THE CENTRAL LIMIT THEOREM FOR MARKOV PROCESSES

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1. Introduction

The central limit theorem has been presented at various levels of generality and application, both with respect to types of processes considered and to types of limit laws considered. The main results of this paper, contained in Section 4, refer to Markov processes, with a time homogeneous law of evolution, whose transition probabilities, averaged in the Cesaro (C-1) sense, converge to a common limit. As indicated in Theorem 2.1, this is equivalent to the assumption that the process has a finite invariant measure and the state space is a final set (in the sense of Doeblin [5]). The results can be extended to processes on indecomposable sets and this generalization is indicated. Conditions for convergence to any infinitely divisible law are given, and special consideration is given to normal and clustering (compound Poisson) distributions.

The method is related to the familiar approach when the state space is countable: look at the interblocks between successive returns to a given state, these being independent with a common distribution when the process is started at this state. For a noncountable state space the single state of the classical approach must be replaced by a uniform state set, a notion first used in [4]. Section 2 provides an introduction to the terminology used regarding Markov processes, in particular to uniform state sets. The interblocks between successive returns to a uniform state set are identically distributed for the process with a suitable starting distribution, but are no longer independent. They do, however, satisfy a pointwise strong mixing condition.

Thus, we need the central limit theorem for pointwise strong mixing stationary sequences. This was first presented in its general form in [3]. In Section 3 a revised and expanded version is developed, which is suitable to the present application.

This study is based upon the comparison of laws of functionals of the process to laws of sums of related independent random variables. This requires the notion of asymptotic equivalence of laws developed by M. Loève; a brief introduction to the subject is provided at the end of Section 2.

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Finally, the results on normal convergence in Section 4 leave the value of the variance of the limit distribution unclear, and several alternative representations of this quantity are given in Section 5.

2. Preliminaries

Let $(\mathscr{X}, \mathscr{A})$ be a measurable space and P be a *Markov kernel* on $(\mathscr{X}, \mathscr{A})$: the domain of P is $\mathscr{X} \times \mathscr{A}$, $P(x, \cdot)$ is a probability on $\mathscr{A}$ for each $x \in \mathscr{X}$, and $P(\cdot, A)$ is $\mathscr{A}$ measurable on $\mathscr{X}$ for each $A \in \mathscr{A}$.

To a Markov kernel P there corresponds a family of Markov kernels $P^0, P^1, P^2, \cdots$: $P^0(x, \cdot) = \delta_x$, the unit measure concentrated at x, for each $x \in \mathscr{X}$; the kernel $P^1 = P$, and the P^n for $n > 1$ are defined inductively by

$$P^n(x, A) = \int P^{n-1}(x, dy)P(y, A). \tag{2.1}$$

The Chapman-Kolmogorov equation then applies: for all $m, n \geqq 0$, $x \in \mathscr{X}$, $A \in \mathscr{A}$,

$$P^{m+n}(x, A) = \int P^m(x, dy)P^n(y, A). \tag{2.2}$$

Let M denote the Banach space of bounded, real valued, measurable functions on $(\mathscr{X}, \mathscr{A})$ with supremum norm, and let Φ denote the Banach space of finite signed measures on $\mathscr{A}$ with total variation norm. Each of these spaces is a vector lattice under its natural ordering, and the Markov kernels $\{P^0, P^1, P^2, \cdots\}$ define a semigroup of positive linear contractions on M by

$$P^n f(x) = \int P^n(x, dy)f(y). \tag{2.3}$$

Similarly, a semigroup of positive linear contractions on Φ is defined by

$$\varphi P^n(A) = \int \varphi(dx)P^n(x, A). \tag{2.4}$$

For any $\varphi \in \Phi$ and real Borel measurable f on $(\mathscr{X}, \mathscr{A})$, we will use both the notations $\int f(x)\varphi(dx)$ and φf, according to convenience, to denote the integral of f with respect to φ whenever this integral exists. Note that the expression $\varphi P f$ is unambiguous since it can be verified that $(\varphi P)f = \varphi(Pf)$.

When $\varphi = \varphi P$, we say that φ is *invariant*.

Elements of $\mathscr{X}$ are *states*, $(\mathscr{X}, \mathscr{A})$ is the *state space*, and sets in $\mathscr{A}$ are *state sets*.

The product measurable space $\Pi_{n=0}^{\infty} (\mathscr{X}^n, \mathscr{A}^n)$, where each $(\mathscr{X}^n, A^n) = (\mathscr{X}, \mathscr{A})$, is the *sample space*; an element of $\Pi_{n=0}^{\infty} \mathscr{X}^n$ is a *sample value*. We denote the projection of $\Pi_{n=0}^{\infty} \mathscr{X}^n$ on its nth coordinate by X_n and call X_n the *sample value* (or *value of the process*) *at time n*.

For any probability $\varphi \in \Phi$, a probability P_φ is determined on $\Pi_{n=0}^\infty \mathscr{A}^n$ by the Markov kernel P through the relations

$$(2.5)\qquad P_\varphi[X_0 \in A_0, \cdots, X_n \in A_n] = \int_{A_0} \varphi(dx_0) \int_{A_1} P(x_0, dx_1) \cdots \int_{A_{n-1}} P(x_{n-2}, dx_{n-1}) P(x_{n-1}, A_n).$$

(See the Ionescu Tulcea theorem [9], p. 137).

We say that φ is the *starting distribution* of the *process probability* P_φ. When $\varphi = \delta_x$, we denote P_φ simply P_x. The term "starting distribution φ" will be used *only* when the measure φ is a probability. Note that $\{X_n\}$ is strictly stationary with respect to P_φ if, and only if, φ is invariant.

The sample space together with the family of process probabilities is a *Markov process*. Any real valued Borel measurable function Z on $\Pi_{n=0}^\infty (X^n, \mathscr{A}^n)$ is a *random variable*. (It is convenient at times to allow complex as well as real valued functions to be random variables. Also it is common to refer to the projections X_n as random variables, whether real valued or not, and to refer to the $\{X_n\}$ as a "process".) We let E_φ, E_x denote the expectations corresponding to P_φ, P_x, respectively, defined on the random variables. A constructive argument verifies that, for any random variable Z, the expectation $E_x Z$ exists on a measurable set and is measurable in x on that set. Moreover, when $E_\varphi Z$ exists for some starting distribution φ, then $E_\varphi Z = \int \varphi(dx) E_x Z$.

For any state set A and $n \geqq 1$, define

$$(2.6)\qquad \tau_A^{(n)} = \min\left\{\infty, k \geqq 1 : \sum_{j=1}^k \chi_A(X_j) = n\right\},$$

where χ_A denotes the indicator function of A ($\chi_A(x) = 1$ or 0 according as $x \in A$ or $x \notin A$). We call $\tau_A^{(n)}$ the nth *entrance time* of A, and we write τ_A for $\tau_A^{(1)}$ and define $\tau_A^{(0)} = 0$. Entrance times play a basic role in the analysis of Markov processes. They are $\Pi_{n=0}^\infty \mathscr{A}^n$ measurable, and hence are random variables (provided we allow the value $+\infty$). They are also Markov times (see Loève [9] for definition and properties). Let

$$(2.7)\qquad P_{(A)}^n(x, B) = P_x[\tau_A^{(n)} < \infty, X_{\tau_A^{(n)}} \in B].$$

Then the $\{P_{(A)}^n\}$ satisfy the Chapman-Kolmogorov equation, and $P_{(A)}^n$ is a Markov kernel if, and only if, $P_x[\tau_A^{(n)} < \infty] = 1$ for every x. Now suppose $P_x[\tau_A < \infty] = 1$ for every $x \in A$. Then we can restrict $P_{(A)}^1$ to $A \times \mathscr{A}_A$, where $\mathscr{A}_A = \{B \in \mathscr{A} : B \subset A\}$. In this case $P_{(A)} = P_{(A)}^1$ is a Markov kernel on $(A, \mathscr{A}_A)$ with the iterates $P_{(A)}^n$ (restricted to $A \times \mathscr{A}_A$). We call the corresponding Markov process the *process restricted to* A.

A state set A is *stochastically closed* (s.cl.) if $A \neq \varnothing$ and $P(x, A) = 1$ for every $x \in A$. If A contains two disjoint s.cl. subsets, then A is *decomposable*. Otherwise A is *indecomposable* (indec.). If A is indec. and contained in no strictly larger indec. set, then A is *maximal* indec.

A state set A is *inessential* if $P_x[\tau_A^{(n)} < \infty] \to 0$ as $n \to \infty$ for every $x \in X$. A countable union of inessential sets is *null*. If a state set A is not null, then it is *positive* (commonly called "absolutely essential").

Doeblin [5] has shown that, if there is a σ-finite measure μ on $\mathscr{A}$ such that μ is positive on every s.cl. set, then $\mathscr{X}$ can be partitioned into a countable family of maximal indec. s.cl. sets and a null set. (For a discussion of related conditions see [4].) Our analysis applies to this situation. To simplify matters, we restrict the process to an indec. subset of $\mathscr{X}$ and hereafter assume $\mathscr{X}$ is indecomposable. The generalization of our results is then immediate. In particular, limit distributions become weighted distributions over the maximal indec. sets.

A positive, indec., s.cl. set A is a *final* set if every null subset of A is *inessential*. *Hereafter we will assume that $\mathscr{X}$ is a final set.* In fact, $\mathscr{X}$ is a final set if, and only if, condition (C) of Harris [7] holds: *there exists a σ-finite measure μ on $\mathscr{A}$ ($\mu \neq 0$) such that $\mu(A) > 0 \Rightarrow P_x[\tau_A^{(n)} < \infty] = 1$ for every n and every $x \in \mathscr{X}$.* (See Theorem 13 of [14] and confer [8].)

When $\mathscr{A}$ is separable (that is, countably generated) and $\mathscr{X}$ is indec., then $\mathscr{X}$ has a final subset differing from $\mathscr{X}$ by a null set. (See Blackwell [1] and Corollary 5 of [4].) Now given any countable class of $\mathscr{A}$ measurable functions, there is always a separable sub-σ-field $\mathscr{A}_0$ of $\mathscr{A}$ such that this class of functions is $\mathscr{A}_0$ measurable and such that the restriction of P to $(\mathscr{X}, \mathscr{A}_0)$ is a Markov kernel (that is, such that $P(\cdot, A)$ is $\mathscr{A}_0$ measurable for $A \in \mathscr{A}_0$) (see Doob [6]). Thus, for most of the results that follow the assumption that $\mathscr{X}$ is a final set is not essential to the results, but the statements of hypotheses and conclusions are considerably complicated without it.

Harris [7] has shown that under his Condition (C) there exists a σ-finite invariant measure π on $\mathscr{A}$, unique up to a multiplicative constant. *The results that follow depend on the additional assumption that π is finite, and hence that $\mathscr{X}$ has a unique invariant starting distribution (probability) π. Moreover, π always denotes this invariant starting distribution.* (In Section 3 of [4] this measure π was denoted φ.)

From Theorems 7 and 5 of [4], we have the following theorem.

THEOREM 2.1. *The set $\mathscr{X}$ is a final set with invariant starting distribution π if, and only if,*

$$\frac{1}{n}\sum_{j=0}^{n-1} P^j(x, A) \to \pi(A) \tag{2.8}$$

as $n \to \infty$ for every $x \in \mathscr{X}$ and $A \in \mathscr{A}$. Moreover, in this case the convergence is uniform in A, that is $\|(1/n)\sum_{j=0}^{n-1} P^j(x, \cdot) - \pi\| \to 0$ as $n \to \infty$ in Φ.

A state set A is *uniform* if A is positive and if

$$\sup_{x \in A, B \in \mathscr{A}} \left| \left(\frac{1}{n}\right) \sum_{j=0}^{n-1} P^j(x, B) - \pi(B) \right| \to 0 \tag{2.9}$$

as $n \to \infty$.

When $\mathscr{A}$ is separable, uniform subsets of $\mathscr{X}$ exist. Moreover, if any uniform subset exists, then $\mathscr{X}$ is a countable union of uniform sets. But it is possible that $\mathscr{X}$ has no uniform subset. (See [4] Section 3.)

For any positive A, let π_A denote the normed restriction of π to $A : \pi_A(B) = \pi(AB)/\pi(A)$ for $B \in \mathscr{A}$. (By [4], Corollary 2, $\pi(A) > 0$ if and only if A is positive.)

For Theorem 14 of [4], we have the following theorem.

