ON POISSON LAWS AND RELATED QUESTIONS

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1. Definitions and some lemmas

We shall consider a class of infinitely divisible laws, which may be called Poisson laws, defined on the Borel sets of a locally compact group. This class of probability measures is arrived at quite naturally by looking at the classical Poisson laws over the Borel sets $\mathscr{B}_1$ of one dimensional Euclidean space R_1. Let $a > 0$. Then the standard Poisson law with mean a can be written in the form

$$\exp\{a(\delta_1 - \delta_0)\} = \delta_0 + \sum_{k=1}^{\infty} \frac{a^k}{k!}(\delta_1 - \delta_0)^k, \tag{1.1}$$

where δ_x is the Dirac measure at $x \in R_1$. Multiplication of measures means convolution. Convergence of the series means convergence in norm. The measure $a(\delta_1 - \delta_0)$ obviously satisfies the following conditions. If f belongs to the set $C(R_1)$ of bounded, continuous functions and fulfills the conditions $f \geqq 0$ and $f(0) = 0$, then $a(\delta_1 - \delta_0)(f) \geqq 0$. Moreover, $a(\delta_1 - \delta_0)(1) = 0$, where 1 denotes the function f identically equal to 1. It is well known that more general probability laws of the Poisson type may be defined along these lines. Let v be any bounded Radon measure defined over $\mathscr{B}_1$ and satisfying the conditions $v(1) = 0$ and $v(f) \geqq 0$ for every $f \in C(R_1)$ with $f \geqq 0$ and $f(0) = 0$. Then e^v is a probability law of Poisson type. Note that 0 is the neutral element of the additive group of R_1, and that the set $\{0\}$ is a compact subgroup of R_1. These considerations lead easily to a generalization of Poisson laws on arbitrary, locally compact groups. To achieve this, some simple definitions are needed.

DEFINITION 1.1. *The set of all bounded Radon measures defined over the Borel sets $\mathscr{B}$ of a locally compact group G is denoted by $\mathscr{R}(G)$, or just by $\mathscr{R}$. The subset of all probability measures is denoted by $Z(G)$, or just by Z. If m is any measure in $\mathscr{R}$, then $S(m)$ denotes its support.*

DEFINITION 1.2. *A measure $\mu \in Z$ is said to be infinitely divisible if for every natural number n there exists a $\mu_{1/n} \in Z$ which satisfies the equation $\mu_{1/n}^n = \mu$. The measure $\mu_{1/n}$ is called an nth root of μ.*

DEFINITION 1.3. *Let H be an arbitrary compact subgroup of G. Then e_H denotes that probability measure belonging to $Z(G)$ whose restriction to $H \cap \mathscr{B}$ is the Haar measure; $\mathscr{R}_H$ denotes the set of all $m \in \mathscr{R}$ which satisfy the equation $e_H m = m e_H = m$. The set $Z \cap \mathscr{R}_H$ is denoted by Z_H.*

For every $m \in \mathscr{R}_H$, one may define $\exp_H m$ by $e_H + \Sigma_{k=1}^{\infty} m^k/k!$, where the convergence of this series is understood as convergence in norm. A similar elementary definition of the logarithm is also available. For every $m \in \mathscr{R}_H$ which satisfies $\|m - e_H\| < 1$, one may define $\log_H m$ by $-\Sigma_{k=1}^{\infty} (m - e_H)^k/k$.

It is easy to see that both $\exp_H m$ and $\log_H m$ belong to $\mathscr{R}_H$. Some simple properties for the mappings $\exp_H$ and $\log_H$ are collected in the following.

