GROWTH RATE OF CERTAIN GAUSSIAN PROCESSES

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1. Introduction

We will be concerned with real, continuous Gaussian processes. In (A) of Theorem 1.1, a result on the growth rate of the supremum as $t \to \infty$ is given. The processes covered by Theorem 1.1 all have stationary increments. The law of the iterated logarithm, given as (C) below, is a consequence of (A).

Theorem 1.1 will be stated and discussed in this section. The proof of this theorem and supporting propositions are given in Section 2. An analogous result for small times is given in Section 3. That the method of proof can also be successfully employed in dealing with certain Gaussian processes not possessing stationary increments is illustrated by Theorem 4.1. The results of Section 1 were announced in [5].

Let $(Y_t, t \geqq 0)$ be a real, separable Gaussian process with $Y_0 = 0, E[Y_t] \equiv 0$, and set

(1.1) $$w(s, t) = E[Y_s Y_t], \quad Q(t) = \tfrac{1}{2} w(t, t).$$

Then let

(1.2) $$X_t = \frac{Y_t}{(2Q(t))^{1/2}}.$$

In this section and the succeeding two sections, (Y_t) will be taken to have stationary increments, so that

(1.3) $$w(s, t) = Q(s) + Q(t) - Q(t - s), \qquad 0 \leqq s \leqq t.$$

THEOREM 1.1. *Suppose there exists a monotone nondecreasing function $v(t)$, defined on the nonnegative reals and vanishing at $t = 0$, and there exist positive constants $s_0, \beta_1, \beta_2, \beta_3$, with $\beta_3 < \tfrac{1}{2}\beta_1 + 1$, such that*

(1.4) $$\lim_{t \to \infty} \frac{Q(s + t) - Q(s)}{v(s + t) - v(s)} = 1 \text{ uniformly in } s,$$

and

(1.5) $$v(t) \geqq \left(\frac{t}{s}\right)^{\beta_1} v(s) > 0, \qquad t \geqq s > s_0,$$

$$(1.6) \qquad v(t) \leqq \left(\frac{t}{s}\right)^{\beta_3} v(s), \qquad\qquad t \geqq s > s_0,$$

$$(1.7) \qquad Q(t) = O(t^{\beta_2}), \qquad\qquad t \downarrow 0.$$

Then with probability one

(A) $\lim\limits_{T \to \infty} \left(\sup\limits_{t \leq T} X_t - (2 \log \log T)^{1/2}\right) = 0,$

and hence also

(B) $\lim\limits_{t \to \infty} \sup \left(X_t - (2 \log \log t)^{1/2}\right) = 0,$

(C) $\lim\limits_{t \to \infty} \sup X_t/(2 \log \log t)^{1/2} = 1.$

Before discussing this theorem, we state a result of [6], to which Theorem 1.1 will, in fact, be reduced.

THEOREM 1.2 (Pickands). *Let $(Z_t, t \geq 0)$ be a real stationary, separable Gaussian process with mean 0 and covariance $\gamma(|t - s|)$ such that for some $\alpha > 0$*

$$(1.8) \qquad 1 - \gamma(t) = O(t^\alpha), \qquad\qquad t \downarrow 0,$$

and

$$(1.9) \qquad \gamma(t) = o\big((\log t)^{-1}\big), \qquad\qquad t \to \infty.$$

Then with probability one

$$(1.10) \qquad \sup\limits_{t \leq T} Z_t - (2 \log T)^{1/2} \to 0.$$

REMARK 1.1. It is only necessary to prove (A); the implications from (A) to (B) and from (B) to (C) are trivial.

REMARK 1.2. Instead of introducing the function v, one could, of course, formulate hypotheses on Q above. However, our hypotheses are convenient; frequently, it is evident that a suitable function v exists, while conditions imposed directly on Q would be harder to check.