THEOREM 2.2. *Let $\mathscr{X}$ be a final set with invariant starting distribution π and let A be positive. Then*

(i) *for every starting distribution φ, as $n \to \infty$,*

$$\frac{\tau_A^{(n)}}{n} \to \frac{1}{\pi(A)} \text{ a.s. } P_\varphi; \tag{2.10}$$

(ii) *for every $x \in A$ and state set B,*

$$\left(\frac{1}{n}\right)\sum_{j=0}^{n-1} P_{(A)}^j(x, B) \to \pi_A(B). \tag{2.11}$$

Thus, A is a final set for the process restricted to A, and π_A is the unique invariant starting distribution for the restricted process. Moreover, if A is uniform, then

$$\sup_{x\in A,\, B\in\mathscr{A}} \left|\left(\frac{1}{n}\right)\sum_{j=0}^{n-1} P_{(A)}^j(x, B) - \pi_A(B)\right| \to 0 \tag{2.12}$$

as $n \to \infty$.

The process restricted to positive A is either aperiodic or has a period c, in which case there is a decomposition of A into c disjoint subsets $A_1, \cdots, A_c$ such that the process moves cyclically through the subsets ($P(x, A_{i+1(\text{mod } c)}) = 1$ for every $x \in A_i$ and $i = 1, \cdots, c$) and $A - \cup A_i$ is null (see Doeblin [5] or Theorem 11 of [4]). Furthermore, the process restricted to any A_i is aperiodic (that is, it has no further cyclic decomposition) and

$$\|P_{(A_i)}^n(x, \cdot) - \pi_{A_i}\| \to 0 \tag{2.13}$$

in Φ as $n \to \infty$ for every $x \in \mathscr{X}$. (See [4] Theorem 12.) We call c the *period* of A (where $c = 1$ if A is aperiodic). Take note that $\mathscr{X}$ can be aperiodic, and yet the process restricted to positive subsets of $\mathscr{X}$ may have period greater than one.

We say that A is *exponential* if A is uniform and aperiodic. In this case

$$\rho_n(A) = \sup_{x\in A} \|P_{(A)}^n(x, \cdot) - \pi_A\| \to 0 \tag{2.14}$$

as $n \to \infty$, and it follows that the convergence of the $\rho_n(A)$ to 0 is exponentially fast (see the exponential convergence case in Loève [9]).

We will have recourse several times to the following result (stated as Proposition 34 and Corollary 7 in [4]).

LEMMA 2.1. *Let f be a real valued monotone nondecreasing function on $\{1, 2, 3, \cdots\}$ and set $\Delta f(n) = f(n + 1) - f(n)$. Then for every positive A,*

$$\pi(A)E_{\pi_A}f(\tau_A) = f(1)\pi(A) + \big(1 - \pi(A)\big)E_{\pi_{\mathscr{X}-A}}\Delta f(\tau_A). \tag{2.15}$$

In particular, $E_{\pi_A}\tau_A = 1/\pi(A)$ *and* $E_{\pi_A}\tau_A^2 = (2E_\pi\tau_A - 1)/\pi(A)$, *and, for any* $r \geqq 0$, $s \geqq 0$,

$$E_\pi\big(\tau_A^r(\log \tau_A)^s\big) < \infty \Leftrightarrow E_{\pi_A}\big(\tau_A^{n+1}(\log \tau_A)^s\big) < \infty. \tag{2.16}$$

A set A is *strongly uniform* if A is uniform and if $\sup_{x\in A} E_x\tau_A < \infty$. It is not clear that strongly uniform sets exist even when uniform sets do, but in [4] it is shown that sufficient conditions for the existence of strongly uniform sets are either that $\mathscr{A}$ has a positive atom (as in the countable state space case) or that there is a uniform set A such that $E_\pi(\log \tau_A) < \infty$ (equivalently, $E_{\pi_A}(\tau_A \log \tau_A) < \infty$). If $\mathscr{X}$ has a strongly uniform set A, then there exist strongly uniform $A_n \uparrow$ such that $\mathscr{X} - \cup A_n$ is null and $\cup A_n$ is s.cl. ([4] Theorem 16).

THEOREM 2.3. (Theorem 17 and Corollary 9 of [4]). *Let* $\mathscr{X}$ *be a final set with invariant starting distribution* π.

(i) *If* $E_\pi\tau_A < \infty$ (*equivalently* $E_{\pi_A}\tau_A^2 < \infty$) *for some strongly uniform* A, *then*

$$\beta(x) = \sup_{n, \|f\| \leqq 1} \left| \sum_{j=0}^{n} \big(P^j f(x) - \pi(f)\big)\right| \tag{2.17}$$

is bounded on strongly uniform sets and finite outside a null set. Moreover, β *is bounded above by a* π *integrable function* $\bar{\beta}$. (*The upper* π *integral of* β *is finite but* β *may not be measurable.*)

(ii) *If* β *is bounded on some positive set, then* $\mathscr{X}$ *contains strongly uniform sets and* $E_\pi\tau_A < \infty$ *for every positive set* A.

In what follows we will study conditions under which sequences of probability laws are asymptotically equivalent, comparing sums of dependent random variables to sums of independent random variables with the same distributions. We will use $\mathscr{L}$, with or without affixes, to denote a probability law. In particular, $\mathscr{L}(Z)$ denotes the law of the random variable Z, which may also be described by the distribution function, $F_Z(x) = P[Z < x]$, $-\infty < x < \infty$, or the characteristic function, $g_Z(u) = E \exp\{iuZ\}$, $-\infty < u < \infty$.

A full discussion of asymptotic equivalence of laws will be found on pp. 371–375 of Loève [9]. Our terminology differs slightly, and we summarize briefly. Let $\{\mathscr{L}_n\}$, $\{\mathscr{L}_n'\}$ be two sequences of probability laws, where F_n, g_n and F_n', g_n' are the distribution function and characteristic function of $\mathscr{L}_n$ and $\mathscr{L}_n'$, respectively. Then we say the laws $\mathscr{L}_n$ are *asymptotically equivalent* to the laws $\mathscr{L}_n'$, written $\mathscr{L}_n \sim \mathscr{L}_n'$, if the two sequences have the same weak limits for the same subsequences of subscripts. (Here $\mathscr{L}_n$ and $\mathscr{L}_n'$ are always probability laws, although their weak limits may have variation less than one. We define $\mathscr{L}_n \sim \mathscr{L}_n'$ only for probability laws. Thus, weak and complete equivalence (as defined in [9]) coincide, since if $\mathscr{L}_n \sim \mathscr{L}_n'$ and $\mathscr{L}_{n'} \to \mathscr{L}$ as $n' \to \infty$, where $\mathscr{L}$ is a probability law, (that is, the convergence is complete), then $\mathscr{L}_{n'}' \to \mathscr{L}$ completely as $n' \to \infty$ also.)

Now we have $\mathcal{L}_n \sim \mathcal{L}_{n'}$ if, and only if, $\int h(x)(dF_n(x) - dF'_n(x)) \to 0$ as $n \to \infty$ for every continuous function h such that $\lim_{x \to \pm\infty} h(x) = 0$. The condition $g_n(u) - g'_n(u) \to 0$ as $n \to \infty$ for every u is sufficient for $\mathcal{L}_n \sim \mathcal{L}'_n$ but not necessary. However, if either sequence $\{\mathcal{L}_n\}$ or $\{\mathcal{L}'_n\}$ is completely compact (that is, bounded in probability) and $\mathcal{L}_n \sim \mathcal{L}'_n$, then both sequences are completely compact and $g_n(u) - g'_n(u) \to 0$ as $n \to \infty$ for every u.

Our comparison of laws in most cases will involve an iterated limit. *All iterated limits are to be understood in the generalized sense.* Thus, $a_{n,\ell} \to a$ as $n \to \infty$ then $\ell \to \infty$ if

$$(2.18) \qquad \limsup_\ell \limsup_n a_{n,\ell} = \liminf_\ell \liminf_n a_{n,\ell} = a.$$

In particular, we say that distribution functions $F_{n,\ell}$ *converge weakly* to the distribution function F, denoted $F_{n,\ell} \xrightarrow{w} F$, as $n \to \infty$ then $\ell \to \infty$, if, for every pair of continuity points x, y of F, $F_{n,\ell}(y) - F_{n,\ell}(x) \to F(y) - F(x)$ as $n \to \infty$ and then $\ell \to \infty$. The convergence is *complete*, denoted $F_{n,\ell} \xrightarrow{c} F$, if, in addition, $\text{Var}(F_{n,\ell}) \to \text{Var}(F)$ as $n \to \infty$ then $\ell \to \infty$. Convergence of laws and of characteristic functions in the iterated limit are defined analogously. The weak compactness of laws, the Helly-Bray theorem, and the Lévy continuity theorem remain true for iterated limits, as the reader may verify.

3. Pointwise strong mixing stationary sequences

Let $(\mathcal{Y}, \mathcal{B})$ be a measurable space, $\Pi_{n=0}^{\infty} (\mathcal{Y}^n, \mathcal{B}^n)$ the corresponding product space, with coordinates $(\mathcal{Y}^n, \mathcal{B}^n) = (\mathcal{Y}, \mathcal{B})$, Y_k be the projection of $\Pi\mathcal{Y}^n$ on the kth coordinate and P be a probability on $\Pi\mathcal{B}^n$. In other words the sequence $\{Y_n\}$ is a stochastic process with values in $(\mathcal{Y}, \mathcal{B})$. We assume the process is stationary:

$$(3.1) \qquad P[Y_0 \in B_0, \cdots, Y_m \in B_m] = P[Y_n \in B_0, \cdots, Y_{m+n} \in B_m]$$

for every m, n; $B_0, \cdots, B_m \in \mathcal{B}$.

Let $\mathcal{F}_n$ be the σ-field generated by $Y_n, Y_{n+1}, \cdots$ (the σ-field of measurable cylinders with base in $\Pi_{k=n}^{\infty} \mathcal{Y}^k$). We assume that a regular conditional probability $P(Y_0, \cdots, Y_n; F)$ exists on $\mathcal{Y}^0 \times \cdots \times \mathcal{Y}^n \times \mathcal{F}_n$ for each n. This is, for each $F \in \mathcal{F}_n$, a measurable function of $(Y_0, \cdots, Y_n)$ which is in the equivalence class of the conditional probability of F given $Y_0, \cdots, Y_n$ and a probability on $\mathcal{F}_n$ for each $Y_0, \cdots, Y_n$. This is always possible if $(\mathcal{Y}, \mathcal{B})$ is a Borel subset of the real line or a finite or denumerable product of real lines (see Doob [6]). In Section 4, where we apply the theory of this section to Markov processes, the regular conditional probabilities arise constructively from the Markov kernel $P(x, A)$.

Let

$$(3.2) \qquad \rho_n = \sup_{m;\, Y_0, \cdots, Y_m;\, F \in \mathcal{F}_{m+n}} |P(Y_0, \cdots, Y_m; F) - PF|.$$

Note that the ρ_n are monotone nonincreasing. The results that follow depend on $\rho_n \to 0$ sufficiently fast as $n \to \infty$.

The next three lemmas parallel Lemmas 7.1 to 7.3 of Chapter V of Doob [6]. The proofs are also similar and we indicate differences only where necessary.

LEMMA 3.1. *Let U, V be real or complex valued, where U and V are measurable functions of $Y_0, \cdots, Y_m$ and $Y_{m+n}, Y_{m+n+1}, \cdots$, respectively. Let $r, s > 1$ with $1/r + 1/s = 1$. Then*

$$|EUV - EU \cdot EV| \leqq 2\rho_n^{1/r} E^{1/r}|U|^r \cdot E^{1/s}|V|^s. \tag{3.3}$$

The right side may be infinite, but then the bound is trivial.

LEMMA 3.2. *Let U be a bounded real or complex valued measurable function of $Y_{m+n}, Y_{m+n+1}, \cdots$. Then*

$$|E(U \mid Y_1, \cdots, Y_m) - EU| \leqq 2\rho_n \sup |U|, \tag{3.4}$$

where the conditional expectation is defined from the regular conditional probabilities introduced above.

LEMMA 3.3. *Let f be a real or complex valued measurable function on $(\mathscr{Y}, \mathscr{B})$ and let $Ef(Y_0) = 0$, and $E|f(Y_0)|^2 < \infty$. Set*

$$\sigma^2 = E|f(Y_0)|^2 + 2\mathscr{R}\left(\sum_{n=1}^{\infty} Ef(Y_0)f^*(Y_n)\right), \tag{3.5}$$

where f^* is the complex conjugate of f and $\mathscr{R}$ denotes the real part of the expression following it.

