda$ for $\lambda'' | p$ a prime of E_g. More precisely, there exists a weight 1 Katz modular eigenform whose q-expansion is equal to that of f modulo p. If we exclude finitely many primes, we may assume that this form lifts to an eigenform $g \in S_1(\Gamma_1(N), \mathbb{C})$.

Since we can make this argument for infinitely many p, an argument, as in (1.2.1) shows that $\rho \sim \rho_{g,\lambda''}$ for some $g \in S_1(\Gamma_1(N), \mathbb{C})$. $\square$

1.4. Modularity of motives of GL_2**-type.** To finish this section, we explain how to extend the above results for Abelian varieties of GL_2-type to Grothendieck motives of GL_2-type.

Let X be a smooth projective variety over $\mathbb{Q}$ of dimension d. We denote by $Z^i(X)$ the group of cycles on X of codimension i, and by $C^i(X)$ the quotient of $Z^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ by the subspace spanned by cycles whose classes in cohomology are trivial. This condition is independent of the cohomology theory used, since any of the standard theories (l-adic, de Rham, crystalline) may be compared with the Betti cohomology $H_B^{2i}(X(\mathbb{C}), \mathbb{Q})$. If X, Y, Z are smooth projective varieties over $\mathbb{Q}$ we have a map

$$C^{\dim X + i}(X \times Y) \times C^{\dim Y + j}(Y \times Z) \to C^{\dim X + i + j}(X \times Z)$$

given by $(U, W) \mapsto p_{13*}(U \times Z \cap X \times W)$, where p_{13} denotes the projection from $X \times Y \times Z$ to $X \times Z$. In particular, $C^{\dim X}(X \times X)$ is a ring.

Recall [**Ja**] that a Grothendieck motive M over $\mathbb{Q}$ is a tuple (X, π, m) where X is a smooth projective variety over $\mathbb{Q}$ of dimension n, $\pi \in$

$C^n(X \times X)$ satisfies $\pi^2 = \pi$, and $m \in \mathbb{Z}$. One defines

$$\mathrm{Hom}((X', \pi', m'), (X, \pi, m)) = \pi C^{\dim X - m + m'}(X \times X')\pi'.$$

Given any cohomology theory H^* on smooth projective varieties over $\mathbb{Q}$, and $M = (X, \pi, m)$ we set $H^i(M) = \pi(H^{i+2m}(X))(m)$. With our conventions this is contravariant in M. We refer to this as the realization of M (in degree i) corresponding to the theory H^*.

Let E be a number field, and consider an embedding $E \hookrightarrow \mathrm{End}\, M$. If $\lambda | p$ is a prime of E, we set $H^i_\lambda(M) = H^i(M, \mathbb{Q}_p) \otimes_{E \otimes \mathbb{Q}_p} E_\lambda$. Here $H^i(M, \mathbb{Q}_p)$ denotes the p-adic étale realization of M.

LEMMA 1.4.1. *Suppose that* $M = (X, \pi, m)$, *and* $E \hookrightarrow \mathrm{End}\, M$. *If* v *is a prime of* $\mathbb{Q}$ *where* X *has good reduction then* $\det_{E_\lambda}(1\text{-}T\mathrm{Frob}_v^{-1} | H^i_\lambda(M))$ *has coefficients in* E, *and is independent of* $\lambda \nmid v$.

PROOF. Let $Q \in E[X]$. It suffices to show that for any Q,

$$t_{Q,\lambda} := \mathrm{tr}_{E_\lambda}(Q(\mathrm{Frob}_v^{-1}) | H^i_\lambda(M)),$$

a priori an element of $\bar{\mathbb{Q}}_p$, is in E and independent of λ. Now

$$\begin{aligned}\sum_{\lambda | p} t_{Q,\lambda} &= \mathrm{tr}_{\mathbb{Q}_p}(Q(\mathrm{Frob}_v^{-1}) | H^i(M, \mathbb{Q}_p)) \\ &= \mathrm{tr}_{\mathbb{Q}_p}(\pi Q(\mathrm{Frob}_v^{-1}) | H^{i+2m}(X)(m)).\end{aligned}$$

The right hand side has coefficients in $\mathbb{Q}$, and is independent of p by [**KM**, Thm 2]. It follows that for any $a \in E$, $\sum_{\lambda | p} \lambda(a) t_{Q,\lambda} \in \mathbb{Q}$, since we may apply the above observations to aQ instead of Q. Hence $\sum_\lambda t_{Q,\lambda} \lambda$ is a $\mathbb{Q}$-linear map, $E \to \mathbb{Q} \subset \bar{\mathbb{Q}}_p$. Since the embeddings $\lambda : E \hookrightarrow \bar{\mathbb{Q}}_p$, are $\bar{\mathbb{Q}}_p$-linearly independent, this implies that there is some $t \in E$ such that $t_{Q,\lambda} = \lambda(t)$ for all λ. So $t_{Q,\lambda} \in E$, and depends only on p.

If $\lambda, \lambda' \nmid v$ are two primes of E, then the above shows that for $a \in E$, we have

$$\mathrm{tr}_{E/\mathbb{Q}}(a t_{Q,\lambda}) = \mathrm{tr}_{E/\mathbb{Q}}(a t_{Q,\lambda'}) \in \mathbb{Q}.$$

Hence $\mathrm{tr}_{E/\mathbb{Q}}(a(t_{Q,\lambda} - t_{\mathbb{Q},\lambda'})) = 0$ for all a, which implies $t_{Q,\lambda} = t_{Q,\lambda'}$. $\square$

1.4.2. Given M and E, as above, we will say that M is of GL_2-type, if the Betti cohomology $H^i_B(M, \mathbb{Q})$ is two dimensional over E. The de Rham realization $H^i_{\mathrm{dR}}(M)$ is then also a two dimensional E-vector space. We call the two degrees in which $\mathrm{gr}^\bullet H^i_{\mathrm{dR}}(M)$ are non-zero the Hodge weights of $h^i(M)$. According to standard (perhaps unfortunate) conventions, these are the negatives of the Hodge-Tate weights of the $G_{\mathbb{Q}_p}$-representation $H^i(M, \mathbb{Q}_p)$.

If $\lambda | p$ is a prime of E, then $H^i_\lambda(M)$ is a two dimensional E_λ-representation of $G_{\mathbb{Q},S}$ where S the union of $\{p, \infty\}$ and the set of primes at which X has bad reduction.

THEOREM 1.4.3. *Let M be of* GL_2*-type with Hodge weights* $r \leqslant s$. *Suppose that* $H^i_\lambda(M)$ *is an absolutely irreducible representation of* $G_{\mathbb{Q},S}$, *and if* $r = s$ *assume that* $\det_{E_\lambda} H^i_\lambda(M)$ *is odd. Then for some* $N \geq 1$, *there exists an eigenform* f *in* $S_{s-r+1}(\Gamma_1(N), \mathbb{C})$, *such that* $\rho_{f,\lambda'} \sim H^i_\lambda(M)(s)$ *for some prime* $\lambda'|p$ *of* E_f.

PROOF. If $r = s$, then the main theorem of citeKiW implies that the action of $G_{\mathbb{Q},S}$ on $H^i_\lambda(M)(s)$ factors through a finite quotient, and hence the theorem follows from (1.3.1). Alternatively, the same argument as in (1.3.1) can be applied directly to show that $H^i_\lambda(M)$ comes from a weight 1 form.

Thus we may assume that $r < s$. Then an argument with the Hodge decomposition, as in the proof of (1.2.1), shows that $\det H^i_\lambda(M)$ is odd. Write $\rho_{M,\lambda}$ for $H^i_\lambda(M)(s)$. Note that the Hodge-Tate weights of $\rho_{M,\lambda}$ as a $G_{\mathbb{Q}_p}$-representation are 0 and $j = s-r$. By a result of Fontaine-Messing, [**FMe**, 2.3], if $s - r < p - 1$, then $\rho_{M,\lambda}$ arises from a weakly admissible module, and using Fontaine-Laffaille theory [**FL**, §8, Thm. 5.3] one sees that $\bar\rho_{M,\lambda}|_{I_{\mathbb{Q}_p}}$ is either of the form $\begin{pmatrix} \omega_2^j & 0 \\ 0 & \omega_2^{pj} \end{pmatrix}$ or $\left(\begin{smallmatrix} \omega^j & * \\ 0 & 1 \end{smallmatrix}\right)$. Moreover, if $j = 1$, then $\bar\rho_{M,\lambda}|_{G_{\mathbb{Q}_p}}$ arises from a finite flat group scheme [**FL**, §9], so in the second case $*$ is peu ramifiée. It follows that $k(\bar\rho_{M,\lambda}) = s - r + 1$.

To bound $N(\bar\rho_{M,\lambda})$, let N_0 be the product of the primes of bad reduction of X. If $g = [E : \mathbb{Q}]$, the image of $\bar\rho_{M,\lambda}$ has order dividing $(p^{2g} - 1)(p^{2g} - p^g)$. There exists a positive integer a, and a class $y \in \mathbb{Z}/N_0^a\mathbb{Z}$ such that $(y^{2g} - 1)(y^{2g} - y^g) \neq 0 \pmod{l^a}$ for any $l|N_0$. Hence, if $p = y \pmod{N_0^a}$, then the order of the image of $\bar\rho_{M,\lambda}$ has l-adic valuation at most $a - 1$, and [**Se 1**, 4.9.4] implies

$$v_l(N(\bar\rho_{M,\lambda})) \leqslant 2(a + \frac{1}{p-1}).$$

It follows that for $\lambda|p$ with $p = y \pmod{N_0^a}$, there exists $f_\lambda \in S_{s-r+1}(\Gamma_1(N), \mathbb{C})$ such that $\bar\rho_{f_\lambda,\lambda'} \sim \bar\rho_{M,\lambda}$ for some prime $\lambda'|p$ of E_{f_λ}, where N is an integer which does not depend on λ. The rest of the proof is identical to that of (1.2.1), using the fact that for $v \notin S$, the characteristic polynomial of $\rho_{M,\lambda}(\mathrm{Frob}_v)$ does not depend on $\lambda \nmid v$.

We could also have used an argument involving a modularity lifting theorem as in (1.2.2). $\square$

COROLLARY 1.4.4. *Keep the above notation and assumptions. Then*

$$L_v(h^i(M), T) := \det{}_{E_\lambda}(1 - T\mathrm{Frob}_v^{-1}|H^i_\lambda(M)^{I_v})^{-1}$$

is the reciprocal of a polynomial with coefficients in E, *which does not depend on* $\lambda \nmid v$. *Moreover, for any embedding* $\sigma : E \hookrightarrow \mathbb{C}$, *the product*

$$L_\sigma(h^i(M), s) = \prod_\ell \sigma(L_\ell(h^i(M), \ell^{-s}))$$

which converges for $\mathrm{Re}\ s$ *sufficiently large, extends to an entire function.*

PROOF. This follows immediately from (1.4.3) and the corresponding properties of the L-function $L(f, s)$. Note that the claim regarding L-factors at primes $\ell | N$ follows from the results of Carayol already sited above [**Ca 1**]. □

2. Modularity lifting theorems

2.1. Statement of results. In this section we explain some of the ideas which go into the proof of Theorem (0.1). For simplicity, we will restrict ourselves to the case $p > 2$. While the case $p = 2$ presents some technical difficulties, the main ideas are similar.

When proving results of the type $\bar{\rho}$ modular $\implies$ ρ modular, it is more convenient to work with Barsotti-Tate representations rather than potentially Barsotti-Tate representations. This is because in the former case, the p-adic Hodge theory, which is used to control the deformation rings appearing in the argument, is better behaved. For the purposes of proving Theorem (0.1), we can reduce ourselves to the case of Barsotti-Tate representations if we replace $\mathbb{Q}$ by a suitable finite, solvable totally real extension F. If one can show that $\rho|_{G_F}$ arises from a Hilbert modular form, then Theorem (0.1) follows by Langlands base change (cf. the end of the proof of [**Ta 4**, Thm. 2.4]).

The modularity of $\rho|_{G_F}$ can be deduced from the following result over totally real fields. (As for modular forms, if π is a Hilbert modular eigenform over F, and λ the prime of its coefficient field, we denote by $\rho_{\pi,\lambda}$ (resp. $\bar{\rho}_{\pi,\lambda}$) the corresponding λ-adic (resp. mod λ) representation. In the following λ will usually be a prime dividing p, and we will write $\rho_{\pi,\lambda}$ without further comment.)

THEOREM 2.1.1. *Let $F/\mathbb{Q}$ be totally real, $p > 2$, S a finite set of primes of F, and $\rho : G_{F,S} \to \mathrm{GL}_2(E)$ a continuous representation. Suppose that*

(1) *ρ is Barsotti-Tate at each prime $\mu | p$ of F, and has cyclotomic determinant.*
(2) *$\bar{\rho} \sim \bar{\rho}_{\pi,\lambda}$ for some Hilbert modular form π over F of parallel weight 2 and prime to p level, such that $\rho_{\pi,\lambda}$ is ordinary at a prime $\mu | p$ of F if and only if ρ is.*
(3) *$\bar{\rho}|_{F(\zeta_p)}$ is absolutely irreducible, and $[F(\zeta_p) : F] > 2$ if $p = 5$.*

Then $\rho \sim \rho_{\pi',\lambda}$ for some Hilbert modular form π' over F.