REMARK 1.3. One class of examples is obtained by taking $Y_t = \int_0^t Y'_s \, ds$, where $(Y'_s, s \geq 0)$ is a real stationary Gaussian process with continuous sample functions. Let $q(|t - s|)$ be the covariance of the Y' process and set $R(t) = \int_0^t q(s) \, ds$. Then $Q(t) = \int_0^t R(s) \, ds$. If $v(t)$ is a differentiable function satisfying conditions (1.5)–(1.7), $v(0) = 0$ and v nondecreasing, and if $R(t) \sim v'(t)$ as $t \to \infty$, then (1.4) will hold. In particular, if $R(t)$ converges to a positive limit K as $t \to \infty$, the hypotheses of the theorem will be satisfied with $v(t) = Kt$, $\beta_1 = \beta_3 = 1$, and $\beta_2 = 2$. This will be the situation whenever $q(t)$ is everywhere positive and $q(t)$ is integrable, for example, if Y'_t is the Ornstein–Uhlenbeck process with covariance $e^{-|t-s|}$. Related to this class of examples is a law of the iterated logarithm given in [3].

REMARK 1.4. Theorem 1.1 applies to the case $Y_t = B_t$, where B_t is Brownian motion. In this case, however, the results are not new, for $B_{e^t} e^{-t/2}$ is the

Ornstein–Uhlenbeck process and (A) is an immediate consequence of Pickands' theorem applied to the Ornstein–Uhlenbeck process. For the Ornstein–Uhlenbeck process Newell [4] and Pickands [6] also proved delicate limit theorems for the distribution of the supremum which translate to interesting statements about $\sup \{t^{-1/2} B_t : 1 \leq t \leq T\}$. Condition (C) in this case is, of course, the classical law of the iterated logarithm of Khintchin, and (B) can be deduced from the Kolmogorov test for upper and lower functions.

2. Key propositions

The plan is to compare $X_t^* = X_{e^t}$ with stationary processes to which Pickands' theorem is applicable. Such a comparison is made possible by Slepian's lemma.

LEMMA 2.1 (Slepian). *Let X_j^-, X_j^+ for $j = 1, 2, \cdots, N$, be real Gaussian sequences of length N, with mean 0 and covariances*

$$(2.1) \qquad E[X_i^- X_j^-] = \gamma^-(i, j), \qquad E[X_i^+ X_j^+] = \gamma^+(i, j),$$

and suppose

$$(2.2) \qquad \begin{aligned} \gamma^-(i, i) &= \gamma^+(i, i), & 1 \leq i \leq N, \\ \gamma^-(i, j) &\leq \gamma^+(i, j), & 1 \leq i \leq N, 1 \leq j \leq N. \end{aligned}$$

Then for any choice of constants a_j, for $j = 1, 2, \cdots, N$,

$$(2.3) \qquad P\left[\bigcap_{j=1}^N [X_j^- \leq a_j]\right] \leq P\left[\bigcap_{j=1}^N [X_j^+ \leq a_j]\right].$$

This extremely useful result is given in [8].
Defining

$$(2.4) \qquad \gamma(s, t) = E[X_s X_t],$$

one obtains from (1.3) that

$$(2.5) \qquad \gamma(s, t) = \frac{Q(s) + Q(t) - Q(t - s)}{2[Q(s) Q(t)]^{1/2}}, \qquad 0 \leq s \leq t.$$

The notation

$$(2.6) \qquad X_t^* = X_{e^t}, \qquad \gamma^*(s, t) = \gamma(e^s, e^t)$$

is to be in force up to the end of this section.

PROPOSITION 2.1. *Under the conditions of Theorem 1.1, except that (1.6) need not be assumed, one has with probability one*

$$(2.7) \qquad \limsup_{T \to \infty} \left(\sup_{t \leq T} X_t - (2 \log \log T)^{1/2}\right) \leq 0.$$

PROOF. The crux of the argument is the establishing of (2.19) which, along with (2.10), provides a suitable *lower bound* for $\gamma^*(s, t)$.

Define $\delta(s, t)$ for nonnegative s and t by

$$(2.8) \qquad Q(s + t) - Q(s) = \big(v(s + t) - v(s)\big)\big(1 + \delta(s, t)\big),$$

so that condition (1.4) of Theorem 1.1 implies that $\delta(s, t)$ tends to 0 uniformly in s as t approaches infinity. Using (1.3), one may write for $t > s$,

$$(2.9) \quad w(s, t) = v(s)\big(1 + \delta(0, s)\big) + \big(v(t) - v(t - s)\big)\big(1 + \delta(t - s, s)\big),$$

and since v is nonnegative and nondecreasing, it is clear that there exists an s_1 such that

$$(2.10) \qquad\qquad\qquad w(s, t) \geqq 0, \qquad\qquad\qquad t \geqq s \geqq s_1.$$