If $\Sigma\rho_n^{1/2} < \infty$, then the series on the right converges, $0 \leqq \sigma^2 < \infty$, and

$$\lim_{n\to\infty} E\left|n^{-1/2}\sum_{k=0}^{n-1} f(Y_k)\right|^2 = \sigma^2. \tag{3.6}$$

If, moreover, either (i) *f is bounded and $\Sigma\rho_n^{1/2} < \infty$ or,* (ii) *$\Sigma\, n\rho_n^{1/2} < \infty$, then*

$$\lim_{n\to\infty}\left\{E\left|\sum_{k=0}^{n-1} f(Y_k)\right|^2 - n\sigma^2\right\} = -2\mathscr{R}\left(\sum_{k=1}^{\infty} kE\big(f(Y_0)f^*(Y_k)\big)\right) \tag{3.7}$$

where the series on the right converges to a finite limit.

PROOF. The finiteness of σ^2 follows from Lemma 3.1. The remainder of the proof is based on the identity

$$\begin{aligned} E\left|\sum_{k=0}^{n-1} f(Y_k)\right|^2 &- n\sigma^2 \\ &= -2\mathscr{R}\sum_{k=1}^{n-1} kE\big(f(Y_0)f^*(Y_k)\big) - 2n\mathscr{R}\sum_{k=n}^{\infty} E\big(f(Y_0)f^*(Y_k)\big). \end{aligned} \tag{3.8}$$

The first limit assertion follows upon dividing by n and applying Lemma 3.1 to the terms in the series on the right. Likewise, the final assertion follows if $\Sigma\, n\rho_n^{1/2} < \infty$ upon applying Lemma 3.1. If, instead, f is bounded and $\Sigma\, \rho_n^{1/2} < \infty$ then the final assertion follows by applying Lemma 3.2 to the terms in the series and noting that $\Sigma\, n\rho_n < \infty$. To establish this observe that the ρ_n

are nonincreasing so

$$n\rho_n^{1/2} \leqq \sum_{k=1}^{n} \rho_k^{1/2} \leqq \sum_{k=1}^{\infty} \rho_k^{1/2} \tag{3.9}$$

and

$$\sum n\rho_n = \Sigma\, \rho_n^{1/2}(n\rho_n^{1/2}) \leqq (\Sigma\, \rho_n^{1/2})^2. \tag{3.10}$$

Let f be a real valued Borel measurable function on $(\mathscr{Y}, \mathscr{B})$ and let

$$S_\ell = \sum_{j=0}^{\ell-1} f(Y_j), \qquad g_\ell(u) = E \exp\{iuS_\ell\}. \tag{3.11}$$

LEMMA 3.4. *For any positive integers k, ℓ, m, real b, c and real $u \neq 0$,*

$$\begin{aligned} &|g_{\ell m}(u) - (g_\ell(u))^m| \\ &\leqq 2|u|E^{1/2}\left(\min\left\{|S_\ell - c|, \frac{2}{|u|}\right\}\right)^2 \\ &\qquad \left(m|u|E^{1/2}\left(\min\left\{|S_k - b|, \frac{2}{|u|}\right\}\right)^2 + 1\right)\Sigma\, \rho_n^{1/2}. \end{aligned} \tag{3.12}$$

PROOF. Let $1 \leqq \nu < m$ and $j = [\ell\nu/k]$ (if $j = 0$, then the summation in the following expansion vanishes by convention and only the last two terms remain). Elementary computations yield

(3.13)

$$\begin{aligned} &|g_{\ell(\nu+1)}(u) - g_\ell(u)g_{\ell\nu}(u)| \\ &= |E \exp\{iu(S_{\ell(\nu+1)} - c)\} - E \exp\{iu(S_\ell - c)\}\, E \exp\{iu(S_{\ell(\nu+1)} - S_\ell)\}| \\ &= \Big|\sum_{i=0}^{j-1} [E(\exp\{iu(S_\ell - c)\} - 1)(\exp\{iu(S_{\ell+(i+1)k} - S_{\ell+ik} - b)\} - 1) \\ &\qquad\qquad \cdot \exp\{iu(S_{\ell(\nu+1)} - S_{\ell+(i+1)k} - (i+1)b)\} \\ &\quad - E(\exp\{iu(S_\ell - c) - 1\}E(\exp\{iu(S_{\ell+(i+1)k} - S_{\ell+ik} - b)\} - 1) \\ &\qquad\qquad \cdot \exp\{iu(S_{\ell(\nu+1)} - S_{\ell+(i+1)k} - (i+1)b)\}] \\ &\quad + E(\exp\{iu(S_\ell - c)\} - 1)\exp\{iu(S_{\ell(\nu+1)} - S_{\ell+jk} - jb)\} \\ &\quad - E(\exp\{iu(S_\ell - c)\} - 1)E \exp\{iu(S_{\ell(\nu+1)} - S_{\ell+jk} - jb)\}\Big|. \end{aligned}$$

Applying Lemma 3.1 to the above expression, while observing that $|\exp\{iuz\} - 1| \leqq |u| \min\{|z|, 2/|u|\}$, we obtain an upper bound of

$$\begin{aligned} &\sum_{i=0}^{j-1} 2\rho_{ik}^{1/2}E^{1/2}(\exp\{iu(S_\ell - c)\} - 1)^2E^{1/2}(\exp\{iu(S_k - b)\} - 1)^2 \\ &\qquad\qquad + 2\rho_{kj}^{1/2}E^{1/2}(\exp\{iu(S_\ell - c)\} - 1)^2 \\ &\leqq 2|u|E^{1/2}\left(\min\left\{|S_\ell - c|, \frac{2}{|u|}\right\}\right)^2 \\ &\qquad \cdot\left(\sum \rho_n^{1/2}|u|E^{1/2}\left(\min\left\{|S_k - b|, \frac{2}{|u|}\right\}\right)^2 + \rho_{jk}^{1/2}\right). \end{aligned} \tag{3.14}$$

Applying this bound to the absolute value of each term in the expansion

$$g_{\ell m}(u) - (g_\ell(u))^m = \sum_{\nu=1}^{m-1} \big(g_{\ell(\nu+1)}(u) - g_\ell(u) g_{\ell\nu}(u)\big)(g_\ell(u))^{m-\nu-1} \tag{3.15}$$

completes the proof.

We are now in a position to transpose the central limit theorem for independent random variables.

Let $f_1, f_2, \cdots$ be real valued Borel measurable functions on $(\mathscr{Y}, \mathscr{B})$ and let $S_{n,\ell} = \Sigma_{j=0}^{\ell-1} f_n(Y_j)$. Let $F_{n,\ell}$ and $g_{n,\ell}$ be the distribution function and characteristic function of $S_{n,\ell}$, respectively. For any constant $\tau > 0$ (arbitrary but fixed) let

$$a_{n,\ell} = \int_{|x|<\tau} x dF_{n,\ell}(x), \qquad \bar{F}_{n,\ell}(x) = F_{n,\ell}(x + a_{n,\ell}), \tag{3.16}$$

$$\alpha_{n,\ell} = a_{n,\ell} + \int \frac{x}{1+x^2} d\bar{F}_{n,\ell}(x),$$

$$\Psi_{n,\ell}(x) = \int_{-\infty}^{x} \frac{y^2}{1+y^2} d\bar{F}_{n,\ell}(y).$$

Here and in what follows, we use the notation $[z]$ for the largest integer in (that is, less than or equal to) a real number z. Also, $\mathscr{L}^{*m}$ denotes the mth convolution of the law $\mathscr{L}$.

The above notations follow closely those for the general case of the central limit theorem as given in Loève [9].

THEOREM 3.1. *Let* $\Sigma \rho_n^{1/2} < \infty$ *and* $f_n(Y_0) \xrightarrow{P} 0$ *as* $n \to \infty$.

(i) *If* $\limsup_\ell \limsup_n (k_n/\ell) \operatorname{Var}(\Psi_{n,\ell}) < \infty$, *then* $\mathscr{L}(S_{n,k_n}) \sim \mathscr{L}^{*[k_n/\ell]}(S_{n,\ell})$ *as* $n \to \infty$ *then* $\ell \to \infty$. *Thus the possible limit distributions of* $\mathscr{L}(S_{n,k_n})$ *are infinitely divisible and* $\mathscr{L}(S_{n,k_n}) \to \mathscr{L}$ *with characteristic function*

$$g(u) = \exp\left\{iu\alpha + \int \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) \frac{1+x^2}{x^2} d\Psi(x)\right\} \tag{3.17}$$

if, and only if $(k_n/\ell)\alpha_{n,\ell} \to \alpha$, $(k_n/\ell)\Psi_{n,\ell} \xrightarrow{c} \Psi$ *as* $n \to \infty$, *then* $\ell \to \infty$.

(ii) *If the laws* $\mathscr{L}^{*[k_n/\ell]}(S_{n,\ell})$ *are completely compact, then* $(k_n/\ell) \operatorname{Var}(\Psi_{n,\ell})$ *is bounded.*

PROOF. Under the hypothesis in Section 2, we have for all $\ell \geqq \ell_0$ sufficiently large that

$$\limsup_n (k_n/\ell) \int \frac{x^2}{1+x^2} d\bar{F}_{n,\ell}(x) = \limsup_n (k_n/\ell) \operatorname{Var}(\Psi_{n,\ell}) \leqq C < \infty. \tag{3.18}$$

Then, for $\ell \geqq \ell_0$,

$$\limsup_n \left(\frac{k_n}{\ell}\right) E\left(\min\left\{|S_{n,\ell} - a_{n,\ell}|, \frac{2}{|u|}\right\}\right)^2 \leqq C\left(1 + \frac{4}{u^2}\right). \tag{3.19}$$

Now applying Lemma 3.4, where $f = f_n$, $m = [k_n/\ell]$, k is the ℓ_0 introduced above, $b = a_{n,\ell_0}$ and $c = a_{n,\ell}$, we obtain

$$g_{n,\ell m}(u) - (g_{n,\ell}(u))^m \to 0 \tag{3.20}$$

as $n \to \infty$ then $\ell \to \infty$, hence $\mathscr{L}(S_{n,\ell m}) \sim \mathscr{L}^{*m}(S_{n,\ell})$.

Since $S_{n,k_n} - S_{n,\ell m}$ contains less than ℓ terms distributed like $f_n(Y_0)$, we have $S_{n,k_n} - S_{n,\ell m} \xrightarrow{P} 0$ as $n \to \infty$ for each ℓ. Hence, $\mathscr{L}(S_{n,k_n}) \sim \mathscr{L}^{*m}(S_{n,\ell})$ as $n \to \infty$ then $\ell \to \infty$. The remaining assertions are immediate consequences of the central limit theorem for independent random variables. The theorem is proved.

The central limit theorem thus applies in full generality for the iterated limit as $n \to \infty$ then $\ell \to \infty$. Under weaker mixing conditions, such as ordinary strong mixing, the limit must be taken as $n \to \infty$ and then $m \to \infty$ (where S_{n,k_n} is approximated by the sum of m independent random variables distributed like S_ℓ and $\ell = [k_n/m]$) and much weaker results are obtainable (see [3], Part II).

Let $\mathscr{N}(\mu, \sigma^2)$ denote the normal distribution with expectation μ and variance σ^2 and let

$$\begin{aligned} \mu_{n,\ell} &= \frac{k_n}{\ell} a_{n,\ell} = \frac{k_n}{\ell} \int_{|x|<\tau} x\,dF_{n,\ell}(x), \\ \sigma^2_{n,\ell} &= \frac{k_n}{\ell} \left(\int_{|x|<\tau} x^2 dF_{n,\ell}(x) - a^2_{n,\ell} \right). \end{aligned} \tag{3.21}$$

(Recall that τ is an arbitrary but fixed positive constant.)

COROLLARY 3.1. *Let* $\Sigma \rho_n^{1/2} < \infty$, $k_n \to \infty$ *as* $n \to \infty$, $\limsup_\ell \limsup_n |\mu_{n,\ell}| < \infty$, $\limsup_\ell \limsup_n \sigma^2_{n,\ell} < \infty$ *and* $k_n P[|f_n(Y_0)| \geqq \varepsilon] \to 0$ *as* $n \to \infty$ *for every* $\varepsilon > 0$. *Then* $\mathscr{L}(S_{n,k_n}) \sim \mathscr{N}(\mu_{n,\ell}, \sigma^2_{n,\ell})$ *as* $n \to \infty$ *then* $\ell \to \infty$.