2.1.2. Here and below, when we say that ρ is *ordinary* at μ we mean that $\rho|_{G_{F_\mu}}$ has a rank 1 quotient on which the action of inertia at μ is trivial. We will say that ρ is *potentially ordinary* at μ if this condition holds on the restriction of ρ to an open subgroup of G_{F_μ}.

Note that the hypothesis (2) is stronger than just asking that $\bar{\rho} \sim \bar{\rho}_{\pi,\lambda}$. Since p may be ramified in F, it may happen that $\bar{\rho}$ arises as the reduction of both ordinary and non-ordinary Barsotti-Tate representations. Hence (0.1) does not follow immediately from (2.1); one needs to show that if ρ arises from a representation of $G_{\mathbb{Q}}$, one can find a π satisfying the stronger condition in (2). Sometimes one can improve this result. The following corollary can be deduced from the theorem using exactly the same methods as in [**Ki 2**, §2].

COROLLARY 2.1.3. *Let $p > 2$, $F/\mathbb{Q}$ a totally real field, S a finite set of primes of F, and $\rho : G_{F,S} \to \mathrm{GL}_2(E)$ a continuous representation. Suppose that*

(1) *ρ is potentially Barsotti-Tate at each prime $\mu|p$ of F, and that if ρ is potentially ordinary at μ then $F_\mu = \mathbb{Q}_p$.*
(2) *$\bar{\rho} \sim \bar{\rho}_{\pi,\lambda}$ for some Hilbert modular form π over F of parallel weight 2.*
(3) *$\bar{\rho}|_{F(\zeta_p)}$ is absolutely irreducible, and $[F(\zeta_p) : F] > 2$ if $p = 5$.*

Then $\rho \sim \rho_{\pi',\lambda}$ for some Hilbert modular form π' over F.

In the remainder of this section we will try to outline the proof of (2.1). Further details may be found in [**Ki 1**] and [**Ki 2**]. More precisely, the theorem is proved there assuming that if $\mu|p$ is a place of F then the residue field at μ is equal to $\mathbb{F}_p$. This assumption has been removed by Gee [**Ge**].

2.2. Barsotti-Tate deformation rings. Suppose that $K/\mathbb{Q}_p$ is a finite extension, $\mathbb{F}$ a finite extension of $\mathbb{F}_p$, and $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ a continuous representation. We will suppose that $\mathrm{End}_{\mathbb{F}[G_K]}\bar{\rho} = \mathbb{F}$, although there is a variant of the theory without this assumption.

Let $R(\bar{\rho})$ denote the universal deformation ring of $\bar{\rho}$. If $E/W(\mathbb{F})[1/p]$ is a finite extension, and $x : R(\bar{\rho}) \to E$ a map of $W(\mathbb{F})$-algebras, then we denote by V_x the two dimensional E-representation of G_K obtained by specializing the universal representation via x.

PROPOSITION 2.2.1. *There exists a p-torsion free quotient $R^{0,1}(\bar{\rho})$ of $R(\bar{\rho})$ with the following properties:*

(1) *If $x : R(\bar{\rho}) \to E$ is a map of $W(\mathbb{F})$-algebras then x factors through $R^{0,1}(\bar{\rho})$ if and only if V_x is Barsotti-Tate with Hodge-Tate weights equal to $0, 1$, and $\det V_x$ is equal to the cyclotomic character.*
(2) *$R^{0,1}(\bar{\rho})[1/p]$ is formally smooth over $W(\mathbb{F})[1/p]$ of dimension $[K : \mathbb{Q}_p]$.*
(3) *Suppose that K has residue field $\mathbb{F}_p$. If $x_1, x_2 : R^{0,1}(\bar{\rho}) \to E$ are two $W(\mathbb{F})[1/p]$-algebra maps, then the images of associated maps of spectra lie on the same component of $\mathrm{Spec}\, R^{0,1}(\bar{\rho})[1/p]$ if and only if V_{x_1} and V_{x_2} are either both ordinary or both non-ordinary. Moreover if $E/\mathbb{Q}_p$ is a finite extension, then the*

connected components of $R^{0,1}(\bar{\rho}) \otimes_{W(\mathbb{F})} E$ *are in bijection with those of* $R^{0,1}(\bar{\rho})[1/p]$.

2.2.2. The proof of (2.2) uses p-adic Hodge theory. To construct the ring $R^{0,1}(\bar{\rho})$ consider first the quotient $R^{\mathrm{fl}}(\bar{\rho})$ of $R(\bar{\rho})$ which corresponds to deformations of $\bar{\rho}$ which are the generic fiber of a finite flat group scheme [**Ram**]. Let $R^{\mathrm{fl},\chi}(\bar{\rho})$ denote the quotient of $R^{\mathrm{fl}}(\bar{\rho})$ corresponding to deformations with cyclotomic determinant. Then we take $R^{0,1}(\bar{\rho})$ to be $R^{\mathrm{fl},\chi}(\bar{\rho})$ modulo its ideal of p-power torsion elements.

It is not hard to see that $R^{0,1}(\bar{\rho})$ satisfies (1). The most delicate point is to show that it satisfies (3). The proof uses a classification of finite flat group schemes initiated by Breuil [**Br 2**]. Rather than studying deformations of $\bar{\rho}$ directly, one studies the finite flat group schemes over $\mathcal{O}_K$ which give rise to them (finite flat models).

We will give some of the ideas below. First we explain how this result implies (2.1.1). We remark that although we use (2.2.1) in the proof of (2.1.1) below, we have stated (2.1.1) without any assumption on the residue fields of F at primes $v|p$. This is because Gee [**Ge**] has proved (a variant of) (2.2.1) without any assumption on the residue field in the case when $\bar{\rho}$ is trivial. The case of trivial image is enough for applications to modularity, since one can always reduce to this case by base change. Of course even to formulate (2.2.1) in this situation requires the use of framed deformation rings, which we have avoided here.

2.3. The modified Taylor-Wiles method.

Sketch of (2.2.1) $\implies$ (2.1.1) : We will explain the proof in the so called "minimal case"; namely when the conductor of $\bar{\rho}$ at any prime $\mu \nmid p$ is equal to that of ρ, and the action of the inertia at μ is unipotent.

Recall the original method of Taylor-Wiles [**TW**] for proving modularity. One wants to check that $\theta : R \xrightarrow{\sim} \mathbb{T}$ where R is a global deformation ring with certain local conditions imposed, and $\mathbb{T}$ is a localized Hecke algebra. The representation ρ corresponds to a map $R \to E$, and we would like to show that this factors through $\mathbb{T}$. By considering deformation rings and Hecke rings with auxiliary primes in the level and applying a patching argument, one finds that the map θ sits in the following diagram

$$\begin{array}{ccccc} W(\mathbb{F})[\![x_1, \dots x_r]\!] & \longrightarrow & R_\infty & \xrightarrow{\theta_\infty} & \mathbb{T}_\infty \\ & & \downarrow & & \downarrow \\ & & R & \xrightarrow{\theta} & \mathbb{T} \end{array}$$

Here r is some non-negative integer. One knows the following information:

(1) θ is obtained from θ_∞ by factoring out by $(x_1, \dots, x_r)$

(2) $\mathbb{T}_\infty$ is finite flat over $W(\mathbb{F})[\![x_1, \dots x_r]\!]$.
(3) There is a surjective map $W(\mathbb{F})[\![z_1, \dots z_r]\!] \to R_\infty$.

The condition (2) is deduced using the geometry of modular curves, while (3) follows from a calculation with Galois cohomology and Poitou-Tate duality.

From (2) it follows that $\mathbb{T}_\infty$ is pure of dimension $r+1$, and hence the composite of the surjective maps

$$W(\mathbb{F})[\![z_1, \dots z_r]\!] \overset{(3)}{\to} R_\infty \to \mathbb{T}_\infty$$

is an isomorphism. It follows that θ_∞ is an isomorphism, and so is θ by (1).

In the situation of Theorem (2.1.1), (3) no longer holds, and one does not know how to prove (2) (it may well be false). Instead one replaces them with the following weaker conditions

(2)$'$ There exists a faithful, finite, rank 1, $\mathbb{T}_\infty$-module M_∞ which is finite flat over $W(\mathbb{F})[\![x_1, \dots, x_r]\!]$.
(3)$'$ Let $R^{0,1}_{v|p} = \widehat{\otimes}_{\mathfrak{v}|p} R^{0,1}(\bar\rho|_{G_{F_v}})$, where the tensor product is taken over $W(\mathbb{F})$. We can assume that $r \geq d = [F : \mathbb{Q}]$, and there exists a surjection

$$R^{0,1}_{v|p}[\![z_1, \dots, z_{r-d}]\!] \to R_\infty.$$

The module M_∞ is built by patching spaces of modular forms at auxiliary level. (The idea of replacing (2), which is a condition involving Hecke algebras, with a condition on modular forms goes back to Diamond [**Di 2**] and Fujiwara.)

We can now finish the argument as follows: From (2)' it still follows that $\mathbb{T}_\infty$ is pure of dimension $r+1$. Consider the map on spectra

$$\operatorname{Spec} \mathbb{T}_\infty \hookrightarrow \operatorname{Spec} R_\infty \hookrightarrow \operatorname{Spec} R^{0,1}_{v|p}[\![z_1, \dots, z_{r-d}]\!] \tag{2.3.1}$$

induced by composing (3)' and θ_∞. We have already observed that the left hand side is pure of dimension $r+1$, and the right hand side has dimension

$$1 + \sum_{\mathfrak{p}|p} [F_\mathfrak{p} : \mathbb{Q}_p] + r - d = r + 1$$

by (2.2.1)(2). The formal smoothness of (2.2.1)(2) implies that the image of (2.3.1) is a union of irreducible components of $\operatorname{Spec} R^{0,1}_{v|p}[\![z_1, ..., z_{r-d}]\!]$. The description of these components in (2.2.1)(3), together with the condition (2.1.1)(2) guarantees that the point corresponding to ρ is in the image of (2.3.1). Hence ρ factors through $\mathbb{T}_\infty$, and hence through $\mathbb{T}$.

Finally let us remark that the argument in the non-minimal case is very similar, but one needs to patch over a tensor product of local rings at the non-minimal primes of $\bar{\rho}$ as well as primes dividing p. (cf. §3 below).

2.4. Moduli of finite flat group schemes. We now explain some of the ideas which go into the proof of (2.2.1). We begin by recalling a construction of finite flat group schemes introduced by Breuil [**Br 2**].

2.4.1. Let k denote the residue field of K, and let $W = W(k)$. Let $\mathfrak{S} = W[\![u]\!]$, and equip the ring $\mathfrak{S}$ with an endomorphism φ which acts as the usual Frobenius on W, and sends u to u^p. Fix a uniformiser $\pi \in \mathcal{O}_K$, and let $E(u)$ be the Eisenstein polynomial of π.

Denote by $'(\mathrm{Mod}/\mathfrak{S})$ the category of $\mathfrak{S}$-modules $\mathfrak{M}$, equipped with a φ-semi-linear map $\varphi : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of $\varphi^*(\mathfrak{M}) \to \mathfrak{M}$, the $\mathfrak{S}$-linear map induced by φ, is killed by $E(u)$. We give $'(\mathrm{Mod}/\mathfrak{S})$ the structure of an exact category induced by that on the abelian category of $\mathfrak{S}$-modules.

Let $(\mathrm{Mod\ FI}/\mathfrak{S})$ be the full subcategory of $'(\mathrm{Mod}/\mathfrak{S})$ consisting of those $\mathfrak{M}$ such that as an $\mathfrak{S}$-module $\mathfrak{M}$ is isomorphic to $\oplus_{i\in I}\mathfrak{S}/p^{n_i}\mathfrak{S}$, where I is a finite set and n_i is a non-negative integer. We denote by $(\mathrm{Mod}/\mathfrak{S})$ the full subcategory of $'(\mathrm{Mod}/\mathfrak{S})$ consisting of objects $\mathfrak{M}$ which are successive extensions of objects whose underlying $\mathfrak{S}$-modules are finite free $\mathfrak{S}/p\mathfrak{S}$-modules. This is equivalent to asking that the $\mathfrak{S}$-module $\mathfrak{M}$ is killed by a power of p, and has projective dimension 1 [**Ki 3**, 2.3.2]. Finally, we denote by $(\mathrm{Mod}/\mathfrak{S})_{\mathbb{Z}_p}$ the full subcategory of $'(\mathrm{Mod}/\mathfrak{S})$ consisting of objects whose underlying $\mathfrak{S}$-modules are finite free.

We will write $(p\text{-Gr}/\mathcal{O}_K)$ for the category of finite flat group schemes over $\mathcal{O}_K$ of p-power order, and $(p\text{-div}/\mathcal{O}_K)$ for the category of p-divisible groups over $\mathcal{O}_K$.