Relation (2.5) and a trivial estimate show that, if $t \geqq s$,

$$(2.11) \qquad 1 - \gamma^*(s, t) = 1 - \frac{Q(e^s) + Q(e^t) - Q(e^t - e^s)}{2\big(Q(e^s)Q(e^t)\big)^{1/2}} \leqq \frac{Q(e^t - e^s)}{2\big(Q(e^s)Q(e^t)\big)^{1/2}}.$$

Using assumption (1.4) and the monotonicity of v, it follows that there exists s_2 such that

$$(2.12) \qquad\qquad 1 - \gamma^*(s, t) \leqq \frac{Q(e^t - e^s)}{v(e^s)}, \qquad\qquad t \geqq s \geqq s_2,$$

and, furthermore, there exists s_3 such that

$$(2.13) \qquad\qquad \frac{Q(e^t - e^s)}{v(e^s)} \leqq \frac{2v(e^t - e^s)}{v(e^s)}, \qquad\qquad e^t - e^s \geqq s_3.$$

Now let $s_4 = s_0 \vee s_3$ and consider the following cases.

Case a: $s_4 \leqq e^t - e^s < e^s, s > s_2$. Then condition (1.5) is applicable to give

$$(2.14) \qquad\qquad v(e^s) \geqq \left(\frac{e^s}{e^t - e^s}\right)^{\beta_1} v(e^t - e^s),$$

and since the left side of (2.12) is less than the right side of (2.13) one obtains

$$(2.15) \qquad\qquad 1 - \gamma^*(s, t) \leqq 2(e^{t-s} - 1)^{\beta_1}.$$

Case b: $0 \leqq e^t - e^s \leqq s_4, s > s_2$. By condition (1.7), Q is continuous at 0, hence everywhere. So it follows from (1.7) that for $0 \leqq \beta \leqq \beta_2$ there exists a finite constant $c(\beta)$ such that

$$(2.16) \qquad\qquad Q(t) \leqq c(\beta)t^\beta, \qquad\qquad 0 \leqq t \leqq s_4.$$

Substituting into (2.12) and using condition (1.5), one obtains

$$(2.17) \qquad 1 - \gamma^*(s, t) \leqq \frac{c(\beta)(e^t - e^s)^\beta}{e^{\beta_1 s}} \frac{s_0^{\beta_1}}{v(s_0)} = \frac{c_1(\beta)e^{\beta s}(e^{t-s} - 1)^\beta}{e^{\beta_1 s}},$$

where $c_1(\beta)$ is a finite constant and $0 \leqq \beta \leqq \beta_2$. Choosing $\beta_0 = \beta_1 \wedge \beta_2$ gives, with $c_1 = c_1(\beta_0)$,

(2.18) $$1 - \gamma^*(s, t) \leqq c_1 (e^{t-s} - 1)^{\beta_0}.$$

From (2.15) and (2.18), it follows that there is a constant c such that

(2.19) $$1 - \gamma^*(s, t) \leqq c(t - s)^{\beta_0}, \qquad s_2 \leqq s \leqq t \leqq s + 1.$$

Let $\gamma^-(t)$ be an even function such that $\gamma^-(s - t)$ is a covariance satisfying

(2.20) $$\gamma^-(0) = 1, \qquad \gamma^-(t) < 1 - ct^{\beta_0}, \qquad 0 \leqq t \leqq 1,$$
$$\gamma^-(t) = 0, \qquad t > 1.$$

By Polya's criterion such functions exist. Let (X_t^-) be a separable stationary Gaussian process with covariance $\gamma^-(s, t)$ and mean 0. Pickands' theorem applies to (X_t^-) and implies

(2.21) $$\limsup_{T \to \infty} \left[\sup_t \{ X_t^- ; s_1 \vee s_2 \leqq t \leqq T \} - (2 \log T)^{1/2} \right] \leqq 0$$

with probability one.
It follows from (2.10), (2.19) and (2.20) that

(2.22) $$\gamma^-(s, t) \leqq \gamma^*(s, t), \qquad t \geqq s \geqq s_1 \vee s_2,$$
$$\gamma^-(s, s) = \gamma(s, s), \qquad s > 0.$$