PROOF. We have

$$P[|S_{n,\ell}| \geqq \varepsilon] \leqq P \bigcup_{j=0}^{\ell-1} \left[|f_n(Y_j)| \geqq \frac{\varepsilon}{\ell} \right] \leqq \ell P\left[|f_n(Y_0)| \geqq \frac{\varepsilon}{\ell} \right]. \tag{3.22}$$

The hypotheses then imply that $(k_n/\ell) P[|S_{n,\ell}| \geqq \varepsilon] \to 0$ as $n \to \infty$ for every ℓ and every $\varepsilon > 0$. The hypotheses also imply that $a_{n,\ell} \to 0$ as $n \to \infty$ for each ℓ, and

$$\begin{aligned} &\left(\frac{k_n}{\ell}\right) \operatorname{Var}(\Psi_{n,\ell}) \\ &\leqq \left(\frac{k_n}{\ell}\right) \left(\left(\int_{|x|<\tau} (x - a_{n,\ell})^2\, dF_{n,\ell} x \right) + P[|S_{n,\ell} - a_{n,\ell}| \geqq \tau] \right) \\ &\leqq \sigma^2_{n,\ell} + \left(\frac{k_n}{\ell}\right) P[|S_{n,\ell}| \geqq \tau - |a_{n,\ell}|]. \end{aligned} \tag{3.23}$$

It follows that $\limsup_\ell \limsup_n (k_n/\ell) \operatorname{Var}(\Psi_{n,\ell}) < \infty$. The corollary is then an immediate consequence of the theorem and the normal convergence criterion (see Loève [9]).

NOTE. The condition $k_n P[|f_n(Y_0)| \geqq \varepsilon] \to 0$ implies that

$$\max_{j \leqq k_n/\ell} P[|S_{n,(j+1)\ell} - S_{n,j\ell}| \geqq \varepsilon] \to 0 \tag{3.24}$$

as $n \to \infty$ for each $\varepsilon > 0$. But even this condition is not necessary (as in the case of independence) for approximation of $\mathscr{L}(S_{n,k_n})$ by normal distributions. In [3] an example of this is given for a stationary Markov process on the real line, where the exponential convergence case applies. The example is easily modified to a countable state space, as well.

COROLLARY 3.2. *Let* $\sum \rho_n^{1/2} < \infty$, *and* $k_n \to \infty$ *as* $n \to \infty$, *and let* $Ef_n(Y_0) = 0$ *for each* n *and* $\lim_{\ell \to \infty} \lim_{n \to \infty} ES_{n,\ell}^2 = 1$. *Then* $\mathscr{L}(S_{n,k_n}) \to \mathscr{N}(0, 1)$ *as* $n \to \infty$ *if, and only if,* $(k_n/\ell) \int_{|x| \geqq \varepsilon} x^2 dF_{n,\ell}(x) \to 0$ *as* $n \to \infty$ *then* $\ell \to \infty$ *for every* $\varepsilon > 0$.

In particular, if $Ef(Y_0)^2 < \infty$ *for some function* f, *then*

$$\sigma^2 = \lim_{n \to \infty} \left(\frac{1}{n}\right) E \left(\sum_{j=0}^{n-1} f(Y_j) - nEf(Y_0)\right)^2 \tag{3.25}$$

exists, where $0 \leqq \sigma^2 < \infty$. *If* $\sigma^2 > 0$, *then*

$$\mathscr{L}\left[\left(\sum_{j=0}^{n-1} f(Y_j) - nEf(Y_0)\right)\Big/\sqrt{n}\right] \to \mathscr{N}(0, \sigma^2) \tag{3.26}$$

as $n \to \infty$. *If* $\sigma^2 = 0$, *then*

$$\left(\sum_{j=0}^{n-1} f(Y_j) - nEf(Y_0)\right)\Big/\sqrt{n} \xrightarrow{P} 0 \tag{3.27}$$

while if, in addition to the above hypotheses, either f *is bounded or* $\sum n\rho_n^{1/2} < \infty$, *then the sums* $\sum_{j=0}^{n-1} f(Y_j) - nEf(Y_0)$ *are uniformly bounded in probability and quadratic norm.*

These assertions all follow directly from Theorem 3.1, the Lindenberg-Feller normal convergence criterion and Lemma 3.3, since

$$\operatorname{Var}(\Psi_{n,\ell}) \leqq ES_{n,\ell}^2 + a_{n,\ell}^2 \leqq 2ES_{n,\ell}^2. \tag{3.28}$$

EXAMPLE. Let $\mathscr{Y} = \{0, \pm 1, \pm 2, \cdots\}$ and $\{Y_n\}$ be a Markov chain with $p_{0,n} = 1/2^{n+1}$ for $n \geqq 0$, $p_{n,-n} = p_{-n,0} = 1$ for $n \geqq 1$. Then $\{Y_n\}$ is stationary with the starting distribution $\pi\{0\} = 1/2$ and $\pi\{\pm n\} = 1/2^{n+2}$ for $n \geqq 1$. Thus $E_\pi Y_0 = 0$ and $E_\pi Y_0^2$ is positive and finite. Also $\rho_n \to 0$ exponentially fast. In this case $\sigma^2 = 0$ and we have $\sum_{j=0}^{n-1} Y_j$ uniformly bounded in quadratic norm. Moreover, $\mathscr{L}_\pi(\sum_{j=0}^{n-1} Y_j)$ converges as $n \to \infty$ to a noninfinitely divisible limit. Now, if $b_n \to \infty$, then $\sum_{j=0}^{kn-1} Y_j/b_n \xrightarrow{P} 0$, but we will have $(k_n/\ell)\Psi_{n,\ell} \xrightarrow{c} 0$ in the iterated limit if and only if $\liminf (b_n/\sqrt{k_n}) > 0$, and this is the condition needed to bound $(k_n/\ell) \operatorname{Var}(\Psi_{n,\ell})$ as $n \to \infty$ then $\ell \to \infty$. Thus the assumption of boundedness of $(k_n/\ell) \operatorname{Var}(\Psi_{n,\ell})$ or some similar condition is required for Theorem 3.1. This assumption is incorrectly omitted from Theorem 17 of [3].

COROLLARY 3.3. *Let* $\sum \rho_n^{1/2} < \infty$, $k_n \to \infty$ *as* $n \to \infty$ *and* $B_n \in \mathscr{B}$ *satisfy*

$$\limsup_n k_n P[Y_0 \in B_n] < \infty. \tag{3.29}$$

Let $S_{n,\ell} = \Sigma_{j=0}^{\ell-1} \chi_{B_n}(Y_j)$. *Then* $\mathscr{L}(S_{n,k_n}) \sim \mathscr{L}_{n,\ell}$ *as* $n \to \infty$ *then* $\ell \to \infty$, *where* $\mathscr{L}_{n,\ell}$ *is the law with characteristic function*

$$h_{n,\ell}(u) = \exp\left\{\sum_{\nu=1}^{\infty} (e^{iu\nu} - 1)q_{\nu;n,\ell}\right\} \tag{3.30}$$

and

$$q_{\nu;n,\ell} = \left(\frac{k_n}{\ell}\right) P[S_{n,\ell} = \nu]. \tag{3.31}$$

Moreover, the $\mathscr{L}(S_{n,k_n})$ *are completely compact.*

PROOF. Choosing $\tau < 1$, we have that

$$\operatorname{Var}(\Psi_{n,\ell}) \leqq P[S_{n,\ell} > 0] \leqq \ell P[Y_0 \in B_n]. \tag{3.32}$$

Thus $(k_n/\ell)\operatorname{Var}(\Psi_{n,\ell})$ is bounded and Theorem 3.1 applies. The hypotheses also imply that $S_{n,\ell} \xrightarrow{P} 0$ as $n \to \infty$ for each ℓ, hence

$$\log g_{n,\ell}(u) = \sum_{\nu=1}^{\infty} (e^{iu\nu} - 1)P[S_{n,\ell} = \nu](1 + o(1)) \tag{3.33}$$

as $n \to \infty$ for each ℓ (recall $g_{n,\ell}$ is the characteristic function of $S_{n,\ell}$). Thus $g_{n,\ell}^{[kn/\ell]} - h_{n,\ell} \to 0$ as $n \to \infty$ for each ℓ. The complete compactness of $\mathscr{L}(S_{n,k_n})$ follows since $ES_{n,k_n} = k_n P[Y_0 \in B_n]$ is bounded.

For any random variable S, let med (S) denote any number in the median interval of S. Let $S_n = \Sigma_{j=0}^{n-1} f(Y_j)$. The following lemma is based on inequalities of P. Lévy.

LEMMA 3.5. *For any integers* ℓ, $m > 0$ *and any real* c,

$$\begin{aligned}(1/2 - \rho_\ell)&\left[P \max_{k \leqq m} \{S_k + \operatorname{med}(S_{m-k})\} \geqq c\right] \\ &\leqq P\left[S_{m+\ell} \geqq \frac{c}{2}\right] + mP\left[S_\ell < \frac{-c}{2}\right]\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}(1/2 - \rho_\ell)&P[\max_{k \leqq m} |S_k + \operatorname{med}(S_{m-k})| \geqq c] \\ &\leqq P\left[|S_{m+\ell}| \geqq \frac{c}{2}\right] + mP\left[|S_\ell| > \frac{c}{2}\right].\end{aligned} \tag{3.35}$$

PROOF. Let $S'_k = S_k + \operatorname{med}(S_{m-k})$ and

$$\begin{aligned} A_1 &= [S'_1 \geqq c] \\ A_k &= [\max_{j<k} S'_j < c, S'_k \geqq c], \qquad k = 2, 3, \cdots \\ B_k &= [S_{m+\ell} - S_{k+\ell} - \operatorname{med}(S_{m-k}) \geqq 0] \\ C_k &= \left[S_{k+\ell} - S_k < \frac{-c}{2}\right]. \end{aligned} \tag{3.36}$$

Then $[\max_{k \leqq m} S'_k \geqq c] = \cup_{k=1}^m A_k$ and

$$\bigcup_{k=1}^{m} A_k B_k \subset \left[S_{m+\ell} \geqq \frac{c}{2} \right] \cup \left(\bigcup_{k=1}^{m} C_k \right). \tag{3.37}$$

By stationarity, med (S_{m-k}) is a median of $S_{m+\ell} - S_{k+\ell}$ and, for $k \leqq m$,

$$P(B_k | Y_0, \cdots, Y_{k-1}) \geqq \tfrac{1}{2} - \rho_\ell. \tag{3.38}$$

Since A_k is determined by $Y_0, \cdots, Y_{k-1}$ and since $PC_k = P[S_\ell < -a/2]$ for each k, we have

$$\begin{aligned}(1/2 - \rho_\ell) P[\max_{k \leqq m} S'_k \geqq a] &\leqq \sum_{k=1}^{m} PA_k B_k \\ &\leqq P\left[S_{m+\ell} \geqq \frac{c}{2} \right] + mP\left[S_\ell < \frac{-a}{2} \right].\end{aligned} \tag{3.39}$$

This establishes the first inequality. Applying this inequality with f replaced by $-f$, so the signs of all the variables are changed, and adding the two resulting inequalities yields the second inequality, which proves the lemma.

THEOREM 3.2. *Let* $\Sigma \rho_n^{1/2} < \infty$, $f_n(Y_0) \xrightarrow{P} 0$ *as* $n \to \infty$ *and*

$$\limsup_{\ell} \limsup_{n} \left(\frac{k_n}{\ell} \right) \mathrm{Var}\,(\Psi_{n,\ell}) < \infty. \tag{3.40}$$

Let v_n *be random times (positive integer valued* $\Pi \mathscr{B}^n$ *measurable functions on* $\Pi \mathscr{Y}^n$*) such that* $v_n/k_n \xrightarrow{P} 1$ *as* $n \to \infty$. *Then* $\mathscr{L}(S_{n, v_n}) \sim \mathscr{L}(S_{n, k_n})$ *as* $n \to \infty$.