THEOREM 2.4.2. *For $p > 2$, there is an exact equivalence*

$$(\mathrm{Mod}/\mathfrak{S}) \xrightarrow{\sim} (p\text{-Gr}/\mathcal{O}_K). \tag{2.4.3}$$

This induces an equivalence between $(\mathrm{Mod\ FI}/\mathfrak{S})$ *and the category of finite flat groups schemes G such that $G[p^n]$ is finite flat for $n \geq 1$ as well as an equivalence*

$$(\mathrm{Mod}/\mathfrak{S})_{\mathbb{Z}_p} \xrightarrow{\sim} (p\text{-div}/\mathcal{O}_K).$$

2.4.4. To explain some of the ideas behind the proof of (2.4.2), we need to introduce another category of modules. Let S denote the p-adic completion of $W[u, E(u)^i/i!]_{i\geq 1}$. We equip S with a Frobenius φ, which is the usual Frobenius on W and sends u to u^p, and we denote by $\mathrm{Fil}^1 S \subset S$, the kernel of the map of W-algebras $S \to \mathcal{O}_K$ which sends u to π.

Let $'(\mathrm{Mod}/S)$ be the category of triples $(\mathcal{M}, \mathrm{Fil}^1\mathcal{M}, \varphi_1)$, where $\mathcal{M}$ is an S-module, $\mathrm{Fil}^1\mathcal{M} \subset \mathcal{M}$ is a submodule containing $\mathrm{Fil}^1 S\mathcal{M}$, and

$\varphi_1 : \mathrm{Fil}^1\mathcal{M} \to \mathcal{M}$ is a Frobenius semi-linear map which satisfies

$$\varphi_1(sx) = (\varphi(E(u))/p)^{-1}\varphi_1(s)\varphi_1(E(u)x)$$

for $s \in \mathrm{Fil}^1 S$ and $x \in \mathcal{M}$. Note that $\varphi(E(u))/p$ is a unit in S, so the formula makes sense. We denote by (Mod FI$/S$) the full subcategory of $'$(Mod$/S$) such that

(1) $\mathcal{M} \xrightarrow{\sim} \oplus_{i\in I} S/p^{n_i}S$ as an S-module, for I a finite set, and n_i non-negative integers.
(2) $\varphi_1(\mathrm{Fil}^1\mathcal{M})$ generates $\mathcal{M}$ as an S-module.

We denote by (Mod$/S$) the full subcategory of $'$(Mod$/S$) consisting of objects which are successive extensions of objects in (Mod FI$/S$) which are killed by p. Finally, we denote by (Mod FI$/S)_{\mathbb{Z}_p}$ the full subcategory of $'$(Mod$/S$) consisting of those objects whose underlying S-modules are finite free, and which satisfy (2) above.

There is a functor

$$(\mathrm{Mod}/\mathfrak{S}) \to (\mathrm{Mod}/S); \quad \mathfrak{M} \mapsto S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M},$$

where $\mathrm{Fil}^1(S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$ is the preimage of $\mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M}$ under

$$S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \overset{1\otimes\varphi}{\to} S \otimes_{\mathfrak{S}} \mathfrak{M},$$

and the map φ_1 is the composite

$$S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \overset{1\otimes\varphi}{\to} \mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M} \overset{\varphi/p\otimes 1}{\to} S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}.$$

2.4.5. *Remarks on the proof of* (2.4.2). The functor in the theorem was constructed by Breuil [**Br 2**], who showed that that it was fully faithful and an equivalence on objects killed by p. This uses another result of Breuil [**Br 1**], which asserts that for $p > 2$ there is an anti-equivalence between (Mod$/S$) and the category of finite flat group schemes over $\mathcal{O}_K$. This equivalence can then be composed with Cartier duality and the functor $(\mathrm{Mod}/\mathfrak{S}) \to (\mathrm{Mod}/S)$ defined above to give the functor of (2.4.2).

The connection between finite flat group schemes and S-modules is via the theory of crystals attached to finite flat group schemes initiated by Grothendieck [**BBM**]. If G is a finite flat group scheme over $\mathcal{O}_K$, there is crystal $\mathbb{D}(G)$ on the crystalline site of $\mathcal{O}_K/W$ attached to G. The values of this crystal on a thickening T in this site are coherent (but not in general free) $\mathcal{O}_T$-modules. Since S is a divided power thickening of $\mathcal{O}_K$, one can evaluate $\mathbb{D}(G)$ on S, and this gives the underlying S-module of the object of (Mod$/S$) corresponding to G.

Breuil's results can be combined with a deformation theoretic argument to obtain the classification of p-divisible groups in (2.4.2) [**Ki 1**, 2.2.22]. From this, one can deduce that (2.4.3) is an equivalence by writing a finite flat group scheme as a kernel of p-divisible groups [**BBM**, 3.3.1].

There is another way of obtaining the classification of p-divisible groups directly, without first constructing the functor (2.4.3) for finite flat group schemes [**Ki 3**, 2.3]: The crystalline theory allows one to describe deformations of p-divisible groups [**Me**]. Using this one can show directly that for $p > 2$ there is an equivalence

$$(p\text{-div}/\mathcal{O}_K) \xrightarrow{G \mapsto \mathbb{D}(G^*)(S)} (\text{Mod FI}/S)_{\mathbb{Z}_p},$$

where G^* denotes the Cartier dual of G.

Then, using a classification of crystalline representations in terms of $\mathfrak{S}$-modules, one shows that the composite

$$(\text{Mod}/\mathfrak{S})_{\mathbb{Z}_p} \to (\text{Mod FI}/S)_{\mathbb{Z}_p} \to (p\text{-div}/\mathcal{O}_K)$$

is an equivalence.

For $p = 2$ the above functor is an equivalence up to isogeny, and is an equivalence if one considers only connected objects [**Ki 5**]. However Breuil conjectured that (2.4.2) should be true for all p. The difficulties at $p = 2$ seem to occur because the crystalline theory no longer works as well, one reason for this being that the divided powers of 2, are not topologically nilpotent ($2^i/i!$ does not tend 2-adically to 0). This suggests that there should be an approach to (2.4.2) which does not go through the crystalline theory. There are some intriguing calculations of Breuil which point in this direction [**Br 2**].

2.4.6. Before applying the above theory to prove (2.2.1), we need to relate the above classification to Galois representations. Fortunately, there is a very simple way of doing so.

Let $\mathcal{O}_{\mathcal{E}}$ denote the p-adic completion of $\mathfrak{S}[1/u]$. The Frobenius φ extends to $\mathcal{O}_{\mathcal{E}}$ by continuity. The ring $\mathcal{O}_{\mathcal{E}}$ is a complete discrete valuation ring, with residue field $k((u))$. We denote by $\mathbf{\Phi M}_{\mathcal{O}_{\mathcal{E}}}$ the category of finite $\mathcal{O}_{\mathcal{E}}$-modules M equipped with an isomorphism $\varphi^*(M) \xrightarrow{\sim} M$. Then one has the following [**Ki 1**, 1.1.13], [**Br 3**, 3.4.3].

PROPOSITION 2.4.7. *There is a commutative diagram of functors*

$$\begin{array}{ccc} (p\text{-Gr}/\mathcal{O}_K) & \xrightarrow{\sim} & (\text{Mod}/\mathfrak{S}) \\ \Big\downarrow{\scriptstyle G \mapsto G(\mathcal{O}_{\bar{K}})} & & \Big\downarrow{\scriptstyle \mathfrak{M} \mapsto \mathfrak{M}[1/u]} \\ (\text{f.fl } G_K\text{-reps}) & \longrightarrow & \mathbf{\Phi M}_{\mathcal{O}_{\mathcal{E}}} \end{array}$$

where (f.fl G_K-reps) *denotes the category of G_K-representations on finite length $\mathbb{Z}_p$-modules, which arise from a finite flat group scheme. Moreover both horizontal functors are fully faithful.*

2.4.8. To describe the ring $R^{0,1}(\bar{\rho})$ one studies the finite flat group schemes which give rise to deformations of $\bar{\rho}$. It turns out that, in general, these have moduli of positive dimension, so that Artin rings are insufficient to describe them. For this reason we replace finite flat group schemes by the $\mathfrak{S}$-modules with Frobenius introduced above.

Let A be a $W(\mathbb{F})$-algebra, set $\mathfrak{S}_A = \mathfrak{S} \otimes_{\mathbb{Z}_p} A$, and extend φ to $\mathfrak{S}_A$ by A-linearity. We denote by $(\mathrm{Mod}\ \mathrm{FI}/\mathfrak{S})_A$ the category of finite projective $\mathfrak{S}_A$-modules $\mathfrak{M}_A$, equipped with a map $\varphi^*(\mathfrak{M}_A) \to \mathfrak{M}_A$, whose cokernel is killed by $E(u)$.

Let $V_{\mathbb{F}}$ denote the underlying $\mathbb{F}$-vector space of $\bar{\rho}$. According to (2.4.7), attached to $V_{\mathbb{F}}$ there is an object $M_{\mathbb{F}}$ of $\mathbf{\Phi M}_{\mathcal{O}_{\mathcal{E}}}$. It is equipped with an action of $\mathbb{F}$ by functoriality, and one can check that it is finite free over $k((u)) \otimes_{\mathbb{F}_p} \mathbb{F}$.

Let $\mathfrak{Aug}_{W(\mathbb{F})}$ denote the category of pairs (A, I) consisting of a $W(\mathbb{F})$-algebra A and a nilpotent ideal $I \subset A$ such that $(p) \subset I$. We define a functor $D_{\mathfrak{S}, M_{\mathbb{F}}}$ on $\mathfrak{Aug}_{W(\mathbb{F})}$ by declaring $D_{\mathfrak{S}, M_{\mathbb{F}}}(A, I)$ to be the set of isomorphism classes of pairs consisting of an object $\mathfrak{M}_A$ in $(\mathrm{Mod}\ \mathrm{FI}/\mathfrak{S})_A$, together with an isomorphism

$$\mathfrak{M}_A \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \xrightarrow{\sim} M_{\mathbb{F}} \otimes_{\mathbb{F}} A/I$$

which is $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{F}_p} A/I$-linear, and compatible with Frobenius.

Theorem 2.4.9. *The functor $D_{\mathfrak{S}, M_{\mathbb{F}}}$ is represented by a projective $R^{\mathrm{fl}}(\bar{\rho})$-scheme*

$$\Theta_{V_{\mathbb{F}}} : \mathscr{GR}_{V_{\mathbb{F}}} \to \operatorname{Spec} R^{\mathrm{fl}}(\bar{\rho}).$$

Moreover, $\Theta_{V_{\mathbb{F}}}$ becomes an isomorphism after inverting p.

Proof. [**Ki 1**, 2.1.11, 2.4.8]. In fact this holds for any (not necessarily two dimensional) $\bar{\rho}$. The statement that $\mathscr{GR}_{V_{\mathbb{F}}}$ represents $D_{\mathfrak{S}, M_{\mathbb{F}}}$ is an abuse of terminology, since $\mathscr{GR}_{V_{\mathbb{F}}}$ is not actually an object of $\mathfrak{Aug}_{W(\mathbb{F})}$. It means that there is a functorial isomorphism

$$D_{\mathfrak{S}, M_{\mathbb{F}}}(A, I) \xrightarrow{\sim} \mathrm{Hom}_{W(\mathbb{F}), I}(\operatorname{Spec} A, \mathscr{GR}_{V_{\mathbb{F}}})$$

where the right hand side means maps of $W(\mathbb{F})$-schemes such that under the composite

$$\operatorname{Spec} A \to \mathscr{GR}_{V_{\mathbb{F}}} \to \operatorname{Spec} R^{\mathrm{fl}}(\bar{\rho})$$

the radical of $\operatorname{Spec} R^{\mathrm{fl}}(\bar{\rho})$ pulls back into I.

The fact that $\Theta_{V_{\mathbb{F}}}$ becomes an isomorphism after inverting p can be thought of as an incarnation of Tate's theorem that the functor which associates to a p-divisible group over $\mathcal{O}_K$ its generic fibre is fully faithful. □

2.4.10. To prove (2.4.2), we need to consider a subfunctor of $D_{\mathfrak{S}, M_{\mathbb{F}}}$. We define $D^{0,1}_{\mathfrak{S}, M_{\mathbb{F}}}(A, I)$ to be the set of $\mathfrak{M}_A$ in $D_{\mathfrak{S}, M_{\mathbb{F}}}(A, I)$ such that the image of the composite

$$\varphi^*(\mathfrak{M}_A) \to \mathfrak{M}_A \to \mathfrak{M}_A/E(u)\mathfrak{M}_A$$

is a maximal isotropic submodule in $\mathfrak{M}_A/E(u)\mathfrak{M}_A$ (that is, locally on $\operatorname{Spec} \mathfrak{S}_A$, it is its own annihilator under a symplectic pairing on

$\mathfrak{M}_A/E(u)\mathfrak{M}_A$, which is locally free of rank 2 over $\mathcal{O}_K \otimes_{\mathbb{Z}_p} A$), and the rank 1 $W(k) \otimes_{\mathbb{Z}_p} A$-module $\det \mathfrak{M}_A/u\mathfrak{M}_A$ is spanned by sections on which φ acts by p.

THEOREM 2.4.11. *The subfunctor $D^{0,1}_{\mathfrak{S},M_{\mathbb{F}}}$ is represented by a closed subscheme $\mathscr{GR}^{0,1}_{V_{\mathbb{F}}} \subset \mathscr{GR}_{V_{\mathbb{F}}}$. The morphism $\Theta_{V_{\mathbb{F}}}$ induces a projective map*

$$\Theta^{0,1}_{V_{\mathbb{F}}} : \mathscr{GR}^{0,1}_{V_{\mathbb{F}}} \to \operatorname{Spec} R^{0,1}(\bar{\rho})$$

which becomes an isomorphism after inverting p.