That (2.21) holds with probability one is equivalent to the existence for every $\varepsilon > 0$ of a $T(\varepsilon) > 0$ such that for every choice of n, $T_n \geqq T_{n-1} \geqq \cdots \geqq T_1 \geqq T(\varepsilon)$,

(2.23) $$P \left\{ \bigcap_{i=1}^{n} \left[\sup_t \{ X_t^- ; s_1 \vee s_2 \leqq t \leqq T_i \} - (2 \log T_i)^{1/2} < \varepsilon \right] \right\} \geqq 1 - \varepsilon,$$

which in turn is equivalent to the assertion that for every choice of m, and $t_{i,j}$, $s_1 \vee s_2 \leqq t_{i,j} \leqq T_i$ for $j = 1, 2, \cdots, m$, and $i = 1, 2, \cdots, n$, one has

(2.24) $$P \left\{ \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} [X_{t_{i,j}}^- < (2 \log T_i)^{1/2} + \varepsilon] \right\} \geqq 1 - \varepsilon.$$

Now because of (2.22), Slepian's lemma can be applied to conclude that the corresponding assertion for X_t^* holds, and so Proposition 2.1 follows.

PROPOSITION 2.2. *The conditions of Theorem 1.1, except that (1.7) need not be assumed, imply that for every $\varepsilon > 0$,*

(2.25) $$\lim_{T \to \infty} P[\sup_{t \leqq T} X_t - (2 \log \log T)^{1/2} < -\varepsilon] = 0.$$

PROOF. The crucial inequality in this proof is (2.32), which provides an *upper bound* for $\gamma^*(s, t)$.

To establish the proposition, it suffices to show that there exists a $\Delta > 0$ such that

(2.26) $$\lim_{N \to \infty} P[\sup \{ X_{n\Delta}^* : n = 0, 1, \cdots, N \} > (2 \log N)^{1/2} - \varepsilon] = 1,$$

where N varies through the integers.

From (2.8), it appears that there exists $s_5 > s_0$ such that

$$(2.27) \qquad w(s, t) \leq \tfrac{3}{2}[v(s) + (v(t) - v(t - s))], \qquad t \geq s \geq s_5.$$

Observe that if β_1 and β_3, satisfying the conditions of Theorem 1.1, exist, then β_3 can always be chosen to be greater to or equal to 1, and we shall assume it to be thus chosen.

Assume now that $t = \lambda s \geq 2s > s_5$ and use (1.6) twice to obtain

$$
\begin{aligned}
(2.28) \quad v(s) + [v(t) - v(t - s)] &= v(s) + [v(\lambda s) - v((\lambda - 1)s)] \\
&\leq v(s) + v((\lambda - 1)s)\left[\left(\frac{\lambda}{\lambda - 1}\right)^{\beta_3} - 1\right] \\
&\leq v(s)\left[1 + (\lambda - 1)^{\beta_3}\left[\left(\frac{\lambda}{\lambda - 1}\right)^{\beta_3} - 1\right]\right] \\
&= v(s)[1 + \beta_3(\lambda - c)^{\beta_3 - 1}],
\end{aligned}
$$

where c is a constant between 0 and 1 and the last step uses the mean value theorem. Therefore,

$$(2.29) \quad w(s, t) \leq \tfrac{3}{2}v(s)[1 + \beta_3(\lambda - c)^{\beta_3 - 1}] \leq 5v(s)\lambda^{\beta_3 - 1}, \quad t = \lambda s \geq 2s \geq 2s_5.$$

Recalling (1.4) and (2.5) and using assumption (1.6), one obtains the existence of a constant s_6 such that

$$(2.30) \qquad \gamma(s, t) \leq \frac{5v(s)\lambda^{\beta_3 - 1}}{(v(s)v(t))^{1/2}} \leq 5\lambda^{\beta_3 - 1 - (\beta_1/2)}, \qquad s_6 < s, \qquad t = \lambda s \geq 2s.$$

Let $\alpha = -[\beta_3 - 1 - (\beta_1/2)]$, so that $\alpha > 0$ by assumption. Then there exists a constant s_7 such that

$$(2.31) \qquad \gamma^*(s, t) \leq 5e^{-\alpha(t - s)}, \qquad t \geq s + \log 2 > s > s_7.$$