PROOF. If $j_n = o(k_n)$, then $(j_n/\ell)\,\mathrm{Var}\,(\Psi_{n,\ell}) \to 0$ as $n \to \infty$ for ℓ sufficiently large, and it follows from Theorem 3.1 that, uniformly in $0 < j \leqq j_n$,

$$S_{n,j} - \frac{j\alpha_{n,\ell}}{\ell} \xrightarrow{P} 0 \tag{3.41}$$

as $n \to \infty$ then $\ell \to \infty$. Also

$$j_n P[|S_{n,\ell} - \alpha_{n,\ell}| \geqq \delta] \leqq j_n \frac{1 + \delta^2}{\delta^2} \mathrm{Var}\,(\Psi_{n,\ell}) \to 0 \tag{3.42}$$

as $n \to \infty$ then $\ell \to \infty$ for every $\delta > 0$. Applying Lemma 3.5 with $f = f_n - \alpha_{n,\ell}/\ell$, we have

$$\begin{aligned}(1/2 - \rho_\ell) P\left[\max_{j \leqq j_n} \left| S_{n,j} - \frac{j\alpha_{n,\ell}}{\ell} + \mathrm{med}\,(S_{n,j_n - j}) \right| \geqq \delta \right] \\ \leqq P\left[\left| S_{n,j_n+\ell} - \frac{(j_n + \ell)\alpha_{n,\ell}}{\ell} \right| \geqq \frac{\delta}{2} \right] + j_n P\left[\left| S_{n,\ell} - \alpha_{n,\ell} \right| > \frac{\delta}{2} \right] \to 0\end{aligned} \tag{3.43}$$

as $n \to \infty$ then $\ell \to \infty$. But also, by (3.4),

$$\max_{j \leqq j_n} \left| \mathrm{med}\,(S_{n,j_n - j}) - \frac{(j_n - j)\alpha_{n,\ell}}{\ell} \right| \to 0 \tag{3.44}$$

thus

$$\max_{j \leq j_n} \left| S_{n,j} - \frac{(j_n - 2j)\alpha_{n,\ell}}{\ell} \right| \xrightarrow{P} 0 \tag{3.45}$$

in the iterated limit.

Now, since $v_n/k_n \xrightarrow{P} 1$, there exist $j_n = o(k_n)$ such that $P[|v_n - k_n| > j_n] \to 0$ as $n \to \infty$. Letting $C_\ell(j) = (j_n - 2_j)\alpha_{n,\ell}/\ell$ for $j > 0$, $C_\ell(0) = 0$ and $C_\ell(j) = -C_\ell(-j)$ for $j < 0$, it follows from stationarity that

$$S_{n,v_n} - S_{n,k_n} - C_\ell(v_n - k_n) \xrightarrow{P} 0 \tag{3.46}$$

as $n \to \infty$ then $\ell \to \infty$. Thus

$$\mathscr{L}\left(S_{n,v_n} - C_\ell(v_n - k_n) - \frac{k_n\alpha_{n,\ell}}{\ell}\right) \sim \mathscr{L}\left(S_{n,k_n} - \frac{k_n\alpha_{n,\ell}}{\ell}\right) \tag{3.47}$$

in the iterated limit. But the $S_{n,k_n} - k_n\alpha_{n,\ell}/\ell$ are completely compact (that is, all weak limits as $n \to \infty$ then $\ell \to \infty$ are complete limits). It follows that, if $\mathscr{L}(S_{n,k_n})$ are completely compact, then the $k_n\alpha_{n,\ell}/\ell$ are bounded in n for ℓ sufficiently large and $C_\ell(v_n - k_n) \xrightarrow{P} 0$ since in this case $\max_{|j| \leq j_n} |C_\ell(j_n)| \to 0$ as $n \to \infty$. On the other hand, if for some subsequence, n', $\mathscr{L}(S_{n',k_{n'}})$ have a weak limit that is not complete, then $|k_{n'}\alpha_{n',\ell}/\ell| \to \infty$ as $n' \to \infty$ then $\ell \to \infty$. But then $|C_\ell(v_{n'} - k_{n'}) + k_{n'}\alpha_{n',\ell}/\ell| \xrightarrow{P} \infty$ in the iterated limit, and both $\mathscr{L}(S_{n',k_{n'}})$ and $\mathscr{L}(S_{n',v_{n'}})$ lose all mass at $\pm\infty$ as $n' \to \infty$. Thus $\mathscr{L}(S_{n,v_n}) \sim \mathscr{L}(S_{n,k_n})$ and the theorem is proved.

4. Markov processes with convergent transition probabilities

We assume that $\mathscr{X}$ is a final set with invariant starting distribution π. For any state set A consider the interblocks between entrance times of A,

$$Y_{A;n} = (X_{\tau_A(n)}, X_{\tau_A(n)+1}, \cdots, X_{\tau_A(n+1)-1}). \tag{4.1}$$

By Theorem 2.2, the $Y_{A;n}$ constitute a stationary process with respect to P_{π_A}. When A is exponential we also have pointwise strong mixing. Using this device we apply the results of Section 3 to Markov processes.

Parallel to the notation in Section 4, we let

$$S_{A;n,\ell} = \sum_{j=0}^{\tau_A^{(\ell)}-1} f_n(X_j), \tag{4.2}$$

and $F_{A;n,\ell}$ and $g_{A;n,\ell}$ be the distribution function and characteristic function of $S_{A;n,\ell}$, respectively, when π_A is the starting distribution. The quantities $a_{A;n,\ell}$, $\bar{F}_{A;n,\ell}$, $\alpha_{A;n\ell}$ and $\Psi_{A;n,\ell}$ are then defined in terms of $F_{A;n,\ell}$ exactly as the corresponding quantities are defined in terms of $F_{n,\ell}$ in Section 3. In particular, $S_{\mathscr{X};n,\ell} = \Sigma_{j=0}^{\ell-1} f_n(X_j)$ and we denote this quantity simply $S_{n,\ell}$ as in Section 3.

The notation $\mathscr{L}_\varphi(Z)$ means the law of the random variable Z (where Z is a function of the Markov process) when φ is the starting distribution.

LEMMA 4.1. *Let $P^j f_n \to 0$ in φ measure and in φ' measure for two starting distributions, φ and φ'. Then $\mathscr{L}_\varphi(S_{n,k_n}) \sim \mathscr{L}_{\varphi'}(S_{n,k_n})$.*

PROOF. Under the hypothesis, for any m fixed,

$$\mathscr{L}_\varphi(S_{n,k_n}) \sim \mathscr{L}_\varphi(S_{n,k_n} - S_{n,m}) = \mathscr{L}_{\varphi P^m}(S_{n,k_n-m}), \tag{4.3}$$

hence, letting $\psi_m = \varphi(1/m)\sum_{k=0}^{m-1} P^k$,

$$\mathscr{L}_\varphi(S_{n,k_n}) \sim \mathscr{L}_{\psi_m}(S_{n,k_n-m}) \tag{4.4}$$

and the same is true for φ'. Since, as $m \to \infty$,

$$\varphi \frac{1}{m}\sum_{k=0}^{m-1} P^k - \varphi' \frac{1}{m}\sum_{k=0}^{m-1} P^k \to 0 \tag{4.5}$$

in Φ, the lemma follows by letting $n \to \infty$ then $m \to \infty$.

THEOREM 4.1. *Let A be uniform with period c, and $f_n \to 0$ in π measure as $n \to \infty$ and*

$$\limsup_{\ell} \limsup_{n} \left(\frac{k_n}{\ell}\right) \mathrm{Var}\,(\Psi_{A;n,\ell}) < \infty. \tag{4.6}$$

(i) *Then $S_{A;n,\ell} \to 0$ in P_{π_A} probability as $n \to \infty$ for each ℓ and*

$$\mathscr{L}_{\pi_A}(S_{n,k_n}) \sim \mathscr{L}_{\pi_A}^{*[k_n\pi(A)/c\ell]}(S_{A;n,c\ell}) \tag{4.7}$$

as $n \to \infty$ then $\ell \to \infty$. Thus the possible limit distributions of $\mathscr{L}_{\pi_A}(S_{n,k_n})$ are infinitely divisible and $\mathscr{L}_{\pi_A}(S_{n,k_n}) \to \mathscr{L}$ with characteristic function

$$g(u) = \exp\left\{iu\alpha + \int\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2}\,d\Psi(x)\right\} \tag{4.8}$$

if, and only if,

$$\left(\frac{k_n}{\ell}\right)\alpha_{A;n,\ell} \to \alpha,\quad \left(\frac{k_n}{\ell}\right)\Psi_{A;n,\ell} \xrightarrow{c} \Psi \tag{4.9}$$

as $n \to \infty$ then $\ell \to \infty$.

Furthermore, for every starting distribution φ such that $P^j f_n \to 0$ in φ measure as $n \to \infty$ for each fixed j, $\mathscr{L}_\varphi(S_{n,k_n}) \sim \mathscr{L}_{\pi_A}(S_{n,k_n})$.

(ii) *If, moreover, either* $\limsup_\ell \limsup_n (k_n/\ell)|\alpha_{A;n,\ell}| < \infty$ *or* $\limsup_\ell \limsup_n (k_n/\ell)|a_{A;n,\ell}| < \infty$, *then*

$$\mathscr{L}_{\pi_A}(S_{n,k_n}) \sim \mathscr{L}_{\pi_A}^{*[k_n\pi(A)/\ell]}(S_{A;n,\ell}) \tag{4.10}$$

as $n \to \infty$ then $\ell \to \infty$.

(iii) *If the laws $\mathscr{L}^{*[k_n\pi(A)/\ell]}(S_{A;n,\ell})$ are completely compact, then (k_n/ℓ)* Var $\Psi_{A;n,\ell})$ *and $(k_n/\ell)|\alpha_{A;n,\ell}|$ are bounded.*

PROOF. (i). Since $\pi_A P^j \leqq (\pi(A))^{-1}\pi$ for each j, we have $f_n(X_j) \to 0$ in P_{π_A} and in P_π probability for each j as $n \to \infty$, and it follows easily that $S_{A;n,\ell} \to 0$ in P_{π_A} and in P_π probability as $n \to \infty$ for each ℓ.

Let $\{A_1, \cdots, A_c\}$ be the cyclic decomposition of A into exponential subsets. Then, for each A_i, $\pi_{A_i} \leqq c\pi_A$ and $S_{A_i;n,\ell} = S_{A;n,c\ell}$ a.s. $P_{\pi_{A_i}}$. It follows that

$$\frac{k_n}{\ell} \operatorname{Var}(\Psi_{A_i;n,\ell}) \leqq c^2 \frac{k_n}{\ell} \operatorname{Var}(\Psi_{A;n,c\ell}). \tag{4.11}$$

Thus the hypothesis of the theorem on Var $(\Psi_{A;n,\ell})$ implies the corresponding hypothesis with A replaced by A_i. By Theorem 3.1,

$$\mathscr{L}_{\pi_{A_i}}(S_{A_i;n,k_n\pi(A_i)}) \sim \mathscr{L}_{\pi_{A_i}}^{*[k_n\pi(A_i)/\ell]}(S_{A_i;n,\ell}). \tag{4.12}$$

Now let $\nu_n = \min\{k: \tau_{A_i}^{(k)} \geqq k_n\}$. Then $\nu_n/k_n\pi(A_i) \to 1$ a.s. $P_{\pi_{A_i}}$ as $n \to \infty$ by Theorem 2.2, and, by Theorem 3.2,

$$\mathscr{L}_{\pi_{A_i}}(S_{A_i;n,\nu_n}) \sim \mathscr{L}_{\pi_{A_i}}(S_{A_i;n,k_n\pi(A_i)}) \tag{4.13}$$

as $n \to \infty$. Let

$$\Delta_n = S_{A_i;n,\nu_n} - S_{n,k_n} = \sum_{j=k_n}^{\tau_{A_i}^{(\nu_n)}-1} f_n(X_j). \tag{4.14}$$

Since $\pi_{A_i}P^{k+nd} \to \pi$ in Φ as $n \to \infty$ for each k, where d is the cyclic period of $\mathscr{X}$, the measures $\pi_{A_i}P^n$ are uniformly absolutely continuous with respect to π by the Vitali-Hahn-Saks theorem, and

$$P_{\pi_{A_i}}[|\Delta_n| \geqq \varepsilon] = P_{\pi_{A_i}P^{k_n}}[|S_{A_i;n,1}| \geqq \varepsilon] \to 0 \tag{4.15}$$

as $n \to \infty$ for every $\varepsilon > 0$ since $|S_{A_i;n,1}| \leqq \max_{\ell \leqq c}|S_{A;n,\ell}| \to 0$ in P_π probability. It follows that

$$\mathscr{L}_{\pi_{A_i}}(S_{n,k_n}) \sim \mathscr{L}_{\pi_{A_i}}^{*[k_n\pi(A_i)/\ell]}(S_{A_i;n,\ell}) = \mathscr{L}_{\pi_{A_i}}^{*[k_n\pi(A)/c\ell]}(S_{A;n,c\ell}). \tag{4.16}$$

Starting the process with distribution π_A is equivalent to selecting one of the A_i at random then starting the process with distribution π_{A_i}. But $S_{A;n,c\ell}$ has the same distribution for each of the P_{A_i}, $i = 1, \cdots, c$, hence

$$\mathscr{L}_{\pi_A}(S_{A;n,c\ell}) = \mathscr{L}_{\pi_A}(S_{A;n,c\ell}). \tag{4.17}$$

Applying Lemma 4.1, we now have, as $n \to \infty$ then $\ell \to \infty$,

$$\mathscr{L}_{\varphi}(S_{n,k_n}) \sim \mathscr{L}_{\pi_A}(S_{n,k_n}) \sim \mathscr{L}_{\pi_A}^{*[k_n\pi(A)/c\ell]}(S_{A;n,c\ell}) \tag{4.18}$$

for any starting distribution φ such that $P^jf_n \to 0$ in φ measure, since $P^jf_n \to 0$ in both π_A and π_{A_i} measure. The assertions in part (i) follow from this and the central limit theorem for independent random variables.