LEMMA 1.1. *Let $p_1, p_2 \in \mathscr{R}_H$ and $p_1 p_2 = p_2 p_1$. Then*

$$\exp_H (p_1 + p_2) = \exp_H p_1 \exp_H p_2. \tag{1.2}$$

If, in addition, $\|p_i - e_H\| < 1$ for $i = 1, 2$, and $\|p_1 p_2 - e_H\| < 1$, then $\log_H(p_1 p_2) = \log_H p_1 + \log_H p_2$. Furthermore, if $m \in \mathscr{R}_H$, then the condition $\|m - e_H\| < 1$ implies $\exp_H \log_H m = m$, and the condition $\|m\| < \log 2$ implies $\log_H \exp_H m = m$.

Use is also made of the following.

LEMMA 1.2 (Böge [2]). *Let $a \in Z_H$ and assume that there exists a natural number ℓ such that $\|a^\ell - e_H\| + 2\|a - e_H\| < 1$. Then $\|a^j - e_H\| + 2\|a - e_H\| < 1$, and, moreover, $\log_H a^j = j \log_H a$ for $1 \leqq j \leqq \ell$.*

Let us introduce some more definitions.

DEFINITION 1.4. *Let $\mathscr{K}_H = \{v \in \mathscr{R}_H : v(1) = 0,\ v(f) \geqq 0$ for all $f \in C(G)$ satisfying $f \geqq 0$ and $f(x) = 0$ for $x \in H\}$.*

This last condition may also be written in the form $f(H) = 0$.

DEFINITION 1.5. *Let $\mu \in \mathscr{R}(G)$. If there exists a compact subgroup H of G and a $v \in \mathscr{K}_H$ such that $\mu = \exp_H v$, then μ is called a Poisson law.*

An immediate consequence of this definition is the following.

LEMMA 1.3. *Every Poisson law belongs to Z_H.*

Let us now formulate the following important, known result.

LEMMA 1.4. (Heyer [5], Pym [8], Wendel [12]). *A measure $\mu \in Z(G)$ is an idempotent if and only if there exists a compact subgroup H of G such that $\mu = e_H$.*

This lemma and Definition 1.5 make it clear that the theory of Poisson laws and the theory of one parameter, strongly continuous, operator semigroups are related to each other. But the former theory and some of the methods used here are of some interest on their own.

We are going to introduce an important concept. Let $\mu \in Z(G)$ and define $H_\mu = \{x \in G : \delta_x \mu = \mu \delta_x = \mu\}$. It is easy to see that H_μ is a closed subgroup of G. Indeed, H_μ is a compact subgroup of G, which may be stated as a lemma.

LEMMA 1.5. *The group H_μ is always a compact subgroup of G.*

PROOF. Suppose that H_μ is not compact. It follows that $H_\mu \cap (G \backslash K) \neq \varnothing$ for every compact subset K of G. Let K_0 be a compact set such that

$$\mu(K_0) > 0. \tag{1.3}$$

Choose $x_1 \in H_\mu$ and $x_n \in H_\mu \cap \left(G \backslash \bigcup_{i=1}^{n-1} x_i K_0 K_0^{-1}\right)$ for $n \geqq 2$. It is easy to see that $x_i K_0 \cap x_j K_0 = \varnothing$ for $i \neq j$ and for $1 \leqq i, j$. Obviously, the equation $\mu(x_i K_0) = \mu(K_0)$ holds for $i = 1, 2, \cdots$. This together with (1.3) leads to a contradiction.

DEFINITION 1.6. *The compact group H_μ (considered in Lemma 1.5) is called the invariance group of μ.*

Obviously, the following equation holds,

$$e_H\mu = \mu e_H = \mu. \tag{1.4}$$

LEMMA 1.6. *Let $a, b \in Z$. Assume that there exists a natural number $n \geqq 1$ such that $a = b^n$. Then $H_b \subseteqq H_a$.*

PROOF. It follows from (1.4) that $e_{H_b}a = e_{H_b}b^n = (e_{H_b}b)^n = a$, and similarly for ae_{H_b}.

Another simple but useful result follows.