Moreover, the complete local rings on $\mathscr{GR}^{0,1}_{V_{\mathbb{F}}}$ are isomorphic to those on integral models of Hilbert modular varieties. In particular, $\mathscr{GR}^{0,1}_{V_{\mathbb{F}}} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is reduced.

PROOF. [**Ki 1**, 2.4.6, 2.4.8]. The integral models in the proposition are those studied by Deligne-Pappas [**DP**]. In more general situations where $\dim V_{\mathbb{F}} > 2$, the local geometry of the analogous schemes is controlled by local models of Shimura varieties. The reason for the relationship is that both moduli problems are controlled by the same linear algebra. We do not know if this has some deeper meaning. □

2.4.12. *Sketch of* (2.4.11) $\implies$ (2.2.1). That $R^{0,1}(\bar{\rho})[1/p]$ satisfies (2) follows from (2.4.11), because the description of the local structure of $\mathscr{GR}^{0,1}_{V_{\mathbb{F}}}$ implies, in particular, that $\mathscr{GR}^{0,1}_{V_{\mathbb{F}}}[1/p]$ is formally smooth over $W(\mathbb{F})[1/p]$.

For a topological space X, write $H_0(X)$ for the set of its connected components. Let $\mathscr{GR}^{0,1}_{V_{\mathbb{F}},0}$ denote the fibre of $\mathscr{GR}^{0,1}_{V_{\mathbb{F}}}$ over the closed point of $\operatorname{Spec} R^{0,1}(\bar{\rho})$, and denote by $\widehat{\mathscr{GR}}^{0,1}_{V_{\mathbb{F}}}$ the completion of $\mathscr{GR}^{0,1}_{V_{\mathbb{F}}}$ along $\mathscr{GR}^{0,1}_{V_{\mathbb{F}},0}$.

Then one has isomorphisms

$$\begin{aligned} H_0(\operatorname{Spec} R^{0,1}(\bar{\rho})[1/p]) &\xrightarrow{\sim} H_0(\mathscr{GR}^{0,1}_{V_{\mathbb{F}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \\ &\xrightarrow{\sim} H_0(\mathscr{GR}^{0,1}_{V_{\mathbb{F}}}) \xrightarrow{\sim} H_0(\widehat{\mathscr{GR}}^{0,1}_{V_{\mathbb{F}}}). \end{aligned}$$

The first isomorphism follows directly from (2.4.11), while the second follows easily from the fact that $\mathscr{GR}^{0,1}_{V_{\mathbb{F}}} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is reduced. The third isomorphism is a consequence of formal GAGA [**GD**, III, 5.5.1].

Finally, the underlying topological space of $\widehat{\mathscr{GR}}^{0,1}_{V_{\mathbb{F}}}$ is the same as that of $\mathscr{GR}^{0,1}_{V_{\mathbb{F}},0}$. In the situation of (3) one can check by a direct computation that two closed points on $\mathscr{GR}^{0,1}_{V_{\mathbb{F}},0}$ are connected by a chain of rational curves, provided they give rise via (2.4.2) to group schemes which are either both ordinary or both non-ordinary. Not surprisingly, the latter case is much more delicate.

It is this final step which was carried out in [**Ki 1**] in the non-ordinary case only when K has residue field $\mathbb{F}_p$. This was the condition removed by Gee in the case when $\bar{\rho}$ has trivial image.

3. The work of Taylor

3.1. The results of this section are contained in [**Ta 1**], [**Ta 2**] and [**KW 3**, §5]. The starting point is a "potential" version of Serre's conjecture:

THEOREM 3.1.1. *Let $p > 2$, and $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be odd and irreducible. Then there exists a Galois totally real number field F in which p is unramified, and a Hilbert modular form π over F of parallel weight 2, such that $\bar{\rho}_{\pi,\lambda} \sim \bar{\rho}|_{G_F}$.*

SKETCH OF PROOF. The idea is to find an abelian variety $\mathcal{A}/F$, equipped with an embedding $\mathcal{O}_L \hookrightarrow \mathrm{End}_F\mathcal{A}$, where $\mathcal{O}_L$ is the ring of integers of a totally real field L such that $[L : \mathbb{Q}] = \dim\mathcal{A}$, and primes $\lambda|p$ and $\lambda' \nmid p$ of L such that

(1) $\mathcal{A}[\lambda] \sim \bar{\rho}|_{G_F}$.
(2) $\mathcal{A}[\lambda']$ is dihedral and not induced from a subfield of $F(\zeta_p)$.

Given such an $\mathcal{A}$, we find that $\mathcal{A}[\lambda']$ is evidently modular, since it arises from a CM form, and hence $\mathcal{A}$ is modular by a modularity lifting theorem of the type in §1. Hence $\mathcal{A}[\lambda]$ is modular and so is $\bar{\rho}|_{G_F}$.

Collections of data of the above type are classified by a twisted Hilbert modular variety $\mathcal{M}$ (once we fix L, λ and λ', and the dihedral extension appearing in (2)). These have points over large enough number fields F, and we want to check that F can be chosen so that F is totally real and p is unramified in F. A general result of Moret-Bailly [**MB**] asserts that F can be so chosen, provided that there are no local obstructions. That is, provided that $\mathcal{M}$ has a point over $\mathbb{R}$ and over some unramified extension of $\mathbb{Q}_p$. This can be proved by giving an explicit construction of the required Hilbert-Blumenthal abelian varieties by using CM abelian varieties. □

3.1.2. In fact, Taylor proves a more precise result: F can be chosen so that p splits in F if $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible, and such that the residue field extensions at primes over p have degree at most 2 if not. This refinement is especially important in the absolutely irreducible case, since it allows one to show that $\bar{\rho}$ arises from a Hilbert modular form over F of parallel weight $k(\bar{\rho})$ - the Serre weight of $\bar{\rho}$ (see §3). This is used in [**Ta 2**] to establish potential modularity theorems for 2-dimensional crystalline representations with distinct Hodge-Tate weights in $[0, p-2]$. That is, Taylor shows that such a representation is modular when restricted to some totally real field.

COROLLARY 3.2. *Suppose that $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(E)$ is potentially Barsotti-Tate at p, and that $\bar{\rho}|_{\mathbb{Q}(\zeta_p)}$ is absolutely irreducible. Then there* *exists a Galois totally real extension $F/\mathbb{Q}$ such that p is unramified in F and $\rho|_{G_F}$ arises from a Hilbert modular form π over F of parallel weight* 2.

Proof. If $\bar{\rho}$ has dihedral image then it is modular, and the theorem follows from (2.1.3). Otherwise we can apply (3.1.1) and choose F totally real, Galois such that $\bar{\rho}|_{G_F}$ is modular, and p is unramified in F. Since $\bar{\rho}(G_F) \subset \bar{\rho}(G_{\mathbb{Q},S})$ is a normal subgroup, $G_{\mathbb{Q},S}$ normalizes $\bar{\rho}(\mathbb{F}[G_F]) \subset M_2(\mathbb{F})$. Since $\bar{\rho}(G_F)$ contains a non-scalar semi-simple element - the image of any complex conjugation - $\bar{\rho}(\mathbb{F}[G_F])$ is either a Borel subalgebra, a Cartan subalgebra, or $M_2(\mathbb{F})$. In the first case $\bar{\rho}(G_{\mathbb{Q},S})$ would be contained in a Borel subgroup, while in the second it would be dihedral. Hence $\bar{\rho}(\mathbb{F}[G_F]) = M_2(\mathbb{F})$, and $\bar{\rho}|_{G_F}$ is absolutely irreducible. Thus the condition (3) of (2.1.3) holds.

If ρ is not potentially ordinary at p, then the theorem follows from (2.1.3). If $\rho|_{G_{\mathbb{Q}_p}}$ is potentially ordinary at p, then the abelian variety $\mathcal{A}$ in the proof of (3.1.1) can be chosen to have either potentially good ordinary reduction or potentially multiplicative reduction at each prime $v|p$ of F (cf. [**Ta 1**, Lem. 1.2] and [**KW 1**, Prop. 2.5]). If π is the Hilbert modular form corresponding to $\mathcal{A}$, then by Hida theory there is a form π' of parallel weight 2 such that π' principal series at all $v|p$, the representation $\rho_{\pi',\lambda}$ is potentially Barsotti-Tate and ordinary, and $\bar{\rho}_{\pi',\lambda} \sim \bar{\rho}_{\pi,\lambda} \sim \bar{\rho}|_{G_F}$. Thus, there is a totally real solvable extension F'/F such that the base change of π' to F' is ordinary and Barsotti-Tate at all $v|p$. Hence $\rho_{G_{F'}}$ is modular by (2.1.1), and $\rho|_{G_F}$ is modular by base change. □

3.3. Let F be a number field, and S a finite set of primes. Suppose that L is a number field, and for each finite place λ of L write $N(\lambda) = N_{L/\mathbb{Q}}(\lambda)$. Let S_λ denote the union of S and the primes v of F such that $v|N(\lambda)$.

A compatible system with coefficients in L, is a collection $\{\rho_\lambda\}$ where λ runs over the finite places of L, and for each such λ, $\rho_\lambda : G_{F,S_\lambda} \to \mathrm{GL}_n(L_\lambda)$ is a continuous representation such that

(1) ρ_λ is unramified outside S_λ.
(2) For $v \notin S_\lambda$, $\mathrm{tr}(\rho_\lambda(\mathrm{Frob}_v)) \in L$.
(3) If λ, λ' are two finite primes of L and $v \notin S_\lambda \cup S_{\lambda'}$ is a finite prime of F, then $\mathrm{tr}(\rho_\lambda(\mathrm{Frob}_v)) = \mathrm{tr}(\rho_{\lambda'}(\mathrm{Frob}_v))$.

We can strengthen these conditions as follows. Recall that for each λ, and any $v \nmid N(\lambda)$ one can attach a semi-simple representation of the Weil-Deligne group at v to $\rho_\lambda|_{G_{F_v}}$. If $v|N(\lambda)$ one can also attach such a representation if ρ_λ is potentially semi-stable at v. The construction uses the theory of weakly admissible modules and is given in [**Fo**]. More precisely, we take the semi-simplification of the representation produced in *loc. cit.*

We call $\{\rho_\lambda\}$ *strongly compatible* if in addition to the above conditions we have

(4) For each λ and any finite prime v of F with $v|N(\lambda)$, ρ_λ is potentially semi-stable at v.
(5) For each λ and v the semi-simple Weil-Deligne representation at v attached to ρ_λ is defined over L.
(6) If λ and λ' are two finite primes of L and v is any finite prime of F, then the Weil-Deligne representations at v attached to ρ_λ and $\rho_{\lambda'}$ are isomorphic (when viewed as L-representations using (5)).

In fact (5) and (6) imply (2) and (3) respectively.

Fix an embedding $\sigma : L \hookrightarrow \mathbb{C}$. Then as in (1.4.4) the Weil-Deligne representation at v given by (5) gives rise to a local L-factor, which the reciprocal of a polynomial with coefficients in L. For any λ, we denote by $L_\sigma(\rho_\lambda, s)$ the formal product of these L-factors.

COROLLARY 3.3.1. *Keep the assumptions of* (3.2)*. Then*

(1) *ρ occurs as part of a strongly compatible system $\{\rho_\lambda\}$ of λ-adic representations with coefficients in a number field L.*
(2) *For any $\sigma : L \hookrightarrow \mathbb{C}$, The L-function $L_\sigma(\rho, s)$ converges for* $\mathrm{Re}\ s$ *sufficiently large and has a meromorphic continuation to* $\mathbb{C}$.
(3) *If there exists a prime $\ell \neq p$ such that $\rho|_{G_{\mathbb{Q}_\ell}} \sim \left(\begin{smallmatrix} \varepsilon\chi & * \\ 0 & \varepsilon \end{smallmatrix}\right)$, where ε is a character of finite order, then $\rho \sim T_\lambda\mathcal{A}$ for some abelian variety $\mathcal{A}$ of dimension* $[L : \mathbb{Q}]$, *equipped with an embedding* $L \hookrightarrow \mathrm{End}_{\mathbb{Q}}\mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Q}$, *where $\lambda|p$ is a prime of L.*

PROOF. Suppose that a prime ℓ as in (3) exists. If ρ has dihedral image then it arises from a CM-form, and the corollary is clear, so we will assume that this is not the case. We know that there exists a Galois totally real number field F such that $\rho|_{G_F}$ is modular. That is $\rho|_{G_F} \sim \rho_{\pi,\lambda}$ for some Hilbert modular form π over F. This implies that $\rho|_{G_F} \sim T_\lambda\tilde{\mathcal{A}}$ for an abelian variety $\tilde{\mathcal{A}}$ over F with multiplication by some field L, since by construction [**Ca 1**] $\rho_{\pi,\lambda}$ is the λ-adic Tate module of a factor of a Jacobian of a Shimura curve. We may assume that the endomorphism ring of $\tilde{\mathcal{A}}$ contains the ring of integers $\mathcal{O}_L$ of L.