(ii) Fix an integer i and let $\ell' = c[\ell/c] - ic$ for $\ell \geqq (i+1)c$. Then $ic \leqq \ell - \ell' < (i+1)c$ and $\ell'/\ell \to 1$ as $\ell \to \infty$ so

$$\mathscr{L}_{\pi_A}^{*[k_n\pi(A)/\ell]}(S_{A;n,\ell'}) \sim \mathscr{L}_{\pi_A}^{*[k_n\pi(A)/\ell']}(S_{A;n,\ell'}) \sim \mathscr{L}_{\pi_A}(S_{n,k_n}) \tag{4.19}$$

as $n \to \infty$ then $\ell \to \infty$ by the central limit theorem and part (i). We must establish that

$$\mathscr{L}_{\pi_A}^{*[k_n\pi(A)/\ell]}(S_{A;n,\ell}) \sim \mathscr{L}_{\pi_A}^{*[k_n\pi(A)/\ell]}(S_{A;n,\ell'}) \tag{4.20}$$

as $n \to \infty$ then $\ell \to \infty$. In fact we will show that

$$\mathscr{L}_{\pi_A}^{*[k_n\pi(A)/\ell]}(S_{A;n,\ell-\ell'}) \to \mathscr{L}_0, \tag{4.21}$$

the law degenerate at 0, which implies the above.

If $\limsup_\ell \limsup_n (k_n/\ell)|\alpha_{n,\ell}| < \infty$, then this latter convergence is a consequence of the central limit theorem, since, for i sufficiently large,

$$\begin{aligned} &\frac{k_n\pi(A)}{\ell} \operatorname{Var}(\Psi_{A;n,\ell-\ell'}) \to 0 \\ &\frac{k_n\pi(A)}{\ell} \alpha_{A;n,\ell} \to 0 \end{aligned} \tag{4.22}$$

as $n \to \infty$ then $\ell \to \infty$.

The hypothesis on $a_{A;n,\ell}$ implies the corresponding hypothesis on $\alpha_{A;n,\ell}$. In fact, elementary computations show that

$$|\alpha_{A;n,\ell} - a_{A;n,\ell}| \leqq \int_{|x|<\tau} x d\bar{F}_{A;n,\ell}(x) + (\tau + \tau^{-1}) \operatorname{Var}(\Psi_{A;n,\ell}) \tag{4.23}$$

and, if $|a_{A;n,\ell}| < \tau/2$, then

$$\int_{|x|<\tau} x d\bar{F}_{A;n,\ell}(x) \leqq (2\tau + 6\tau^{-1}) \operatorname{Var}(\Psi_{A;n,\ell}). \tag{4.24}$$

The hypotheses imply that $a_{A;n,\ell} \to 0$ as $n \to \infty$ for each ℓ, and it follows that

$$\limsup_n \left(\frac{k_n}{\ell}\right) |\alpha_{A;n,\ell}| \leqq \limsup_n \left(\frac{k_n}{\ell}\right) (|a_{A;n,\ell}| + \gamma \operatorname{Var}(\Psi_{A;n,\ell})) \tag{4.25}$$

where γ is a constant depending only on τ. The assertions in part (ii) follow.

(iii) The assertions in part (iii) follow from the central limit theorem for independent random variables, and the theorem is proved.

Let

$$\begin{aligned} Q_A^n(x, B) &= P_x[\tau_A \geqq n, X_n \in B] \\ Q_A^*(x, B) &= \sum_{n=1}^{\infty} Q_A^n(x, B). \end{aligned} \tag{4.26}$$

The $Q_A^n(x, B)$ is the n-step transition probability from x to B with a taboo on A.

LEMMA 4.2. *Let A be positive. Then $\pi = \pi(A)\pi_A Q_A^*$, hence, for any π integrable f,*

$$E_{\pi_A}\left(\sum_{j=0}^{\tau_A - 1} f(X_j)\right) = \frac{\pi f}{\pi A}. \tag{4.27}$$

PROOF. Consider the identity

$$(4.28)\qquad \sum_{j=1}^{n} Q_A^j(x,B) = P(x,B) + \int_{\mathscr{X}-A} P(x,dy)\left(\sum_{j=1}^{n-1} Q_A^j(y,B)\right).$$

Integrating with respect to π, we have

$$(4.29)\qquad \pi\left(\sum_{j=1}^{n} Q_A^j B\right) = \pi(B) - \big(1 + \pi(A)\big)\pi_{\mathscr{X}-A}\left(\sum_{j=1}^{n-1} Q_A^j B\right)$$

or

$$(4.30)\qquad \pi(B) = \pi(A)\pi_A\left(\sum_{j=1}^{n} Q_A^j B\right) + \big(1 - \pi(A)\big)\pi_{\mathscr{X}-A} Q_A^n B.$$

Since $Q_A^n(x,B) \leqq P_x[\tau_A \geqq n] \to 0$ as $n \to \infty$ for every x, the last term converges to 0 by the dominated convergence theorem. Thus $\pi = \pi(A)\pi_A Q_A^*$.

Now, since $\pi_A P^{\tau_A} = \pi_A$ by Theorem 2.2, the monotone convergence theorem implies for $f \geqq 0$,

$$(4.31)\qquad \begin{aligned} E_{\pi_A}\left(\sum_{j=0}^{\tau_A-1} f(X_i)\right) &= E_{\pi_A}\left(\sum_{j=1}^{\tau_A} f(X_j)\right) \\ &= \int \pi_A(dx)\left(E_x \sum_{j=1}^{\infty} \chi_{[\tau_A \geqq j]} f(X_j)\right) \\ &= \int \pi_A(dx) Q_A^* f(x) = \frac{\pi f}{\pi A}. \end{aligned}$$

The last assertion then follows from the linearity of E_{π_A} and π, and the lemma is proved.

Proceeding as in Section 3, we let

$$(4.32)\qquad \begin{aligned} \mu_{A;n,\ell} &= \frac{k_n}{\ell} a_{A;n,\ell} = \frac{k_n}{\ell}\int_{|x|<\tau} x\,dF_{A;n,\ell}(x) \\ \sigma^2_{A;n,\ell} &= \frac{k_n}{\ell}\left(\int_{|x|<\tau} x^2 dF_{A;n,\ell}(x) - a^2_{A;n,\ell}\right). \end{aligned}$$

COROLLARY 4.1. *Let A be uniform with period c,*

$$(4.33)\qquad \limsup_{\ell}\limsup_{n} |\mu_{A;n,\ell}| < \infty, \qquad \limsup_{\ell}\limsup_{n} \sigma^2_{A;n,\ell} < \infty$$

and $k_n P_{\pi_A}[|S_{A;n,1}| > \varepsilon] \to 0$ as $n \to \infty$ for each $\varepsilon > 0$. Then, for every starting distribution φ such that $P^j f_n \to 0$ in φ measure for each fixed j, $\mathscr{L}_\varphi(S_{n,k_n}) \sim \mathscr{N}(\mu_{A;n,\ell}, \sigma^2_{A;n,\ell})$ as $n \to \infty$ then $\ell \to \infty$.

As in the proof of Corollary 3.1, it follows from the hypotheses that $(k_n/\ell)\operatorname{Var}(\Psi_{A;n,\ell})$ is bounded for ℓ sufficiently large.

The conclusion then follows from parts (i) and (ii) of Theorem 4.1.

COROLLARY 4.2. (i) *Let A be uniform,* $k_n \to \infty$ *as* $n \to \infty$, $\pi f_n = 0$ *for each* n, *and*

$$\lim_{\ell \to \infty} \lim_{n \to \infty} \frac{k_n \pi(A)}{\ell} E_{\pi_A} S^2_{A;n,\ell} = 1. \tag{4.34}$$

Then $\mathscr{L}_{\pi_A}(S_{n,k_n}) \to \mathscr{N}(0, 1)$ *as* $n \to \infty$ *if, and only if,*

$$\left(\frac{k_n}{\ell}\right) \int_{|x| \geqq \varepsilon} x^2 dF_{A;n,\ell}(x) \to 0 \tag{4.35}$$

as $n \to \infty$ *then* $\ell \to \infty$ *for every* $\varepsilon > 0$.

(ii) *In particular, if A is uniform and f is* π *integrable and satisfies* $E_{\pi_A}(\sum_{j=0}^{\tau_A - 1} (f(X_j) - \pi f))^2 < \infty$, *then*

$$\sigma_N^2(f) = \lim_{\ell \to \infty} \frac{\pi(A)}{\ell} E_{\pi_A}\left[\sum_{j=0}^{\tau_A^{(\ell)}-1} (f(X_j) - \pi f)\right]^2 \tag{4.36}$$

exists, where $0 \leqq \sigma_N^2(f) < \infty$, *and, for every starting distribution* φ,

$$\mathscr{L}_\varphi\left[\frac{1}{\sqrt{n}}\left(\sum_{j=0}^{n-1} f(X_j) - n\pi f\right)\right] \to \mathscr{N}(0, \sigma_N^2(f)), \tag{4.37}$$

as $n \to \infty$ (*where* $\eta(0, 0)$ *denotes the law degenerate at* 0).

PROOF. As in the proof of Corollary 3.2, the hypotheses imply that $(k_n/\ell) \operatorname{Var}(\Psi_{A;n,\ell})$ is bounded for ℓ sufficiently large. Since $\pi f_n = 0$,

$$\begin{aligned} \left(\frac{k_n}{\ell}\right) |a_{n,\ell}| &= \left(\frac{k_n}{\ell}\right) \left| \int_{|x| > \tau} x dF_{A;n,\ell} \right| \\ &\leqq \left(\frac{k_n}{\ell}\right) E_{\pi_A}\left(\frac{S^2_{A;n,\ell}}{\tau}\right). \end{aligned} \tag{4.38}$$

The assertions in part (i) then follow from Theorem 4.1 and the classical Lindeberg-Feller theorem.

Now if A has period c, let $A_1, \cdots, A_c$ be its exponential subsets.

The hypotheses in part (ii) of the corollary imply

$$E_{\pi_{A_i}}\left[\sum_{j=0}^{\tau_{A_i}-1} (f(X_j) - \pi f)\right]^2 \leqq c E_{\pi_A}\left[\sum_{j=0}^{\tau_A^{(c)}-1} (f(X_j) - \pi f)\right]^2 < \infty. \tag{4.39}$$

Then Lemma 3.3 implies that

$$\sigma^2 = \lim_{\ell \to \infty} \frac{\pi(A_i)}{\ell} E_{\pi_{A_i}}\left[\sum_{j=0}^{\tau_{A_i}^{(\ell)}-1} (f(X_j) - \pi f)\right]^2 \tag{4.40}$$

exists, where $0 \leqq \sigma^2 < \infty$. But

$$(4.41)\qquad \frac{\pi(A)}{\ell c} E_{\pi_A}\left[\sum_{j=0}^{\tau_A^{(\ell c)}-1} (f(X_j) - \pi f)\right]^2 = \frac{\pi(A_i)}{\ell} E_{\pi_{A_i}}\left[\sum_{j=0}^{\tau_A^{(\ell)}-1} (f(X_j) - \pi f)\right]^2$$

and, letting $\ell' = c[\ell/c]$,

$$(4.42)\qquad E_{\pi_A}\left[\sum_{j=0}^{\tau_A^{(\ell)}-1} (f(X_j) - \pi f)\right]^2 - E_{\pi_A}\left[\sum_{j=0}^{\tau_A^{(\ell')}-1} (f(X_j) - \pi f)\right]^2 \leqq (c-1)^2 E_{\pi_A}\left[\sum_{j=0}^{\tau_A-1} (f(X_j) - \pi f)\right]^2$$

It follows that $\sigma^2 = \sigma_N^2(f)$ (as defined in the corollary).