LEMMA 1.7. *Let G be any locally compact group and let H be a compact subgroup of G. Assume that $\mu \in Z_H$, that $\nu \in Z$, and that*

$$\mu\nu = e_H. \tag{1.5}$$

Then there exists an $x \in G$ which belongs to the normalizer of H such that $\mu = e_H\delta_x$. If, in addition, $\nu \in Z_H$, then $\nu \in \delta_{x^{-1}}e_H$.

PROOF. It is well known that (1.5) implies

$$S(\mu)S(\nu) \subseteqq H. \tag{1.6}$$

There exists therefore a $z_1 \in S(\nu)$ such that $S(\mu)z_1 \subseteqq H$ and $S(\mu)$ is compact. Therefore, equality holds in (1.6) and there exists an $x \in S(\mu)$ such that $x^{-1} \in S(\nu)$ and $S(\mu) \subseteqq Hx$. Using $\mu \in Z_H$, it follows that $e_H = e_H\mu\delta_{x^{-1}} = \mu\delta_{x^{-1}}$. Hence, $Hx = S(\mu)$. Similarly, one finds $S(\mu) = yH$ for some $y \in G$. Therefore, there exists an $h \in H$ such that $yh = x$, and consequently, $Hx = xH$.

The last remark of Lemma 1.7 is proved in exactly the same manner.

This lemma shows that the solutions of the equation (1.5) which belong to Z_H are all trivial. The equation may have nontrivial solutions which do not belong to Z_H. A very simple example, communicated to me by H. Carnal and W. Hazod, follows. Let $S_3 = \{e, x_1, x_2, \cdots, x_5\}$ be the permutation group of three objects where e is the neutral element and $x_1 = (2, 3, 1)$, $x_2 = (3, 2, 1)$, $x_3 = (1, 2)$, $x_4 = (2, 3)$, and $x_5 = (1, 3)$.

Consider the measure

$$m = \tfrac{1}{6}[(\delta_{x_1} + \delta_{x_3}) - (\delta_{x_2} + \delta_{x_4})]. \tag{1.7}$$

It is easy to see that $m^2 = 0$ and that $m_1 = (e_{S_3} + m) \in Z(S_3)$. Moreover, it follows that $m_1^2 = e_{S_3}$; but obviously $H_{m_1} \neq S_3$ and $S(m_1) \neq S_3$. This shows that the equality in Lemma 1.6 may not hold, and that equation (1.5) may have nontrivial solutions if the assumptions of Lemma 1.7 are not satisfied. It will be seen (in Section 3) that the existence of nilpotent measures in the algebra $\mathscr{R}$ and the question of whether equality holds in Lemma 1.6 are related, even from a more general point of view.

The following simple result is related to Lemma 1.7.

LEMMA 1.8. *Let $\mu \in Z_H$. Suppose there exists a $\nu \in \mathscr{R}$ with $\mu\nu = e_H$. Then $H_\mu = H$.*

The above follows from $e_{H_\mu}\mu\nu = \mu\nu = e_{H_\mu}e_H = e_H$.

Still another important definition is needed.

DEFINITION 1.7. *Let R_+ be the additive semigroup of all real positive numbers. Let $\{r_1\}$ be a sequence of elements R_+. Suppose that there exists a sequence $\{n_i\}$ of natural numbers such that $r_{i-1} = n_i r_i$ for $i \geqq 1$. Let N_i be the semigroup generated by r_i in R_+. Then $S = \cup_{i \geqq 0} N_i$ is a semigroup, which will be called a real submonogeneous semigroup; every homomorphic image of such a real semigroup will be called a submonogeneous semigroup.*

This has the following immediate consequence.

