

*NSF-CBMS Regional Conference Series  
in Probability and Statistics  
Volume 4*

**HIGHER  
ORDER  
ASYMPTOTICS**

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*Regional Conference Series  
in Probability and Statistics*

Supported by the  
National Science Foundation

The production of the *NSF-CBMS Regional Conference Series in Probability and Statistics* is managed by the Institute of Mathematical Statistics: John Collins, IMS Managing Editor/Statistics; Patrick Kelly, IMS Production Editor; Jessica Utts, IMS Treasurer; and Barbara J. Lindeman, IMS Business Manager.

Library of Congress Catalog Card Number: 94-75429

International Standard Book Number: 0-940600-31-5

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Printed in the United States of America

*To My Parents and Wife*



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# Preface

This monograph is based on my CBMS–NSF lectures in August, 1991 at Chapel Hill, North Carolina. The flow of the chapters mirrors the flow of the lectures. While I have rearranged the material I distributed during the lectures, I have not added or deleted very much, except for adding remarks or details by way of clarification or resolution of issues raised by my very lively audience at Chapel Hill.

Most of the material is taken from my own work done jointly with many students and friends. One of the pleasures of lecturing or writing about all this has been the reliving of that experience of working and discovering together. My tastes and beliefs have evolved over time; some of that is reflected here too. Higher order asymptotics itself, like all mathematical tools, is philosophically neutral, and can be effectively used by both Bayesians and frequentists. The frequentist results here concentrate on optimality, but the theory can, in principle, be applied to parametric robustness studies also. I first learnt this from Kei Takeuchi. However, neither he nor I have followed this up. I hope someone else will. There are also open questions regarding noninformative priors, application of the new likelihoods to Neyman–Scott problems, higher order admissibility and minimaxity and Edgeworth expansions.

I am indebted to Pranab Sen for organizing these lectures and providing support in various forms. His merriment in company of friends and FOE's alike was infectious. Subhobrata Das' help during and after the lectures was indispensable. I would also like to put on record my appreciation of lectures by Masafumi Akahira, Rabi Bhattacharya, Malay Ghosh, Andrew Rukhin, Soumendra Lahiri and Bimal K. Sinha, which supplemented those of mine on higher order asymptotics and inference.

Most of the writing was done while I was visiting the Department of Statistics, Purdue University. The department's hospitality is gratefully acknowledged. Many people, specially R. V. Ramamoorthi, Rabi Bhattacharya and Bimal K. Sinha, helped with the bibliography. Finally, it is a pleasure to

thank Mary Epperson and Teena Seele for typing a mess into a manuscript. Only people who have struggled with my writing know what a heroic task that is.

No words can express my debt to John R. Collins, whose editorial help is gratefully acknowledged.

Jayanta K. Ghosh

West Lafayette, September 1992