The convergence to a normal or degenerate limit follows directly from Theorem 4.1 and the classical cases, or (when $\sigma_N^2(f) > 0$) by applying part (i) of the corollary with $k_n = n$ and $f_n = (n\sigma_N^2(f))^{-1/2}(f - \pi f)$.

The next result provides several conditions under which the hypothesis of part (ii) of Corollary 4.2 may be satisfied.

THEOREM 4.2. *Let $\pi f^2 < \infty$ and A be positive. Then each of the following conditions is sufficient to insure that $E_{\pi_A}(\Sigma_{j=0}^{\tau_A-1} (f(X_j) - \pi f))^2 < \infty$:*

(i) *f vanishes outside A and $\pi f = 0$,*

(ii) *$E_\pi \tau_A < \infty$ (equivalently, $E_{\pi_A}\tau_A^2 < \infty$) and $\sup_{x\notin A}|f(x)| < \infty$.*

(iii) *there exist $r, s > 1$ with $1/r + 1/s = 1$ and there exists a function g on $\{1, 2, 3, \cdots\}$ to $(0, \infty)$ such that* (a) $\int_{\mathscr{X}-A} |f(x)|^{2r}\pi(dx) < \infty$ *and,* (b) $\Sigma_1^\infty 1/g(n) < \infty$ *and,* (c) $E_\pi g(\tau_A)^{2s-1} < \infty$.

In particular, (b) *and* (c) *are satisfied (by some g) if,* (d) $E_{\pi_A}(\tau_A \log \tau_A)^{2s} < \infty$.

PROOF. Let $S = \Sigma_{j=0}^{\tau_A-1} (f(X_j) - \pi f)$. Under condition (i), $S = f(X_0)$ and $E_{\pi_A}S^2 = \pi f^2/\pi A < \infty$.

Under condition (ii), τ_A as well as $f(X_0) - \pi f$ have finite second moments with respect to E_{π_A}, and

$$(4.43)\qquad |S| \leqq |f(X_0) - \pi f| + (\tau_A - 1)\big(\sup_{x\notin A}|f(x)| + |\pi f|\big).$$

The assertion follows in this case upon applying Minkowski's inequality to the right side.

Now let (iii) hold and let $h = \chi_{\mathscr{X}-A}(f - \pi f)$. Then

$$(4.44)\qquad S = f(X_0) - \pi f + \sum_{n=1}^{\infty} \chi_{[\tau_A > n]}h(X_n).$$

Since $f(X_0) - \pi f$ has finite second moment with respect to E_{π_A}, it remains to show that

$$(4.45)\qquad \delta = E_{\pi_A}^{1/2}\left(\sum_{n=1}^{\infty} \chi_{[\tau_A > n]}h(X_n)\right)^2 < \infty.$$

Applying Minkowski's and then Hölder's inequalities,

$$\begin{aligned}\delta &\leqq \sum_{n=1}^{\infty} \{E_{\pi_A}(\chi_{[\tau_A > n]} h^2(X_n))\}^{1/2} \\ &\leqq \sum_{n=1}^{\infty} P_{\pi_A}[\tau_A > n]^{1/2s}(E_{\pi_A}|h(X_n)|^{2r})^{1/2r} \\ &\leqq \left(\frac{\pi|h|^{2r}}{\pi(A)}\right)^{1/2r} \sum_{n=1}^{\infty} P_\pi[\tau_A = n]^{1/2s}\end{aligned} \tag{4.46}$$

where the least inequality follows since $P_{\pi_A}[\tau_A > n] = P_\pi[\tau_A = n]$ (see Lemma 2.1) and since

$$\begin{aligned}E_{\pi_A}|h(X_n)|^{2r} &= (1/\pi(A)) \int_A E_x|h(X_n)|^{2r}\pi(dx) \\ &\leqq (1/\pi(A)) \int E_x|h(X_n)|^{2r}\pi(dx) \\ &= \frac{\pi|h|^{2r}}{\pi(A)}.\end{aligned} \tag{4.47}$$

Condition (iii) (a) insures that $\pi|h|^{2r} < \infty$. Moreover,

$$\begin{aligned}&\sum_1^{\infty} (P_\pi[\tau_A = n])^{1/2s} \\ &= \sum_1^{\infty} (g(n))^{1/2s-1}(P_\pi[\tau_A = n]g(n)^{2s-1})^{1/2s} \\ &\leqq \left[\sum_1^{\infty} (g(n))\right]^{1-1/2s} \left[\sum_1^{\infty} P_\pi[\tau_A = n]g(n)^{2s-1}\right]^{1/2s}\end{aligned} \tag{4.48}$$

by Hölder's inequality. Conditions (iii) (a), (b), and (c) are therefore sufficient to establish that $\delta < \infty$.

Finally, if (d) holds, then, by Lemma 2.1,

$$E_\pi(\tau_A^{2s-1} \log^{2s}\tau_A) = E_\pi(\tau_A \log^{2s/(2s-1)}\tau_A)^{2s-1} < \infty. \tag{4.49}$$

Thus (b) and (c) hold with $g(n) = n \log^{2s/(2s-1)} n$.

COROLLARY 4.3. *Let $E_\pi \tau_A < \infty$ (equivalently, let $E_{\pi_A}\tau_A^2 < \infty$) for some uniform set A. Then for every positive state set B,*

$$\sigma_N^2(B) = \pi(B)^{-3} \lim_{\ell} \frac{\pi(A)}{\ell} E_{\pi_A}\left[\sum_0^{\tau_A^{(\ell)}-1} (\chi_B(X_j) - \pi(B))\right]^2 \tag{4.50}$$

exists, where $0 \leqq \sigma_N^2(B) < \infty$, and, for every starting distribution ψ,

$$\mathscr{L}_\psi\left(\frac{1}{\sqrt{n}}\left(\tau_B^{(n)} - \frac{n}{\pi(B)}\right)\right) \to \mathscr{N}(0, \sigma_N^2(B)). \tag{4.51}$$

Moreover,

$$\sigma_N^2(A) = \lim_{\ell} \frac{1}{\ell} E_{\pi_A}\left(\tau_A^{(\ell)} - \frac{\ell}{\pi(A)}\right)^2. \tag{4.52}$$

PROOF. For any positive B,

$$\begin{aligned}
&\left[\frac{1}{\sqrt{n}}\left(\tau_B^{(n)} - \frac{n}{\pi(B)}\right) \leqq a\right] \\
&\quad= \left[\sum_{j=1}^{n/\pi(B)+a\sqrt{n}} \chi_B(X_j) \geqq n\right] \\
&\quad= \left[\left(\frac{\pi(B)}{n}\right)^{1/2} \sum_{j=1}^{n/\pi(B)+a\sqrt{n}} \left(\chi_B(X_j) - \pi(B)\right) \geqq -a(\pi(B))^{3/2}\right].
\end{aligned} \tag{4.53}$$

Since $E_{\pi_A}[\Sigma_0^{\tau_A-1}(\chi_B(X_j) - \pi(B))]^2 < \infty$ by (ii) of Theorem 4.2, the convergence assertions for $\tau_B^{(n)}$ follow from Corollary 4.2, the variance of the limit distribution being $\pi(B)^{-3}$ times the variance of the limit distribution of $n^{-1/2}\,\Sigma_0^{n-1}(\chi_B(X_j) - \pi(B))$. If $B = A$, then this variance is

$$\pi(A)^{-3} \lim_{\ell} \frac{\pi(A)}{\ell} E_{\pi_A}\left[\sum_{0}^{\tau_A^{(\ell)}-1} \left(\chi_A(X_j) - \pi(A)\right)\right]^2 = \lim_{\ell} \frac{1}{\ell} E_{\pi_A}\left(\frac{\ell}{\pi A} - \tau_A^{(\ell)}\right)^2. \tag{4.54}$$

COROLLARY 4.4. *Let $B_n \in \mathscr{A}$ satisfy* $\limsup k_n\pi(B_n) < \infty$, *where $k_n \to \infty$ as $n \to \infty$ and let $S_{n,\ell} = \Sigma_{j=0}^{\ell-1}\chi_{B_n}(X_j)$. Then, for any uniform set A and any starting distribution φ such that $\varphi P^j B_n \to 0$ as $n \to \infty$ for each fixed j, $\mathscr{L}_\varphi(S_{n,k_n}) \sim \mathscr{L}_{n,\ell}$ as $n \to \infty$ then $\ell \to \infty$ where $\mathscr{L}_{n,\ell}$ is the law with characteristic function*

$$\begin{aligned}
h_{n,\ell}(u) &= \exp\left\{\sum_{\nu=1}^{\infty} (e^{iu\nu} - 1) q_{A;\nu;n,\ell}\right\}, \\
q_{A;\nu;n,\ell} &= \left(\frac{k_n\pi(A)}{\ell}\right) P_{\pi_A}[S_{A;n,\ell} = \nu].
\end{aligned} \tag{4.55}$$

PROOF. Choosing $\tau < 1$ in the formulae defining $\Psi_{A;n,\ell}$, we have

$$\operatorname{Var}(\Psi_{A;n,\ell}) \leqq P_{\pi_A}[S_{A;n,\ell} > 0] \leqq E_{\pi_A}S_{A;n,\ell} = \frac{\ell\pi(B_n)}{\pi(A)} \tag{4.56}$$

by Lemma 4.2, for any positive A. The hypotheses then imply that (k_n/ℓ) $\operatorname{Var}(\Psi_{A;n,\ell})$ is bounded and that

$$\left(\frac{k_n}{\ell}\right)\alpha_{A;n,\ell} \leqq \left(\frac{k_n}{\ell}\right) E_{\pi_A}S_{A;n,\ell} \leqq k_n \frac{\pi(B_n)}{\pi(A)} \tag{4.57}$$

is also bounded. The assertion then follows from Theorem 4.1 in the same way that Corollary 3.3 follows from Theorem 3.1, on considering the characteristic functions $g_{A;n,\ell}$ of $S_{A;n,\ell}$.

5. The variance of the normal limit distribution

In this section f is a real valued measurable function on the Markov state space $(\mathscr{X}, \mathscr{A})$ and $S_n = \Sigma_{j=0}^{n-1} f(X_j)$. Corollary 4.2 gives conditions for convergence in law of $(S_n - n\pi f)/\sqrt{n}$ to a normal distribution as $n \to \infty$. The variance of the limit distribution, $\sigma_N^2(f)$, is the limit variance of $(S_{\tau_A^{(n)}} - \tau_A^{(n)} \cdot \pi f)(\pi(A)/n)^{1/2}$ with respect to P_{π_A}, where A is any uniform set such that these variances are finite. In this section we give conditions under which the variance of $(S_n - n\pi f)/\sqrt{n}$ with respect to P_{π_A} or P_π converges to $\sigma_N^2(f)$.

Consider the following hypotheses:

($E1$) *there exists a strongly uniform set A such that $E_\pi \tau_A < \infty$;*

($E2$) *$E_\pi \tau_A < \infty$ for every positive A and $\mathscr{X}$ has uniform subsets;*

($E3$) *$E_{\pi_A} \tau_A^2 < \infty$ for every positive A and $\mathscr{X}$ has uniform subsets.*

We have shown that these three hypotheses are equivalent in [4] and hereafter refer to them simply as (E).

LEMMA 5.1. *Let (E) hold and $\pi f^2 < \infty$. Let φ be any starting distribution such that $\varphi \ll \pi$ and $d\varphi/d\pi$ is bounded. Then, as $n \to \infty$, $E_\varphi S_n = n\pi f + o(\sqrt{n})$.*

PROOF. Let $d\varphi/d\pi \leqq b$. Then we have for any number c

$$E_\varphi S_n - n\pi f = \left\{\varphi \sum_{k=0}^{n-1} P^k(f\chi_{[|f| \leqq c]}) - n\pi(f\chi_{[|f| \leqq c]})\right\} + \left\{\varphi \sum_{k=0}^{n-1} P^k(f\chi_{[|f| > c]}) - n\pi(f\chi_{[|f| > c]})\right\}. \tag{5.1}$$

Now by Theorem 2.3, the first bracket is bounded by $c\varphi\bar{\beta} \leqq bc\pi\bar{\beta} < \infty$ and the second by $n(b + 1)\pi(|f|\chi_{[|f| > c]})$. Since $\pi f^2 < \infty$ implies $\pi(|f|\chi_{[|f| > c]}) = o(1/c)$ as $c \to \infty$, the lemma follows by choosing $c = c(n)$ so $c(n) = o(\sqrt{n})$ and $\pi(|f|\chi_{[|f| > c]}) = o(1/\sqrt{n})$.