Let $\mathcal{A}'$ denote the Weil restriction of $\tilde{\mathcal{A}}$ to $\mathbb{Q}$. Note that $\mathcal{O}_L$ still acts on $\mathcal{A}'$ and we denote by $T_\lambda\mathcal{A}'$ its λ-adic Tate module. Since ρ is not dihedral, we have $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}(\rho|_{G_F}) \sim \rho \oplus \rho'$ where ρ does not occur as a subquotient of ρ'. By Faltings' isogeny theorem there exists a factor $\mathcal{A}$ of $\mathcal{A}'$ which is stable under the action of $\mathcal{O}_L$, and such that $T_\lambda\mathcal{A} \sim \rho$.

To prove (1) and (2) it will be useful to fix an embedding λ of the algebraic numbers in $\mathbb{C}$ into $\bar{\mathbb{Q}}_p$. Then for any Hilbert modular form π' we will consider the representation $\rho_{\pi',\lambda}$ corresponding to this choice of λ.

Note that (1) follows from (3) when the latter is applicable. In general, we may still choose F and π as above such that $\rho|_{G_F} \sim \rho_{\pi,\lambda}$. Using Brauer's theorem write the trivial representation of $\mathrm{Gal}(F/\mathbb{Q})$ as a virtual sum

$$\mathbf{1} = \sum_j n_j \mathrm{Ind}\,_{G_{F_j}}^{G_{\mathbb{Q}}} \chi_j$$

where $n_j \in \mathbb{Z}$ and χ_j is a complex abelian character of a subfield $F_j \subset F$ such that F/F_j is solvable. By solvable base change π descends to a Hilbert modular form π_j over F_j such that $\rho_{\pi_j,\lambda} \sim \rho|_{G_{F_j}}$ (cf. the final part of the proof of [**Ta 4**, Thm. 2.4]). We now take L to be the subfield of $\mathbb{C}$ generated by the coefficient fields of π and the π_j, as well as the values of the characters χ_j. For λ' a prime of L we form the family of virtual representations $\rho_{\lambda'} = \sum_j n_j \mathrm{Ind}\,_{G_{F_j}}^{G_{\mathbb{Q}}} (\lambda' \circ \chi_j \otimes \rho_{\pi_j,\lambda'})$. Then $\rho_\lambda \sim \rho$, as virtual representations, so ρ_λ is a true representation and (1) is proved by showing that the multiplicity of the trivial representation in the virtual representation $\rho_{\lambda'} \otimes \rho^*_{\lambda'}$ is 1 (cf. [**Ta 3**, §5.3.3]).

To prove (2), note that $L \subset \mathbb{C}$ by definition, and extend σ to an automorphism of $\mathbb{C}$. Then we have

$$L_\sigma(\rho, s) = \prod_j L_\sigma(\mathrm{Ind}\,_{G_{F_j}}^{G_{\mathbb{Q}}} \rho|_{G_{F_j}} \otimes \chi_j, s)^{n_j} = \prod_j L(\pi_j^\sigma \otimes \chi_j^\sigma, s)^{n_j}.$$

The L-functions $L(\pi_j^\sigma \otimes \chi_j^\sigma, s)$ have analytic continuation [**BL**], which proves (2). □

4. The work of Khare-Wintenberger

4.1. Presentations of global deformation rings. We begin with a general result about presentations of global deformation rings over local ones. This is a refinement of a result of Böckle [**Bö**]. We go into a little more detail in this section in order to state the results in a natural level of generality. However, for the level 1 case of Serre's conjecture only the original results of Böckle are needed.

In this subsection only we allow $p = 2$. Let F be a number field, S a finite set of primes of F containing the primes dividing p, and the infinite primes, and let $\Sigma \subset S$. Let $\bar{\rho} : G_{F,S} \to \mathrm{GL}(V)$ be a continuous representation on a finite dimensional $\mathbb{F}$-vector space V and fix a finite extension E of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$, and a continuous character $\psi : G_{F,S} \to \mathcal{O}^\times$, whose composite with the projection $\mathcal{O}^\times \to \mathbb{F}^\times$ is equal to $\det V$.

We denote by $\mathrm{ad}^0 V \subset \mathrm{End}_{\mathbb{F}}(V)$ the subspace of endomorphisms having trace 0. When $p | \dim_{\mathbb{F}} V$, $\mathrm{ad}^0 V$ is not a direct summand in $\mathrm{ad} V$ as a $G_{F,S}$-module, and moreover, in this case, the scalars $\mathbb{F} \subset \mathrm{ad}^0 V$. If G is $G_{F,S}$ or G_{F_v} for $v \in \Sigma$, we denote by $H^1(G, \mathrm{ad}^0 V)'$ the image of $H^1(G, \mathrm{ad}^0 V) \to H^1(G, \mathrm{ad} V)$ and we set $H^2(G, \mathrm{ad}^0 V)' = H^2(G, \mathrm{ad}^0 V)$.

To begin with we will assume that for any $v \in \Sigma$, the commutator of $\bar{\rho}(G_{F_{\mathfrak{v}}})$ is equal to $\mathbb{F}$, and (in case $\Sigma = \emptyset$) that the commutator of $\bar{\rho}(G_{F,S})$ is equal to $\mathbb{F}$. This condition implies that $V|_{G_{F_v}}$ admits a universal deformation to a complete local $\mathcal{O}$-algebra R_v.

Denote by R_v^ψ the quotient of R_v corresponding to deformations with determinant ψ. Set $R_\Sigma^\psi = \widehat{\otimes}_{v\in\Sigma} R_v^\psi$. Similarly, we denote by $R_{F,S}$ the universal deformation $\mathcal{O}$-algebra of V, and by $R_{F,S}^\psi$ the quotient corresponding to deformations of determinant ψ.

Lemma 4.1.1. *For $i = 1, 2$, denote by $h^i_{\Sigma'}$ and $c^i_{\Sigma'}$ the dimensions of the kernel and cokernel of*

$$\theta^i : H^i(G_{F,S}, \mathrm{ad}^0 V)' \to \prod_{v\in\Sigma} H^i(G_{F_v}, \mathrm{ad}^0 V)'.$$

Then $R_{F,S}^\psi$ is a quotient of a power series ring over R_Σ^ψ in $h^1_{\Sigma'}$-variables by at most $c^1_{\Sigma'} + h^2_{\Sigma'}$ relations.

Proof. Write $H^i_{\Sigma'}(G_{F,S}, \mathrm{ad}^0 V)'$ for the kernel of θ^i. Let $\mathfrak{m}_\Sigma$ denote the maximal ideal of R_Σ^ψ. The tangent space of $R_{F,S}^\psi/\mathfrak{m}_\Sigma$ is naturally dual to $H^1_{\Sigma'}(G_{F,S}, \mathrm{ad}^0 V)'$, and this proves the claim about the number of generators.

The proof for the bound on the number of relations is also similar to the standard result for deformation rings [**Ma**, 1.6]. Write $\mathfrak{m}_{F,S}$ for the maximal ideal of $R_{F,S}^\psi$. Let I be the kernel of the map of reduced tangent spaces

$$\mathfrak{m}_\Sigma/(\mathfrak{m}_\Sigma^2, \pi_E) \to \mathfrak{m}_{F,S}/(\mathfrak{m}_{F,S}^2, \pi_E).$$

Then there exists a surjection $\tilde{R} := R_\Sigma^\psi[\![x_1, \dots x_{h^1_{\Sigma'}}]\!] \to R_{F,S}^\psi$ which induces a surjection on reduced tangent spaces with kernel isomorphic to I. Let J denote the kernel of this surjection, and write $\tilde{\mathfrak{m}}$ for the maximal ideal of $\tilde{R}$. Let

$$\rho_{R_{F,S}^\psi} : G_{F,S} \to \mathrm{GL}_n(R_{F,S}^\psi)$$

denote the universal deformation (here $n = \dim_{\mathbb{F}} V$), and consider a set theoretic lifting $\tilde{\rho} : G_{F,S} \to \mathrm{GL}_n(\tilde{R}/\tilde{\mathfrak{m}}\cdot J)$ of $\rho_{R_{F,S}^\psi}$, such that $\det\tilde{\rho}(\gamma) = \psi(\gamma)$ for $\gamma \in G_{F,S}$. The existence of such a lifting follows from the fact that fibres of the map $\det : \mathrm{GL}_n \to \mathrm{GL}_1$ are torsors under the smooth group SL_n. Define a 2-cocycle

$$c : G_{F,S}^2 \to J/\tilde{\mathfrak{m}}\cdot J \otimes_{\mathbb{F}} \mathrm{ad}^0 V; \quad c(g_1, g_2) = \tilde{\rho}(g_1 g_2)\tilde{\rho}(g_2)^{-1}\tilde{\rho}(g_1)^{-1}.$$

Here we identify $J/\tilde{\mathfrak{m}}\cdot J \otimes_{\mathbb{F}} \mathrm{ad} V$ with the kernel of $\mathrm{GL}_n(\tilde{R}/\tilde{\mathfrak{m}}\cdot J) \to \mathrm{GL}_n(\tilde{R}/J)$.

The class $[c]$ of c in $H^2(G_{F,S}, \mathrm{ad}^0 V) \otimes_{\mathbb{F}} J/\tilde{\mathfrak{m}}J$ depends only on $\rho_{R_{F,S}^\psi}$ and not on $\tilde{\rho}$, and it vanishes if and only if $\tilde{\rho}$ can be chosen to be a homomorphism. Since $\rho_{R^{\psi}_{F,S}}|_{G_{F_v}}$ obviously lifts to $GL_n(\tilde{R})$ for $v \in \Sigma$, the image of c in $H^2(G_{F_v}, \mathrm{ad}^0 V) \otimes_{\mathbb{F}} J/\tilde{\mathfrak{m}}J$ is 0, so that $[c] \in H^2_{\Sigma'}(G_{F,S}, \mathrm{ad}^0 V)' \otimes_{\mathbb{F}} J/\tilde{\mathfrak{m}}J$. Thus if $(J/\tilde{\mathfrak{m}}J)^*$ denotes the $\mathbb{F}$-dual of $J/\tilde{\mathfrak{m}}J$, then we obtain a map

$$(J/\tilde{\mathfrak{m}}J)^* \to H^2_{\Sigma'}(G_{F,S}, \mathrm{ad}^0 V)'; \quad u \mapsto \langle [c], u \rangle. \tag{4.1.2}$$

Now note that $(J/\tilde{\mathfrak{m}}J)$ surjects onto $I \subset \tilde{\mathfrak{m}}/(\tilde{\mathfrak{m}}^2, \pi_E)$, and hence we get an inclusion $I^* \subset (J/\tilde{\mathfrak{m}}J)^*$. We claim that I^* contains the kernel of (4.1.2). Suppose that $0 \neq u \in (J/\tilde{\mathfrak{m}}J)^*$ maps to 0 under (4.1.2). Let $\tilde{R}_u$ denote the push-out of $\tilde{R}/\tilde{\mathfrak{m}} \cdot J$ by u, so that $R^{\psi}_{F,S} = \tilde{R}_u/I_u$, where $I_u \subset \tilde{R}_u$ is an ideal of square 0, which is isomorphic to $\mathbb{F}$ as an $\tilde{R}_u$-module. Since $\langle [c], u \rangle = 0$, $\rho_{R^{\psi}_{F,S}}$ lifts to a representation $\rho_u : G_{F,S} \to \mathrm{GL}_n(\tilde{R}_u)$ with determinant ψ, so the map $\tilde{R}_u \to R^{\psi}_{F,S}$ has a section by the universal property of $R^{\psi}_{F,S}$. Hence $\tilde{R}_u \xrightarrow{\sim} R^{\psi}_{F,S} \oplus I_u$, and $\tilde{R}_u/\pi_E\tilde{R}_u \xrightarrow{\sim} R^{\psi}_{F,S}/\pi_E R^{\psi}_{F,S} \oplus I_u$. In particular, the map $\tilde{R}_u \to R^{\psi}_{F,S}$ does not induce an isomorphism on reduced tangent spaces, so that the composite

$$\ker(J/\tilde{\mathfrak{m}}J \to I) \to J/\tilde{\mathfrak{m}}J \to I_u$$

is not surjective, and hence must be the zero map. In other words, u factors through I.

Hence we find that

$$\dim_{\mathbb{F}}(J/\mathfrak{m}J)^* \leqslant \dim_{\mathbb{F}} I + h^2_{\Sigma'} = c^1_{\Sigma'} + h^2_{\Sigma'}.$$

$\square$

4.1.3. In applications, the assumption that for $v \in \Sigma$ the commutator of $\bar{\rho}(G_{F_v})$ is $\mathbb{F}$, is too strong. For example if $d > 1$, it implies that Σ does not contain any infinite primes. We now drop this assumption, but we assume that Σ is non-empty.