LEMMA 1.9 (Hoffman [6]). *Let the x_i for $i \geqq 0$ be elements of a multiplicatively written semigroup. Suppose there exists a sequence $\{n_i\}$ of natural numbers such that $x_{i-1} = x_i^{n_i}$ for $i \geqq 1$. Then the set $\{x_i\}_{i \geqq 0}$ generates a submonogeneous semigroup.*

LEMMA 1.10. *Let G be a locally compact group and let $b \in Z(G)$. Suppose that b is infinitely divisible. Assume, furthermore, that there exists an infinite sequence of roots b_{1/m_i} of b and a compact subgroup H of G such that the following conditions are satisfied:*

(i) *$\{m_i\}$ is a strictly increasing sequence of natural numbers;*

(ii) *$b_{1/m_i} \in Z_{H_i}$ for $H_i \supseteqq H, i = 1, 2, \cdots$;*

(iii) *b_{1/m_i} generates a submonogeneous semigroup in Z (according to Lemma 1.9);*

(iv) *$\|b_{1/m_i} - e_H\| \to 0$ as $i \to \infty$.*

Then b is a Poisson law and $H_b = H$. Moreover, these conditions are also necessary for b to be a Poisson law.

PROOF. To show the necessity of the above conditions is trivial. To show that the conditions are sufficient, let us make the following obvious remark. If the sequence $\{b_{1/m_i}\}$ contains only a finite set of different elements, then $b = e_H$. Therefore, we may assume that $\{b_{1/m_i}\}$ contains infinitely many different elements. Let the natural number $m_j^{(0)}$ be chosen in such a way that $\|b_{1/m_j^{(0)}} - e_H\| < 1$, and define $c_0 = \log_H b_{1/m_1^{(0)}}$. It follows immediately that $b = \exp_H c$, where $v = m_j^{(0)} v_0$. We have to show that $v \in \mathscr{K}_H$. Since b is an element of Z, it is obvious that $v(1) = 0$, and it is enough to show that $v(f) \geqq 0$ when $f \in C(G), f \geqq 0$, and $f(H) = 0$. Now, for every natural i there exists an $n_i \geqq 1$ such that $b_{1/m_i} = b_{1/m_{i+1}}^{n_i}$. Using (iv), it follows from Lemmas 1.1 and 1.2 that

$$b_{1/m_i} = \exp_H \left\{ \frac{v_0}{n_j^{(0)} \cdots n_{i-1}} \right\} \tag{1.8}$$

for all sufficiently large $m_j > m_j^{(0)}$. Furthermore, $M_i = n_j^{(0)} \cdots n_{i-1} \to \infty$ as $i \to \infty$. For any f, as described above, we have $0 \leqq b_{1/m_i}(f) = v_0(f)/M_i + O(1/M_i^2)$. The assumption that $v_0(f) < 0$ (or equivalently, that $v(f) < 0$) would therefore lead to a contradiction.

The fact that $H_b = H$ follows immediately from $\exp_H \{v\} \exp_H \{-v\} = e_H$ and from Lemma 1.8.

Condition (iii) of Lemma 1.10 may be replaced by another condition.

LEMMA 1.11. *Suppose the conditions of Lemma 1.10 are satisfied, with the exception of* (iii) *and* (iv). *Replace the latter by the condition that*

$$(1.9) \qquad \sup_i m_i \|b_{1/m_i} - e_H\| < \infty.$$

Then b is again a Poisson law of the form $b = \exp_H v$ for $v \in \mathscr{K}_H$.

PROOF. We may assume (changing the notation when necessary) that $m_i(b_{1/m_i} - e_H)$ converges vaguely to a measure $v \in \mathscr{R}_H$, which is nonnegative when restricted to the Borel sets of $G\backslash H$. Let $K > 0$ be a real number such that $m_i\|b_{1/m_i} - e_H\| \leqq K$ for $i \geqq 1$. Introducing the notation

$$(1.10) \qquad c_{m_i} = \sum_{k=2}^{\infty} (b_{1/m_i} - e_H)^k/k!,$$

we have,

$$\begin{aligned}(1.11) \quad \|b - \exp_H m_i(b_{1/m_i} - e_H)\| &= \|b - [e_H + (b_{1/m_i} - e_H) + c_{m_i}]^{m_i}\| \\ &= \sum_{k=1}^{m_i} \left\|\binom{m_i}{k} b_{1/m_i}^k c_{m_i}^{m_i - k}\right\| \\ &\leqq \sum_{k=1}^{m_i} \binom{m_i}{k} \|c_{m_i}\|^{m_i - k} = (1 + \|c_{m_i}\|)^{m_i} - 1.\end{aligned}$$