The right side of (2.31) is less than $\exp\{-\tfrac{1}{2}\alpha(t - s)\}$ when $t - s > 2\alpha^{-1}\log 5$. So choose $\Delta > 2\alpha^{-1}\log 5$. Then $\Delta > \log 2$, and so for any integer N_0 exceeding s_7,

$$
\begin{aligned}
(2.32) \quad \gamma^*(m\Delta, n\Delta) &\leq \exp\{-\tfrac{1}{2}\alpha|n - m|\}, \\
n &= N_0, N_0 + 1, \cdots; m = N_0, N_0 + 1, \cdots.
\end{aligned}
$$

The right side of (2.32) defines a covariance $\gamma^+(|n - m|)$ belonging to a stationary Gaussian sequence (X_n^+) to which Pickands' theorem applies. Actually, we need here only a discrete parameter version of Pickands' theorem, which follows already from the results of Berman [1]. Hence,

$$(2.33) \qquad \lim_{N \to \infty} P[\sup_{n \leq N} X_n^+ - (2 \log N)^{1/2} > -\varepsilon] = 1$$

for $\varepsilon > 0$, and again Slepian's lemma may be applied to deduce (2.26).

PROOF OF THEOREM 1.1.   Conclusion (B) is an easy consequence of Propositions 2.1 and 2.2. As for (A), half of this assertion is contained in Proposition 2.1,

but Proposition 2.2 does not quite give the other half. What is needed is the following proposition.

PROPOSITION 2.3. *Under the condition of Proposition 2.2,*

$$(2.34) \qquad \liminf_{T \to \infty} \left(\sup_{t \le T} X_t - (2 \log \log T)^{1/2} \right) \ge 0$$

with probability one.

PROOF. To establish this proposition, it does not suffice simply to apply Pickands' theorem; rather, one must retrace a small part of the argument used in the proof.

Let G_n, $n = 0, 1, \cdots$, be a real Gaussian sequence with mean 0. Then

$$(2.35) \qquad P\left[\liminf_{N \to \infty} \left(\sup_{k \le N} G_k - (2 \log N)^{1/2} \right) \ge 0 \right] = 1$$

is an easy consequence of

$$(2.36) \qquad \lim_{N \to \infty} \left(\log N P\left[\sup_{k \le N} G_k \le (2 \log N)^{1/2} - \varepsilon \right] \right) = 0, \qquad \varepsilon > 0.$$

If (G_k) is stationary and the covariance $g(m, n) = g(|m - n|)$ is such that $g(0) = 1, g(n) = o[(\log n)^{-1}]$ as $n \to \infty$, then Pickands shows that (2.36) holds. Returning now to the last part of the proof of Proposition 2.2, (2.36) holds with X_k^+ in place of G_k. Slepian's lemma is available to allow one to conclude (2.36) for $X_{k\Delta}^*$ in place of G_k, and therefore, finally, (2.35) holds with $X_{k\Delta}^*$ in place of G_k. The truth of the proposition follows immediately.

3. Behavior for small times

We continue to assume that (Y_t) has stationary increments, but now we look at small values of t.

THEOREM 3.1. *Suppose there exist positive constants β_1 and β_3 and T_0 with $\beta_3 < \frac{1}{2}\beta_1 + 1$ such that*

$$(3.1) \qquad Q(T) \ge \left(\frac{T}{S} \right)^{\beta_1} Q(S), \qquad 0 < S < T < T_0,$$

$$(3.2) \qquad Q(T) \le \left(\frac{T}{S} \right)^{\beta_3} Q(S), \qquad 0 < S < T < T_0.$$

Then with probability one

$$(A') \quad \lim_{T \to 0} \left(\sup_{T \le t \le 1} X_t - (2 \log |\log T|)^{1/2} \right) = 0,$$

and hence also

$$(B') \quad \limsup_{t \to 0} \left(X_t - (2 \log |\log t|)^{1/2} \right) = 0,$$

$$(C') \quad \limsup_{t \to 0} \left(X_t (2 \log |\log t|)^{-1/2} \right) = 1.$$

REMARK 3.1.   Recently, a number of studies have been published showing
continuity properties of Gaussian sample functions. Some of these contain
results partly overlapping with Theorem 3.1. (See, for instance, [2], [7].) There
are also interesting examples in [2] of instances when (C') fails, that is, the
iterated logarithm is no longer the correct normalization.