The next result is a small generalization of a lemma of Chung and Robbins (see [2], pp. 84, 85). Its proof is easy and will be omitted.

LEMMA 5.2. *Let $U_1, U_2, \cdots$ be uniformly integrable. Then, as $n \to \infty$, $E(\max_{k \leqq n} |U_k|) = o(n)$.*

Let $\bar{f} = f - \pi f$.

For any state set A and $k \geqq 1$ let

$$Z_{A;k} = \sum_{j=\tau_A^{(k-1)}}^{\tau_A^{(k)} - 1} \bar{f}(X_j), \qquad U_{A;k} = \sum_{j=\tau_A^{(k-1)}}^{\tau_A^{(k)} - 1} |\bar{f}(X_j)|. \tag{5.2}$$

Let $\lambda_n = \max\{k : \tau_A^{(k)} \leqq n\}$ and $Z'_{A;n} = \Sigma_{j=\tau_A^{(\lambda_n)}}^{n-1} \bar{f}(X_j)$ where here and in what follows we use the convention that for a sum $\Sigma_{j=k}^{\ell} t_j$ where $k > \ell$, we set $\Sigma_{j=k}^{\ell} t_j = 0$ if $k = \ell + 1$ and

$$\sum_{j=k}^{\ell} t_j = -\sum_{j=\ell+1}^{k-1} t_j \quad \text{if} \quad k > \ell + 1. \tag{5.3}$$

THEOREM 5.1. *Let A be uniform and $E_{\pi_A}(U_{A;1})^2 < \infty$. Then*

$$\lim_n \left(\frac{1}{n}\right) E_{\pi_A}(S_n - n\pi f)^2 = \sigma_N^2(f). \tag{5.4}$$

If, moreover, (E) holds, then

$$\lim_n \left(\frac{1}{n}\right) E_{\pi_A}(S_n - E_{\pi_A}S_n)^2 = \sigma_N^2(f). \tag{5.5}$$

PROOF. To simplify notation we write Z_k, Z'_n, U_k for $Z_{A;k}$, $Z'_{A;n}$, $U_{A;k}$, respectively. We prove the first assumption with the restriction that A be exponential then indicate how to extend to A uniform. By Lemma 5.2

$$E_{\pi_A}(Z'_n)^2 \leqq E_{\pi_A}(\max_{k \leqq n} \{U_k^2\}) = o(n). \tag{5.6}$$

We will show that

$$E_{\pi_A}\left(\sum_{k=\lambda_n+1}^{[n\pi(A)]} Z_k\right)^2 = o(n). \tag{5.7}$$

The first assertion will follow since, by Corollary 4.2,

$$\lim_n \left(\frac{1}{n}\right) E_{\pi_A}\left(\sum_{k=1}^{[n\pi(A)]} Z_k\right)^2 = \sigma_N^2(f). \tag{5.8}$$

Now the Z_k are a stationary sequence with respect to P_{π_A} and square integrable. Thus the $Z_j Z_k$ are uniformly square integrable for all j, k. Let $H_{n,\varepsilon} = [|\lambda_n/n - \pi(A)| > \varepsilon]$. Now in squaring $\Sigma_{\lambda_n+1}^{[n\pi(A)]} Z_k$, there are at most $2|n\pi(A) - \lambda_n|m$ terms $Z_j Z_k$ with $|j - k| < m$. Since $\lambda_n \leqq n$, we have

$$E_{\pi_A}\left[\sum_{\lambda_n+1}^{[n\pi(A)]} Z_k\right)^2 \leqq 2 \sum_{j=1}^{n} \sum_{k=m}^{\infty} |E(Z_j Z_{j+k})| + \sum_{j=1}^{n} E(Z_j^2 \chi_{H_{n,\varepsilon}}) + 2 \sum_{j=1}^{n} \sum_{k=1}^{m-1} |E(Z_j Z_{j+k} \chi_{H_{n,\varepsilon}})| + 2\varepsilon n m E_{\pi_A} Z_1^2. \tag{5.9}$$

But $\lambda_n/n \to \pi(A)$ in P_{π_A} probability, and it follows that the second and third terms in the above expansion are $o(n)$ for each fixed m, $\varepsilon > 0$. By Lemma 3.1, the first term is bounded by $2n \sum_{k=m}^{\infty} \rho_k^{1/2}(A)$, where the $\rho_k(A)$ converge exponentially to 0. The relation (5.7) is thus proved for A exponential by letting $n \to \infty$ then $\varepsilon \to 0$ then $m \to \infty$ in the above expansion.

If A is uniform, then it follows easily that, for any exponential subset A_i of A, $E_{\pi_{A_i}}((U_{A_i;1})^2 < \infty$, and the first assertion holds with A replaced by A_i. That it also holds for A then follows by the same kind of argument as in Corollary 4.2.

The last assertion follows from the first and Lemma 5.1, and the theorem is proved.

THEOREM 5.2. *Let (E) hold and f be bounded. Then*

$$\lim_n \left(\frac{1}{n}\right) E_{\pi}(S_n - n\pi f)^2 = \sigma_N^2(f). \tag{5.10}$$

PROOF. By (*E*) uniform sets exists, hence there exist uniform $A_m \uparrow \mathscr{X}$ (see [4]), and, for each A_m,

$$E_{\pi_{A_m}}(U^2_{A_m;1}) \leqq \|\bar{f}\|^2 E_{\pi_{A_m}}(\tau^2_{A_m}) < \infty. \tag{5.11}$$

Thus

$$\lim_n \left(\frac{1}{n}\right) E_{\pi_{A_m}}(S_n - n\pi f)^2 = \sigma^2_N(f), \tag{5.12}$$

for each m by Theorem 5.1. Also, by Theorem 2.3,

$$\left|\bar{f} \sum_{k=1}^{n} P^k \bar{f}\right| \leqq 2\|f\|^2 \beta \tag{5.13}$$

for all n, so the $\bar{f}\,\Sigma_{k=1}^n P^k \bar{f}$ are uniformly integrable with respect to π. Now

(5.14)

$$\left(\frac{1}{n}\right) E_\pi(S_n - n\pi f)^2 - \left(\frac{\pi(A_m)}{n}\right) E_{\pi_{A_m}}(S_n - n\pi f)^2$$
$$= \left(\frac{1}{n}\right) \int_{\mathscr{X} - A_m} E_x\left(\sum_{k=0}^{n-1} \bar{f}(X_k)\right)^2 \pi(dx)$$
$$= \int_{\mathscr{X} - A_m} \left(\frac{1}{n}\right)\left(\sum_{k=0}^{n-1} P^k \bar{f}^2\right) d\pi + 2 \int_{\mathscr{X} - A_m} \left(\frac{1}{n}\right)\left[\sum_{j=0}^{n-2} P^j \bar{f}\left(\sum_{k=1}^{n-j-1} P^k \bar{f}\right)\right] d\pi.$$

The integrands in the first integral are uniformly bounded and those in the second integral are uniformly integrable, since π is invariant. It follows that the difference converges to 0 as $m \to \infty$ uniformly in n and the theorem follows by letting $n \to \infty$ then $m \to \infty$.

Let $r_n = \Sigma_{k=1}^n \pi(\bar{f} P^k \bar{f})$. Then

$$\left(\frac{1}{n}\right) E_\pi(S_n - n\pi f)^2 = \pi(\bar{f}^2) + \left(\frac{2}{n}\right) \sum_{k=1}^{n-1} r_k. \tag{5.15}$$

Under the conditions of the preceding theorem this expression converges to $\sigma^2_N(f)$ as $n \to \infty$. When, moreover, $\mathscr{X}$ is aperiodic, the r_n have a limit and the expression simplifies:

COROLLARY 5.1. *Let (E) hold, f be bounded and $\mathscr{X}$ be aperiodic. Then*

$$\sigma^2_N(f) = \pi(\bar{f}^2) + 2 \sum_{n=1}^{\infty} \pi(\bar{f} P^n \bar{f}). \tag{5.16}$$

PROOF. We have to show $r_n \to \Sigma_{n=1}^\infty \pi(\bar{f} P^n \bar{f})$ as $n \to \infty$. But by Theorem 2.3, the quantities $|\Sigma_{k=0}^m P^k \bar{f}|$ are bounded by a π integrable function, $\bar{\beta}$, for all m. Thus, for $m \leqq n$ and $c > 0$,

$$\left|\sum_m^n P^k \bar{f}\right| = \left|P^m \sum_0^{n-m} P^k \bar{f}\right| \tag{5.17}$$
$$\leqq c\|P^m(x, \cdot) - \pi\| + P^m(\bar{\beta}\chi_{[\bar{\beta} > c]}) + \pi\bar{\beta}\chi_{[\bar{\beta} > c]}$$

since $\pi(\Sigma_0^{n-m} P^k \bar{f}) = 0$. Since $\mathscr{X}$ has uniform sets by (E), the (possibly non-measurable) functions $\|P^m(x, \cdot) - \pi\|$ converge to 0 almost uniformly in x as $m \to \infty$, hence are bounded above by measurable functions $g_m \leqq 2$ converging to 0. Thus

$$(5.18) \qquad r_n - r_m = \left|\pi \bar{f} \sum_m^n P^k \bar{f}\right| \leqq \|\bar{f}\| \, \pi \left|\sum_m^n P^k \bar{f}\right| \leqq \|\bar{f}\|(c\pi g_m + 2\pi\beta\chi_{[\beta > c]}) \to 0$$

as $m \to \infty$ then $c \to \infty$ uniformly in $n \geqq m$.

The convergence of the r_n follows and the corollary is proved.

THEOREM 5.3. *Let (E) hold, $f_b = f\chi_{[|f| \leqq b]}$ and $\bar{f}_b = f_b - \pi f_b$ for any $b > 0$. Let $\pi(f^2) < \infty$ and $E_{\pi_A}(\Sigma_{k=0}^{\tau_A} |\bar{f}(X_k)|)^2 < \infty$ for some uniform A. Then $\sigma_N^2(f) = \lim_{b \to \infty} \sigma_N^2(f_b)$. In particular, if $\mathscr{X}$ is aperiodic, then*

$$(5.19) \qquad \sigma_N^2(f) = \pi(\bar{f}^2) + 2 \lim_{b \to \infty} \sum_{n=1}^{\infty} \pi(\bar{f}_b P^n \bar{f}_b).$$

PROOF. If A is not exponential, then for any exponential subset A_i of A, the hypothesis $E_{\pi_{A_i}}(\Sigma_{k=0}^{\tau_{A_i}-1} |\bar{f}(X_k)|)^2 < \infty$ still applies. Thus we assume A exponential in what follows. Now by (E) and Theorem 4.2, $E_{\pi_A}(\Sigma_{k=0}^{\tau_A} \bar{f}_b(X_k))^2 < \infty$ for every finite b. Thus

$$(5.20) \qquad \mathscr{L}\left[n^{-1/2} \sum_{k=0}^{n-1} \bar{f}_b(X_k)\right] \to \mathscr{N}(0, \sigma_N^2(\bar{f}_b))$$

as $n \to \infty$.

Similarly, $E_{\pi_A}[\Sigma_{k=0}^{\tau_A - 1} (\bar{f}(X_k) - \bar{f}_b(X_k))]^2 < \infty$ and

$$(5.21) \qquad \mathscr{L}\left[n^{-1/2} \sum_{k=0}^{n-1} (\bar{f}(X_k) - \bar{f}_b(X_k))\right] \to \mathscr{N}(0, \sigma_N^2(\bar{f} - \bar{f}_b)).$$

Now by Lemma 3.3 and 3.1,

$$(5.22) \qquad \sigma_N^2(\bar{f} - \bar{f}_b) \leqq E_{\pi_A}\left[\sum_0^{\tau_A - 1} (\bar{f}(X_k) - \bar{f}_b(X_k))\right]^2 \left(1 + 4 \sum_{n=1}^{\infty} \rho_n^{1/2}(A)\right).$$

But the right side converges to 0 as $b \to \infty$. It follows that $\sigma_N^2(f_b) \to \sigma_N^2(f)$ as $b \to \infty$, and the last assertion follows from this and Corollary 5.1.

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