It follows from the definition of c_{m_i} that

$$(1.12) \qquad \|c_{m_i}\| \leqq \sum_{k=2}^{\infty} \frac{1}{k!} \|b_{1/m_i} - e_H\|^k \leqq \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{K}{m_i}\right)^k \leqq \frac{1}{m_i^2} \sum_{k=2}^{\infty} \frac{1}{k!} K^k.$$

Equation (1.12) implies that $m_i\|c_{m_i}\| \to 0$ as $m_i \to \infty$, which in turn implies that $(1 + \|c_{m_i}\|)^{m_i} - 1 \to 0$ as $m_i \to \infty$. It follows from (1.11) that $\|b - \exp_H m_i(b_{1/m_i} - e_H)\| \to 0$, and we can conclude that $b = \exp_H v$ and that $v(1) = 0$.

2. Characterization of Poisson laws

First, consider finite groups. We shall show that in this case all infinitely divisible laws are Poisson laws. Before making this statement more precise note that the group ring of a finite group G and $\mathscr{R}(G)$ are algebraically isomorphic (in a very obvious sense). Therefore, whenever desirable, one may identify the elements of $\mathscr{R}(G)$ with the corresponding elements of the group ring.

THEOREM 2.1 (Böge [1]). *Let G be any finite group of order n with neutral element e. Then $b \in Z$ is an infinitely divisible law if and only if there exists a subgroup H of G and a $v \in \mathscr{K}_H$ such that $b = \exp_H v$.*

PROOF. It is obvious that the measure $\exp_H v$ is infinitely divisible for every subgroup H and every $v \in \mathscr{K}_H$. Assume, henceforth, that b is infinitely divisible. We may suppose that $n \geqq 2$. Our first aim is to show that there exists a sequence of roots c_i for $i \geqq 0$ of b such that

$$(2.1) \qquad \begin{aligned} b &= c_0, \\ c_{i-1} &= c_i^n, \qquad 1 \leqq i < \infty. \end{aligned}$$

Now, Z is compact in the norm topology, so $\Pi_{j=0}^{\infty} Z_j$, where $Z_j = Z$ for $j \geqq 0$, is compact with respect to the product topology. Furthermore, there exists a root b_{1/n^k} of b for every $k \geqq 1$, such that $b_{1/n^k}^{n^k} = b$. The sets

$$(2.2) \qquad \bigcup_{j \geqq k} \{b_{1/n^j}^{n^j}\} \times \{b_{1/n^j}^{n^j-1}\} \times \cdots \times \{b_{1/n^j}\} \times \prod_{i=j}^{\infty} Z_i, \qquad k = 1, 2, \cdots,$$

form a filter base in $\Pi_{0 \leqq j < \infty} Z_j$ which has an accumulation point $(c_0, c_1, \cdots)$ in the latter set. Conditions (2.1) are obviously satisfied. According to Lemma 1.9 the set $\{c_j\}$ generates a submonogeneous semigroup in Z.

Continuing the argument, there exists a subsequence $\{d_i\}$ of $\{c_j\}$ which converges to a limit $c \in Z$. Furthermore, for every nonnegative integer ℓ the relation $c_\ell = d_k^{r_k}$ holds for all sufficiently large k, where $r_k \geqq 2$ is appropriately chosen. Consider, for a fixed $\ell \geqq 0$ the equation

$$(2.3) \qquad c_\ell = d_k d_k^{r_k - 1} = d_k^{r_k - 1} d_k.$$

When $k \to \infty$ (consider, if necessary, a subsequence), one obtains from (2.3) the relation