For each $v \in \Sigma$ fix a basis β_v of V. The functor which assigns to a local Artinian $\mathcal{O}$-algebra A with residue field $\mathbb{F}$, the set of isomorphism classes of pairs $(V_A, \beta_{v,A})$ where V_A is a deformation of the G_{F_v}-representation V to A with determinant ψ, and $\beta_{v,A}$ is a basis of V_A lifting β_v, is representable by a complete local $\mathcal{O}$-algebra $R^{\square,\psi}_v$. We set $R^{\square,\psi}_{\Sigma} = \widehat{\otimes}_{v\in\Sigma,\mathcal{O}} R^{\square,\psi}_v$ and we denote by $\mathfrak{m}^{\square}_{\Sigma}$ the radical of $R^{\square,\psi}_{\Sigma}$.

Similarly, we obtain an $\mathcal{O}$–algebra $R^{\square,\psi}_{F,S}$ representing the functor which assigns to A the set of isomorphism classes of tuples $(V_A, \{\beta_{v,A}\}_{v\in\Sigma})$, where V_A is a deformation of the $G_{F,S}$-representation V having determinant ψ, and $\beta_{v,A}$ is as before. We denote by $\mathfrak{m}^{\square}_{F,S}$ the radical of $R^{\square,\psi}_{F,S}$.

Then we have the following variant of (4.1.1).

PROPOSITION 4.1.4. *Let*

$$\eta : \mathfrak{m}_{\Sigma}^{\square}/(\mathfrak{m}_{\Sigma}^{\square 2}, \pi_E) \to \mathfrak{m}_{F,S}^{\square}/(\mathfrak{m}_{F,S}^{\square 2}, \pi_E)$$

be the map on reduced tangent spaces induces by the natural map $R_{\Sigma}^{\square,\psi} \to R_{F,S}^{\square,\psi}$. *Then* $R_{F,S}^{\square,\psi}$ *is a quotient of a power series in* $\dim_{\mathbb{F}}\mathrm{coker}\,\eta$ *variables by at most* $\dim_{\mathbb{F}}\ker\eta + h^2_{\Sigma'}$ *relations, where* $h^2_{\Sigma'}$ *is defined as in* (4.1.1).

PROOF. This is proved just as in (4.1.1). $\square$

PROPOSITION 4.1.5. *Keeping the above notation, suppose that* Σ *contains the primes dividing* p, *and that the map*

$$(\dagger) \quad H^0(G_{F,S}, (\mathrm{ad}^0 V)^*(1)) \to \prod_{v|\infty} \widehat{H}^0(G_{F_v}, (\mathrm{ad}^0 V)^*(1)) \times \prod_{v\in(S\backslash\Sigma)_f} H^0(G_{F_v}, (\mathrm{ad}^0 V)^*(1))$$

is injective, where $(S\backslash\Sigma)_f$ *denotes the finite primes, and* $\widehat{H}^0(G_{F_v}, (\mathrm{ad}^0 V)^*(1))$ *denotes* $H^0(G_{F_v}, (\mathrm{ad}^0 V)^*(1))$ *modulo the subgroup of norms. In particular, this condition holds if* $(S\backslash\Sigma)_f$ *is non-empty, or* $H^0(G_{F,S}, (\mathrm{ad}^0 V)^*(1))$ *is trivial.*

Let $s = \sum_{v|\infty, v\notin\Sigma} \dim_{\mathbb{F}} H^0(G_{F_v}, \mathrm{ad}^0 V)$. *Then for some non-negative integer* r, *there is an isomorphism*

$$R_{F,S}^{\square,\psi} \xrightarrow{\sim} R_{\Sigma}^{\square,\psi}[\![x_1, \ldots, x_{r+|\Sigma|-1}]\!]/(f_1, \ldots, f_{r+s}),$$

where $f_1, \ldots, f_{r+s} \in R_{\Sigma}^{\psi}[\![x_1, \ldots, x_{r+|\Sigma|-1}]\!]$,

PROOF. Note that (cf. [**Ki 1**, 3.2.2])

(4.1.6)
$$\begin{aligned}\dim_{\mathbb{F}}\mathfrak{m}_{F,S}^{\square}/(\mathfrak{m}_{F,S}^{\square 2}, \pi_E) - d^2|\Sigma| &= h^1(G_{F,S}, \mathrm{ad}^0 V)' - h^0(G_{F,S}, \mathrm{ad} V) \\ &= h^1(G_{F,S}, \mathrm{ad}^0 V) - h^0(G_{F,S}, \mathrm{ad}^0 V) - 1,\end{aligned}$$

where a lower case h denotes the $\mathbb{F}$-dimension of the space obtained by replacing "h" by "H", and the final equality follows from the exact sequence

$$0 \to (\mathrm{ad}^0 V)^{G_{F,S}} \to (\mathrm{ad} V)^{G_{F,S}} \to \mathbb{F} \to H^1(G_{F,S}, \mathrm{ad}^0 V) \to H^1(G_{F,S}, \mathrm{ad}^0 V)' \to 0.$$

Similarly, we have

(4.1.7)
$$\dim_{\mathbb{F}}\mathfrak{m}_{\Sigma}^{\square}/(\mathfrak{m}_{\Sigma}^{\square 2}, \pi_E) - d^2|\Sigma| = \sum_{v\in\Sigma} [h^1(G_{F_v}, \mathrm{ad}^0 V) - h^0(G_{F_v}, \mathrm{ad}^0 V) - 1].$$

Using Local Tate duality [**Mi**, Cor. 2.3], together with the final three terms of the Poitou-Tate sequence [**Mi**, Thm. 4.10], one sees that $(\dagger)$

implies that the map θ^2 is surjective so that

$$h^2_{\Sigma'} = h^2(G_{F,S}, \mathrm{ad}^0 V) - \sum_{v \in \Sigma} h^2(G_{F_v}, \mathrm{ad}^0 V). \tag{4.1.8}$$

Combining (4.1.6), (4.1.7) and (4.1.8) with (4.1.4) we find that the proposition holds with s satisfying

$$\begin{aligned} |\Sigma| - 1 - s &= (\dim_{\mathbb{F}} \mathrm{coker}\, \eta - \dim_{\mathbb{F}} \ker \eta) - h^2_{\Sigma'} \\ &= -\chi(G_{F,S}, \mathrm{ad}^0 V) + \sum_{v \in \Sigma} \chi(G_{F_v}, \mathrm{ad}^0 V) + |\Sigma| - 1, \end{aligned}$$

where χ denotes the Euler characteristic as $\mathbb{F}$-vector spaces. So

$$s = \sum_{v \in \Sigma} \chi(G_{F_v}, \mathrm{ad}^0 V) - \chi(G_{F,S}, \mathrm{ad}^0 V).$$

Since

$$\begin{aligned} \chi(G_{F,S}, \mathrm{ad}^0 V) &= \sum_{v|\infty} (h^0(G_{F_v}, \mathrm{ad}^0 V) - [F_v : \mathbb{R}] \dim_{\mathbb{F}} \mathrm{ad}^0 V) \\ &= \sum_{v|\infty} h^0(G_{F_v}, \mathrm{ad}^0 V) - [F : \mathbb{Q}] \dim_{\mathbb{F}} \mathrm{ad}^0 V \end{aligned}$$

by [**Mi**, Thm. 5.1], while

$$\begin{aligned} \sum_{v \in \sigma} \chi(G_{F_v}, \mathrm{ad}^0 V) &= \sum_{v|\infty, v \in \Sigma} h^0(G_{F_v}, \mathrm{ad}^0 V) - \sum_{v|p} [F_v : \mathbb{Q}_p] \dim_{\mathbb{F}} \mathrm{ad}^0 V \\ &= \sum_{v|\infty, v \in \Sigma} h^0(G_{F_v}, \mathrm{ad}^0 V) - [F : \mathbb{Q}] \dim_{\mathbb{F}} \mathrm{ad}^0 V \end{aligned}$$

by [**Mi**, Thm. 2.8], the proposition follows. □

4.1.7. *Remarks.*

(1) There is a version of the proposition even when (†) is not injective; one simply needs to add the dimension of the kernel of (†) to s.

(2) Assuming that (†) is a surjection, the proof of (4.1.5) shows that

$$s = h^2_{\Sigma'} + c^1_{\Sigma'} - h^1_{\Sigma'}$$

which makes (4.1.1) more explicit when the hypotheses of the proposition apply.

(3) The proofs of (4.1.1) and (4.1.5) show that when (†) is surjective, $(J/\tilde{\mathfrak{m}}J)^*$ is an extension of $H^2_{\Sigma'}(G_{F,S}, \mathrm{ad}^0 V)'$ by $\mathrm{coker}\, \theta^1$. When $\Sigma = S$, the Poitou-Tate sequence shows that $H^1(G_{F,S}, \mathrm{ad}^0 V(1))$ is also such an extension. It would be interesting to construct an isomorphism between these two spaces. [1]

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[1] This is in fact done in [**KW 2**, §4.1]; see the remark following the proof of Prop. 4.4.

(4) If we assume only that $\mathrm{End}_{\mathbb{F}[G_{F,S}]}V = \mathbb{F}$, then $R^{\psi,\square}_{F,S}$ is formally smooth over $R^{\psi}_{F,S}$ of relative dimension $d^2|\Sigma| - 1$. Hence we have

$$R^{\psi}_{F,S} \xrightarrow{\sim} R^{\square,\psi}_{\Sigma}[\![x_1, \dots, x_{r+|\Sigma|(1-d^2)}]\!]/(f_1, \dots, f_{r+s}).$$

4.2. Existence of liftings with given type. Suppose that $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ is a continuous, odd representation, and fix a two dimensional representation $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ with open kernel. Here $I_{\mathbb{Q}_p} \subset W_{\mathbb{Q}_p}$ denotes the inertia subgroup of the Weil group of $\mathbb{Q}_p$. As before, E will denote a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, and residue field $\mathbb{F}$.

We are interested in liftings of $\bar{\rho}$ to characteristic 0 which are potentially Barsotti-Tate at p of "Galois type" τ. Recall that if V is a two dimensional, potentially Barsotti-Tate E-representation of $G_{\mathbb{Q}_p}$, then its Galois type is defined as follows: Let $K/\mathbb{Q}_p$ be a finite Galois extension such that $\rho|_{G_K}$ is Barsotti-Tate. Set $W = (B_{\mathrm{cris}} \otimes V)^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)}$. If $K_0 \subset K$ denotes the maximal unramified subfield of K, then W is a finite free $E \otimes_{\mathbb{Q}_p} K_0$-module of rank 2, equipped with a linear action of $I_{\mathbb{Q}_p}$, where $g \in I_{\mathbb{Q}_p}$ acts by $g(b \otimes v) = g(b) \otimes g(v)$. We say that W is of type τ if the resulting representation of $I_{\mathbb{Q}_p}$ is equivalent to τ. This is only possible if τ extends to a representation of $W_{\mathbb{Q}_p}$, so we will assume this from now on.

We will suppose for simplicity that $\mathrm{End}_{\mathbb{F}[G_{\mathbb{Q}_p}]}\bar{\rho} = \mathbb{F}$. Then the $G_{\mathbb{Q}_p}$-representation $\bar{\rho}$ admits a universal deformation ring R_p. Finally we fix a character $\varepsilon : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ of finite order such that ε extends $\det\tau$, and $\det\bar{\rho}$ is equal to the composite of $\psi := \chi\varepsilon$ with the projection $\mathcal{O}^\times \to \mathbb{F}^\times$.

Using the classification of finite flat group schemes given in [**Ki 3**, §3], one can prove the following result [**Ki 4**]:

LEMMA 4.2.1. *There exists a (possibly trivial) quotient $R^{\psi}_p(\tau)$ of $R_p \otimes_{W(\mathbb{F})} \mathcal{O}$ such that*

(1) *$R^{\psi}_p(\tau)$ is p-torsion free. If $R^{\psi}_p(\tau)[1/p]$ is non-zero, then it is formally smooth over $W(\mathbb{F})[1/p]$ of dimension* 1.
(2) *For any finite extension E' of E a map of $W(\mathbb{F})$-algebras $x : R_p \to E'$, factors through $R^{\psi}_p(\tau)$ if and only if the two dimensional E'-representation V_x, obtained by specializing the universal representation by x is potentially Barsotti-Tate of type τ, and* $\det V_x = \psi$.

4.2.2. In fact we will use (4.2.1) only in a very simple case, namely when $\bar{\rho}|_{I_{\mathbb{Q}_p}} \sim \left(\begin{smallmatrix} \omega^{k-1} & * \\ 0 & 1 \end{smallmatrix}\right)$, $\tau \sim \tilde{\omega}^{k-2} \oplus 1$, and $\varepsilon = \tilde{\omega}^{k-2}$. Here $k \in [2, p-1]$, and $\tilde{\omega} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ denotes the Teichmüller lifting of ω.

The ring $R^{\psi}_p(\tau)$ can then be described as the quotient of $R_p \otimes_{W(\mathbb{F})} \mathcal{O}$ corresponding to deformations of determinant ψ, which have the form

$\left(\begin{smallmatrix} \tilde{\omega}^{k-2}\chi & * \\ 0 & 1 \end{smallmatrix}\right)$ on $I_{\mathbb{Q}_p}$, where $*$ is peu ramifiée if $k=2$. In this situation the ring $R_p^{\psi}(\tau)$ is formally smooth [**Ta 4**, E4].