The proof is similar to that in Section 2, only now we let $S = e^{-s}$ and
$T = e^{-t}$, and consider $X_s^* = X_S$ with corresponding covariance $\gamma^*(s, t) =
\gamma(S, T)$. The details are so similar to those of the proof of Theorem 1.1 that
we will not supply them.

4. Nonstationary increments

The argument in the preceding sections may on occasion be fruitfully employed
when dealing with Gaussian processes which do not possess stationary incre-
ments. We proceed to a class of examples.

Let $(Y_t^0, t \geq 0)$ be Brownian motion, and define

$$(4.1) \quad Y_t^{(n)} = \int_0^t Y_s^{(n-1)} \, ds, \; X_t^{(n)} = Y_t^{(n)}/\{E[(Y_t^{(n)})^2]\}^{1/2}, \quad t > 0, n = 1, 2, \cdots.$$

THEOREM 4.1.   *For every nonnegative integer n, $(X_t^{(n)})$ satisfies conclusion* (A)
(hence, also (B) *and* (C)*) of Theorem* 1.1, *and also conclusion* (A'), *(hence also*
(B') *and* (C')*) of Theorem* 3.1.

PROOF.   Let $w^{(n)}(s, t)$, $\gamma^{(n)}(s, t)$ be the covariance of $(Y_t^{(n)})$ and $(X_t^{(n)})$,
respectively. Note

$$(4.2) \qquad\qquad w^{(1)}(s, t) = -\tfrac{1}{6}s^3 + \tfrac{1}{2}s^2 t, \qquad\qquad 0 < s < t,$$

and use induction to verify that

$$(4.3) \qquad w^{(n)}(s, t) = \sum_{k=0}^n a_k^{(n)} s^{2n+1-k} t^k, \qquad a^{(n)} = \sum_{k=0}^n a_k^{(n)} > 0.$$

Of course,

$$(4.4) \qquad\qquad \gamma^{(n)}(s, t) = \frac{w^{(n)}(s, t)}{[w^{(n)}(s, s)w^{(n)}(t, t)]^{1/2}}, \qquad\qquad s > t.$$

Consider the proof of (A). In the proof of Theorem 1.1 the important
inequalities are the lower bounds given by (2.10) and (2.19) and the upper
bound (2.32). The positivity of $w^{(n)}(s, t)$ is easily proved by induction. In dis-
cussing the other two inequalities, let us fix n and write $w(s, t)$, $\gamma(s, t)$ in place
of $w^{(n)}(s, t)$, $\gamma^{(n)}(s, t)$, respectively. Note that $\gamma(s, t)$ is differentiable with respect
to t off the diagonal $t = s$. So by the mean value theorem for $s < t$, there
exists a t^* between s and t such that, with $\gamma^*(s, t) = \gamma(e^s, e^t)$, one has

$$(4.5) \qquad\qquad \gamma^*(s, t) = 1 + \gamma_{0,1}(e^s, e^{t^*})e^s(e^{t-s} - 1),$$

and so a lower bound of the form (2.19) with $c > 0$ and $\beta_0 = 1$ follows from

the existence of a finite K such that

$$(4.6) \qquad \gamma_{0,1}(e^s, e^t)e^s \leq K, s < t < s + 1.$$

The existence of such a K follows from (4.3) and (4.4).

To obtain an upper bound corresponding to (2.32), substitute (4.3) into (4.4) and observe that the resulting expression is bounded by a constant times $(t/s)^{-1/2}$, where $t > s$. This leads at once to the desired bound. The proof of (A) then proceeds as in Theorem 1.1.

Again the proof of (A′) proceeds in the same manner, making the change of variables t into e^{-t}.

REMARK 4.1. Conclusion (C′) of Theorem 4.1 for the case of $n = 1$ can also be deduced from results of [9]. An example in that paper uses the result developed there to obtain information about the lim sup behavior of $\int_0^t |Y_s^{(0)}|^\alpha ds$ for $\alpha > 0$.

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