Suppose now that $\Sigma \subset S$ is a finite set of primes containing p, and satisfying the conditions of (4.1). We use the notation of (4.1) in our situation. For $v \in \Sigma\backslash\{p\}$, fix a representation $\tau_v : I_{\mathbb{Q}_v} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ with open kernel. Then, there there is a (possibly zero) $\mathcal{O}$-flat quotient $R_v^{\psi}(\tau_v)$ of R_v^{ψ} such that for any finite E-algebra A, a map of $\mathcal{O}$-algebras $x : R_v^{\psi} \to A$ factors through $R_v^{\psi}(\tau_v)$ if and only if the corresponding rank 2 A-representation V_x satisfies $V_x|_{I_{\mathbb{Q}_v}} \sim \tau_v$. (That is the two representation are isomorphic over $A \otimes_E \bar{\mathbb{Q}}_p$.)

For each $v \in \Sigma\backslash\{p\}$, we fix some finite collection C_v, of such τ_v and we write $\bar{R}_v^{\psi}$ for the image of $R_v^{\psi} \to \prod_{\tau_v \in C_v} R_v^{\psi}(\tau_v)$. Set $R_{\Sigma}^{\psi}(\tau) = R_p^{\psi}(\tau_p)\widehat{\otimes}_{v\in\Sigma\backslash\{p\}}\bar{R}_v^{\psi}$, and $R_{\mathbb{Q},S}^{\psi}(\tau) = R_{\mathbb{Q},S}^{\psi}\widehat{\otimes}_{R_\Sigma}R_{\Sigma}^{\psi}(\tau)$.

Lemma 4.2.3. *If $v \in \Sigma$, and $v \neq p$, then $\bar{R}_v^{\psi}[1/p]$ is a product of finite extensions of E.*

Proof. It suffices to consider the case where $C_v = \{\tau_v\}$.

Let E' be a finite extension of E, and $x : R_v^{\psi}(\tau_v) \to E'$ a map of $\mathcal{O}$-algebras corresponding to a closed point of $R_v^{\psi}(\tau_v)$ with residue field E'. Then x corresponds to an E'-representation V_x of $G_{\mathbb{Q}_v}$, and the tangent space of $\operatorname{Spec} R_v^{\psi}(\tau)[1/p]$ at x can be identified with

$$H^1(G_{\mathbb{Q}_v}/I_{\mathbb{Q}_v}, (\mathrm{ad}^0 V_x)^{I_{\mathbb{Q}_v}}) = \ker(H^1(G_{\mathbb{Q}_v}, \mathrm{ad}^0 V_x) \to H^1(I_{\mathbb{Q}_v}, \mathrm{ad}^0 V_x)).$$

Since we are assuming $(\mathrm{ad}^0 V_x)^{G_{\mathbb{Q}_v}} = 0$, a topological generator $\mathrm{Frob}_v \in G_{\mathbb{Q}_v}/I_{\mathbb{Q}_v}$ has no eigenvalues on $(\mathrm{ad}^0 V_x)^{I_{\mathbb{Q}_v}}$ equal to 1. It follows that

$$H^1(G_{\mathbb{Q}_v}/I_{\mathbb{Q}_v}, (\mathrm{ad}^0 V_x)^{I_{\mathbb{Q}_v}}) = (\mathrm{ad}^0 V_x)^{I_{\mathbb{Q}_v}}/(\mathrm{Frob}_v - 1) = 0.$$

$\square$

Proposition 4.2.4. *Assume that the map* (†) *of* (4.1.5) *is injective for V the underlying $\mathbb{F}$-vector space of $\bar{\rho}$. If $R_{\Sigma}^{\psi}(\tau)$ is non-zero then for some $r \geq 0$ there is an isomorphism*

$$R_{\mathbb{Q},S}^{\psi}(\tau) \xrightarrow{\sim} R_{\Sigma}^{\psi}(\tau)[\![x_1, \ldots x_r]\!]/(f_1, \ldots, f_{r+1}).$$

In particular, we have $\dim R_{\mathbb{Q},S}^{\psi}(\tau) \geq 1$.

Proof. Note that the integer s in (4.1.7)(2) is equal to 1, since $\bar{\rho}$ is odd. By (4.2.1) and (4.2.3) $\dim R_{\Sigma}(\tau) = 2$, so the proposition follows. $\square$

4.2.5. Similarly, if we drop the assumption that $\bar{\rho}|_{G_{\mathbb{Q}_v}}$ has trivial endomorphisms, then we have an analogue of (4.2.1) for framed deformations and we define analogous rings $\bar{R}_v^{\square,\psi}$, $R_p^{\square,\psi}(\tau)$, $R_{\Sigma}^{\square,\psi}(\tau)$, and $R_{\mathbb{Q},S}^{\square,\psi}(\tau)$ using framed deformations (cf. (4.1.3)). Moreover, if we still assume that the commutator of $\bar{\rho}(G_{\mathbb{Q},S})$ is trivial, then we set $R^{\psi}_{\mathbb{Q},S}(\tau)$ equal to the image of $R^{\psi}_{\mathbb{Q},S}$ in $R^{\square,\psi}_{\mathbb{Q},S}(\tau)$. One can prove the following.

PROPOSITION 4.2.6. *Assume that the map* (†) *of* (4.1.5) *is injective for V the underlying $\mathbb{F}$-vector space of $\bar{\rho}$. If $R^{\square,\psi}_{\Sigma}(\tau)$ is non-zero then for some $r \geq 0$ there is an isomorphism*

$$R^{\square,\psi}_{\mathbb{Q},S}(\tau) \xrightarrow{\sim} R^{\square,\psi}_{\Sigma}(\tau)[\![x_1, \dots x_{r+|\Sigma|-1}]\!]/(f_1, \dots, f_{r+1}).$$

In particular, we have $\dim R^{\square,\psi}_{\mathbb{Q},S}(\tau) \geq 4|\Sigma|$, *and if the commutator of $\bar{\rho}(G_{\mathbb{Q},S})$ consists of scalars then* $\dim R^{\psi}_{\mathbb{Q},S}(\tau) \geq 1$.

PROOF. The proof of the first claim is analogous to that of (4.2.4), using (4.1.5). In this case the ring $R^{\square,\psi}_{\Sigma}$ has dimension $3|\Sigma| + 2$, and so one finds that $R^{\square,\psi}_{\mathbb{Q},S}(\tau)$ has dimension at least $4|\Sigma|$.

If $\bar{\rho}$ has only scalar endomorphisms, then the morphism $R^{\psi}_{\mathbb{Q},S}(\tau) \to R^{\square,\psi}_{\mathbb{Q},S}(\tau)$ is smooth of relative dimension $4|\Sigma| - 1$, and the second claim follows. □

4.2.7. Proposition (4.2.6) gives a general lower bound on the size of $R^{\psi}_{\mathbb{Q},S}(\tau)$. One of the key insights of Khare-Wintenberger is that using the results of Taylor on the potential version of Serre's conjecture, explained in the previous section, one can get an upper bound, and hence show that $R^{\psi}_{\mathbb{Q},S}(\tau)$ is finite over $\mathcal{O}$, of rank ≥ 1. This implies the existence of minimal liftings, and then of strongly compatible systems using Taylor's results once again. More precisely, we have

THEOREM 4.2.8. *If $p > 2$, $\bar{\rho}|_{\mathbb{Q}(\zeta_p)}$ is absolutely irreducible and $R^{\psi}_{\Sigma}(\tau)$ is non-zero, then $R^{\psi}_{\mathbb{Q},S}(\tau)$ is a finite $\mathcal{O}$-module of rank at least* 1.

PROOF. It suffices to show that $R^{\psi}_{\mathbb{Q},S}(\tau)$ is a finite $\mathcal{O}$-algebra. Indeed, once we know this, if $R^{\psi}_{\mathbb{Q},S}(\tau)$ has rank 0, then it is an Artin ring, which contradicts (4.2.6).

By (3.2) we can find a totally real, finite, Galois extension F of $\mathbb{Q}$, in which p is unramified, and such that any characteristic 0 lift of $\bar{\rho}$ corresponding to a point of $R^{\psi}_{\mathbb{Q},S}(\tau)$ becomes modular over F. After replacing F by a larger field, we may assume that any such lift also has cyclotomic determinant and is Barsotti-Tate at any prime $v|p$ of F. Moreover, we may choose this final extension so that the conditions of (2.1.1)(3) continue to hold.

We now put ourselves in the situation of (2.3), with the conditions (1), (2)$'$ and (3)$'$, and we use the notation of that section. In particular, we will assume that $\bar{\rho}|_{G_{F_v}}$ is absolutely irreducible for $v|p$ a prime of F. As usual, this assumption can be lifted using framed deformations. Let us remark that since we are following the notation of (2.3), the argument below is again valid only in the minimal case. The argument for the non-minimal case is identical, if one replaces $R_p^{\psi}(\tau)$ by the ring $R_{\Sigma}^{\psi}(\tau)$ introduced in (4.2.2), for a suitably chosen collection of quotients $\bar{R}_v^{\psi}$ of R_v^{ψ} with $v \in \Sigma\backslash\{p\}$.

For $v|p$ we denote by R_v the universal deformation $\mathcal{O}$-algebra of $\bar{\rho}|_{G_{F_v}}$ and we set $R_{v|p} = \widehat{\otimes}_{v|p} R_v$. As above, we write R_p for the universal deformation ring of $\bar{\rho}|_{G_{\mathbb{Q}_p}}$. We denote by $\tilde{S}$ the primes of F which lie over a prime in S.

We saw in (2.3) that there exists a quotient $R_{v|p}^{0,1\prime}$ of $R_{v|p}^{0,1}$ (itself a quotient of $R_{v|p}$) corresponding to a union of components, such that the map θ_{∞} induces an isomorphism $R_{v|p}^{0,1\prime}[\![z_1, \dots z_{r-d}]\!] \xrightarrow{\sim} \mathbb{T}_{\infty}$. The composite $R_{v|p} \to R_p^{\psi} \to R_p^{\psi}(\tau)$ factors through $R_{v|p}^{0,1}$. The explicit description of the components of $\operatorname{Spec} R_{v|p}[1/p]$ given in (2.2.1) implies that the image of the induced map $\operatorname{Spec} R_p^{\psi}(\tau)[1/p] \to \operatorname{Spec} R_{v|p}^{0,1}[1/p]$ is contained in at most two components of $\operatorname{Spec} R_{v|p}^{0,1}[1/p]$, namely those which correspond to deformations which are ordinary at all $v|p$ or non-ordinary at all $v|p$. After possibly increasing F, we may assume that both these components are contained in $\operatorname{Spec} R_{v|p}^{0,1\prime}[1/p]$. This follows from the argument in the final paragraph of the proof of (3.2).

On the other hand, $R_{\mathbb{Q},S}$ is a finite $R_{F,\tilde{S}}$-algebra. To see this let $\mathfrak{m}_{F,\tilde{S}}$ denote the maximal ideal of $R_{F,\tilde{S}}$. Let $\rho_{\mathbb{Q},F}$ denote the $R_{\mathbb{Q},S}/\tilde{\mathfrak{m}}_{F,S}R_{\mathbb{Q},S}$-representation of $G_{\mathbb{Q},S}$ obtained by specializing the universal representation of $G_{\mathbb{Q},S}$ over $R_{\mathbb{Q},S}$. Then $\rho_{\mathbb{Q},F}|_{G_F}$ is equivalent to $\bar{\rho}$. In particular, if F' denotes the composite of F and the fixed field of $\ker \bar{\rho}$, then $\rho_{\mathbb{Q},F}$ factors through $\mathrm{Gal}(F'/\mathbb{Q})$. The finiteness now follows from the absolute irreducibility of $\bar{\rho}$ and the argument in the second paragraph of [**deJ**, 3.14] (cf. [**KW 1**, Lemma 2.4]).

Hence it suffices to show that $R_{F,\tilde{S}} \otimes_{R_{v|p}} R_{v|p}^{0,1\prime}$ is finite over $\mathcal{O}$. However this ring is isomorphic to

$$R_{v|p}^{0,1\prime}[\![z_1, \dots, z_{r-d}]\!]/(x_1, \dots, x_r) \xrightarrow{\sim} \mathbb{T}_{\infty}/(x_1, \dots x_r),$$

and the right hand side is a finite $\mathcal{O}$-module because $\mathbb{T}_{\infty}$, which acts faithfully on M_{∞}, is a finite $\mathcal{O}[\![x_1, \dots x_r]\!]$-module. □

4.3. The proof of Serre's conjecture. Given $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ as above, we have the invariants $N(\bar{\rho})$ and $k(\bar{\rho})$ introduced in §1. Using (4.2.8) we find.

COROLLARY 4.3.1. *With the assumptions of* (4.2.8) *there exists a lifting ρ of $\bar{\rho}$ which is potentially Barsotti-Tate of type τ with determinant ψ, and such that the prime to p part of the conductor of ρ is $N(\bar{\rho})$.*

More precisely, given a choice of component of $\operatorname{Spec}\bar{R}_v^{\psi}[1/p]$, *for each* $v \in \Sigma\backslash\{p\}$ *we can choose* ρ *so that the corresponding point of* $\operatorname{Spec}\bar{R}_v^{\psi}[1/p]$ *lies on the given component for each* $v \in \Sigma\backslash\{p\}$, *and such that* ρ *is potentially Barsotti-Tate of type* τ *at* p, *with determinant* ψ.

Moreover, there exists a strongly compatible system $\{\rho_\lambda\}$ *with* $\rho \sim \rho_p$.

PROOF. The claims in the first two paragraphs follow from (4.2.8). For the first part one takes S to be the set of primes of ramification of $\bar{\rho}$, together with p and ∞ and Σ the set of primes ℓ such that not every lift of $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ has conductor $N(\bar{\rho})$. At these primes one takes the quotient $\bar{R}_v^{\psi}$ (4.2.2) to be the one corresponding to lifts with conductor $N(\bar{\rho})$.

The final claim follows from (3.3.1). □

4.3.2. Suppose now that $\bar{\rho}$ has Serre weight $k(\bar{\rho})$. After twisting we may assume that $k(\bar{\rho}) \in [2, p+1]$. We have the following variant of (4.3.1) [**KW 1**, Thm 2.1].

THEOREM 4.3.3. *Suppose that* $k(\bar{\rho}) \in [2, p+1]$ *and* $k(\bar{\rho}) \neq p$. *Then there exists a lifting of* ρ *of* $\bar{\rho}$ *which is crystalline with Hodge-Tate weights* $0, k(\bar{\rho}) - 1$, *at* p, *and such that the prime to* p *part of the conductor of* ρ *is* $N(\bar{\rho})$. *If* $k(\bar{\rho}) = p+1$ *there exists a lifting* ρ *which is semi-stable non-crystalline, with Hodge-Tate weights* $0, 1$ *at* p, *and prime to* p *conductor* $N(\bar{\rho})$.

Moreover there exists a strongly compatible system $\{\rho_\lambda\}$ *with* $\rho_p \sim \rho$.

PROOF (SKETCH). The proof is analogous to that of (4.3.1) above, but in fact somewhat easier since we are dealing with liftings which are already semi-stable over $\mathbb{Q}_p$. First one shows an analogue of (3.2), but where one requires π to have parallel weight $k(\bar{\rho}) - 1$. This uses the refinement mentioned in (3.1.2) (see also [**KW 1**, Prop. 2.5]). One also has an analogue of the rings $R_p^{\psi}(\tau)$ for crystalline representation with Hodge-Tate weights in $[0, p-2]$, as well as an analogue of the modularity lifting theorem (2.1.1). When $k(\bar{\rho}) = p+1$ one considers potentially semi-stable, ordinary liftings with Hodge-Tate weights $0, 1$. . The argument is then similar to that given in (4.2); one uses the general result (4.1.1) and (4.1.5) to bound the global deformation ring from below, and its relationship with Hecke algebras to bound it from above. □

THEOREM 4.3.4 (Khare). *Let* $\bar{\rho} : G_{\mathbb{Q},\{p,\infty\}} \to \mathrm{GL}_2(\mathbb{F})$ *be odd. Then* $\bar{\rho}$ *is modular.*

PROOF (SKETCH). When $p = 2$, this is due to Tate, so we may assume that p is odd [**Tat**].

We may assume that $2 \leqslant k(\bar{\rho}) \leqslant p+1$, and we use induction on the pair $(k(\bar{\rho}), p)$ ordered lexicographically. In the following we will apply various modularity lifting theorems such as those of Skinner-Wiles [SW 1], [SW 2], which applies to ordinary liftings, or (2.1.1). Although these theorems require some mild hypotheses on $\bar{\rho}$, we will ignore these. In practice one of course has to check that they are satisfied. We also remark that $\det \bar{\rho}$ is equal to $\chi^{k(\bar{\rho})-1}$ on inertia at p, and is unramified outside p. Hence $\det \bar{\rho} = \chi^{k(\bar{\rho})-1}$, and since this is an odd character, $k(\bar{\rho})$ is even.

It will be convenient to fix embeddings $\lambda_l : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_l$ for each finite prime l of $\mathbb{Q}$. Given a strongly compatible system $\{\rho_\lambda\}$ we will write $\rho_l = \rho_{\lambda_l}$.

Consider first the case $k(\bar{\rho}) = 2$. We lift $\bar{\rho}$ to a Barsotti-Tate representation with cyclotomic determinant, and put it in a strongly compatible family $\{\rho_\lambda\}$, using (4.3.1) or (4.3.3). Each ρ_λ is unramified outside $\{\lambda, \infty\}$ and Barsotti-Tate at λ. We denote by $\bar{\rho}_\lambda$ the reduction of each of these. Then $\bar{\rho}_3^{\mathrm{ss}} \sim \mu_3 \oplus \mathbf{1}$ by a result of Serre. Hence the theorem of Skinner-Wiles implies that ρ_3 is modular and hence that ρ_p and $\bar{\rho} = \bar{\rho}_p$ are modular (and hence reducible, since the modular form is of level 1 and weight 2.).

More generally if $(\bar{\rho}, p)$ is the smallest counterexample, then we may assume that the largest odd prime $l < p$ satisfies $l+1 < k(\bar{\rho})$, since otherwise we could use (4.3.3) and an argument as in the previous paragraph, and reduce the modularity of $\bar{\rho}$ to a mod l representation $\bar{\rho}_l$ of Serre weight $k(\bar{\rho})$ (and conductor 1). The corresponding modularity lifting theorems are due to Skinner-Wiles *loc. cit*, and Fujiwara (see [**Ta 2**]).

Now we consider two cases. Suppose first that $\bar{\rho}|_{I_{\mathbb{Q}_p}}$ is absolutely irreducible or decomposable (i.e., is semi-simple). After twisting, we may assume that $k(\bar{\rho}) \leqslant \frac{p+1}{2}$, and hence the prime l above (if it exists) is at most $\frac{p-1}{2}$. This is impossible, and so $p = 3$, in which case one may invoke Serre's result above.

Suppose then that $\bar{\rho}|_{I_{\mathbb{Q}_p}}$ is reducible and indecomposable. After twisting, we may assume that $\bar{\rho}|_{I_{\mathbb{Q}_p}} \sim \left(\begin{smallmatrix} \omega^i & * \\ 0 & 1 \end{smallmatrix}\right)$, where $i \in [1, p-2]$. Suppose that there exists an odd prime $\ell | p-1$. We use (4.3.1) to lift $\bar{\rho}$ to a characteristic 0 representation ρ which is potentially Barsotti-Tate at p of type $\tilde{\omega}^{i-1} \oplus \mathbf{1}$, where $\tilde{\omega}$ denotes the Teichmüller lift of ω. This is possible because $R_p^\psi(\tau)$ is non-zero by results of Savitt [**Sa**]. Place ρ in a strongly compatible system $\{\rho_\lambda\}$ using (4.3.1). Then ρ_ℓ is unramified outside $\{p, \ell, \infty\}$, is Barsotti-Tate at ℓ, and $\rho_\ell|_{I_{\mathbb{Q}_p}} \sim \begin{pmatrix} \tilde{\omega}_\ell^{i-1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$ as $\bar{\mathbb{Q}}_\ell$-representations, where $\tilde{\omega}_\ell$ denotes $\tilde{\omega}$ viewed as a $\bar{\mathbb{Q}}_\ell$-valued character via our chosen embeddings $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$ and $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_\ell$. We denote by ω_ℓ its mod ℓ reduction.

Note that ω_ℓ has order $\frac{p-1}{\ell^r}$ where $r = v_\ell(p-1) > 0$. Using (4.3.1) we may lift $\bar{\rho}_\ell$ to an ℓ-adic representation ρ'_ℓ which is unramified outside

$\{p, \ell, \infty\}$, is Barsotti-Tate at ℓ, and such that $\rho'_\ell|_{I_{\mathbb{Q}_p}} \sim \begin{pmatrix} \tilde{\omega}_\ell^j & 0 \\ 0 & 1 \end{pmatrix}$ where $j = i$ modulo $\frac{p-1}{\ell^r}$ and $j \in [\frac{\ell^r-1}{2\ell^r}(p-1), \frac{\ell^r+1}{2\ell^r}(p-1)]$. We also put ρ'_ℓ into a strongly compatible family $\{\rho'_\lambda\}$. Then ρ'_p is unramified outside p, and potentially Barsotti-Tate of tame type $\tau' = \tilde{\omega}^j \oplus \mathbf{1}$ at p. The results of Savitt and Breuil-Mézard [**BM**] now imply that $k(\bar{\rho}) \in \{j+2, p+1-j\}$.

If $\bar{\rho}'_p$ were modular then using suitable modularity lifting theorems we would find that

$$\bar{\rho}'_p \text{ modular } \implies \bar{\rho}_\ell \text{ modular } \implies \bar{\rho}_p \text{ modular.}$$

Since $(k(\bar{\rho}), p)$ is our smallest putative counterexample to modularity, we must have $k(\bar{\rho}) \leqslant \frac{\ell^r+1}{2\ell^r}(p-1)+2 \leqslant 2/3(p-1)+2$. Hence the smallest odd prime $l < p$, satisfies $l \leqslant 2/3(p-1)$. This implies that $p \leqslant 5$.

If $p = 3$ we are done as before. If $p = 5$, then we still have $3 + 1 < k(\bar{\rho}) \leqslant 6$, and $2|k(\bar{\rho})$, so $k(\bar{\rho}) = 6$. In this case, one considers a semi-stable weight 2 (that is with Hodge-Tate weights 0 and 1) lifting of $\bar{\rho}$ (see [**KW 1**, 4.3]). If $\bar{\rho}$ is absolutely irreducible, a variant of (3.3.1) for semi-stable non-crystalline weight 2 representations shows that $\bar{\rho}_5$ occurs in the torsion of a semi-stable abelian variety, having good reduction outside 5. A result of Brumer-Kramer [**BK**] (see also [**Sc**]) says that such a variety cannot exist. It follows that $\bar{\rho}$ is reducible, and hence modular.

We have used the assumption that $p-1$ is not a power of 2. If this is the case then one can pass to a slightly larger prime $p' > p$, and use the above arguments. Khare checks in [**Kh 1**] that the Fermat primes are sufficiently sparse that this succeeds. $\square$

4.3.5. To end this article let us say a word about the argument of Khare-Wintenberger in the case when $N(\bar{\rho}) > 1$. The idea is the following: Given a $\bar{\rho}$ lift it to a p-adic representation, and place it in a strongly compatible family $\{\rho_\lambda\}$. If $\ell | N(\bar{\rho})$ then $N(\bar{\rho}_\ell)$ will be prime to ℓ. Using this one can try to reduce the number of primes dividing $N(\bar{\rho})$.

Of course one has to proceed so as to be able to use known modularity lifting theorems to deduce the modularity of $\bar{\rho}$ from that of $\bar{\rho}_\ell$. The argument is therefore a little more involved: First if $N(\bar{\rho})$ is odd, then by a technique similar to that in the case of $N(\bar{\rho}) = 1$, using weight 2 liftings and induction on $(k(\bar{\rho}), p)$, one reduces to the case $p = 2$, $k(\bar{\rho}) = 2$. In the argument for $N(\bar{\rho}) = 1$, we effectively started the induction at $p = 3, 5$ by using the results of Serre and Brummer-Kramer. When $N(\bar{\rho}) > 1$, one reduces the cases $p = 3, 5$ to $p = 2$ by using congruences between 2-adic representations of $G_{\mathbb{Q}_3}$ and $G_{\mathbb{Q}_5}$. These congruences are not of exactly the kind used in the proof of (4.3.4), but the idea is similar.

When $p = 2$ and $k(\bar{\rho}) = 2$, one can implement the above strategy by lifting $\bar{\rho}$ in weight 2 to a strongly compatible system $\{\rho_\lambda\}$. For an odd prime $\ell | N(\bar{\rho})$, ρ_ℓ is potentially Barsotti-Tate at ℓ, and $N(\bar{\rho}_\ell) < N(\bar{\rho})$.

Hence $\bar{\rho}_\ell$ is modular by induction on $N(\bar{\rho})$ and Theorem (0.1) implies that ρ_ℓ is modular.

If $N(\bar{\rho})$ is even one reduces to the odd case by lifting $\bar{\rho}$ in weight 2 to a strongly compatible system $\{\rho_\lambda\}$. By using 3-adic congruences one can reduce to the case where $\rho_p(I_2)$ is not unipotent, so that ρ_2 is potentially Barsotti-Tate. By definition $N(\bar{\rho}_2)$ is odd and hence $\bar{\rho}_2$ is modular. Then Theorem (0.1) implies ρ_2 is modular.

In this argument (0.1) gets used in its full strength, since the ℓ-part of $N(\bar{\rho})$ may be arbitrarily large, and this corresponds to ρ_ℓ becoming Barsotti-Tate only over an extension with arbitrarily large conductor.

A final difficulty which we have ignored above (but which of course has to be dealt with) is that to apply (0.1) one needs to assume that $\bar{\rho}|_{\mathbb{Q}(\zeta_p)}$ is absolutely irreducible and has non-solvable image of $p = 2$. To do this Khare-Wintenberger first reduce to a situation where $\bar{\rho}$ is ramified at some suitably chosen large auxiliary prime q. This prime is chosen so that throughout the argument the image of inertia at q guarantees that $\bar{\rho}$ has non-solvable image. Finally q is "removed" at the end of the argument by the method explained above.

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Department of Mathematics, University of Chicago
E-mail address: kisin@math.uchicago.edu