$$(2.4) \qquad c_\ell = c a_\ell = a_\ell c,$$

where a_ℓ is a certain element of Z. Note that (2.4) implies

$$(2.5) \qquad c_\ell \in cZ \cap Zc$$

for every nonnegative integer ℓ, which in turn implies $d_i \in cZ \cap Zc$ for $i \geqq 0$, and $d_k^{r_k - 1} \in cZ \cap Zc$ for all k. The compactness of $cZ \cap Zc$ therefore entails the relation $a_\ell \in cZ \cap Zc$ for $\ell \geqq 0$. Considering relation (2.4) for the subsequence $\{d_i\}$ only, one obtains

$$(2.6) \qquad c = ca = ac,$$

where

$$(2.7) \qquad a \in cZ \cap Zc.$$

It follows immediately that a is an idempotent. According to Lemma 1.4, there exists a subgroup H of G such that

$$(2.8) \qquad a = e_H,$$

and so c belongs to Z_H. Relations (2.6), (2.7), and (2.8) imply

$$(2.9) \qquad Z_H = cZ \cap Zc.$$

Furthermore, using (2.7) and (2.8), one obtains from Lemma 1.7 that $c = \delta_x e_H$, where x belongs to the normalizer of H. Consider the subsequence $\{c_{j_k}\}$ of $\{c_\ell\}$ whose elements satisfy the condition $c_{j_k} = d_k^n$ for every k. Then d_k^n converges to $(\delta_x e_H)^n = e_H$, and so $\|c_{j_k} - e_H\| \to 0$. This relation together with (2.1), (2.5), and (2.9) allows the application of Lemma 1.10. The sufficiency clause of the theorem is thereby proven.

It is trivial that there cannot be a similar result for the case of an arbitrary locally compact group. If G is a Lie group, an analogue of the Lévy-Khintchine representation formula has been established [7]. Theorem 2.1 is not an immediate consequence of this formula. We shall give a simple characterization of Poisson laws on general, locally compact groups.

THEOREM 2.2 (Hazod and Schmetterer [4]). *Let G be a locally compact group, and let b be an infinitely divisible law belonging to $Z(G)$. Then b is a Poisson law if and only if the following conditions are satisfied. There exists a compact subgroup H of G, and for every $n \geqq 1$ there exists an* nth *root $b_{1/n}$ of b with $H_{b_{1/n}} \supseteqq H$. The sequence $\{b_{1/n}\}$ contains a subsequence $\{b_{1/m_i}\}$ which satisfies conditions* (i) *and* (iii) *of Lemma* 1.10. *There is a sequence $\{k_n\}$ of nonnegative numbers, with $k_n \leqq 1$ as $n \geqq 1$, and another nonnegative sequence $\{w_n\}$ such that*

$$w_n k_n \to \infty, \tag{2.10}$$

$$\|b_{1/n} - e_H\| \leqq 2(1 - k_n), \qquad n \geqq 1, \tag{2.11}$$

and

$$\sup_n w_n \|b_{1/n}\tilde{b}_{1/n} - e_H\| < \infty. \tag{2.12}$$

(If $\mu \in \mathscr{R}$, then $\tilde{\mu}$ denotes, as usual, the measure defined by $f \to \mu(\tilde{f})$, where $\tilde{f}(x) = f(x^{-1})$ for $x \in G$ and $f \in C(G)$.)

PROOF. It is trivial to show that these conditions are necessary; to show that they are also sufficient one has to observe that

$$\|\nu - e_H\| = 2\nu(G \backslash H) \tag{2.13}$$

and

$$\nu\tilde{\nu}(G \backslash H) \geqq \nu(G \backslash H)\nu(H) \tag{2.14}$$

whenever $\nu \in Z_H$. An application of (2.13) and (2.14) yields

$$w_n \|b_{1/n}\tilde{b}_{1/n} - e_H\| \geqq 2w_n b_{1/n}(G \backslash H) b_{1/n}(H) = w_n \|b_{1/n} - e_H\| b_{1/n}(H). \tag{2.15}$$

An application of (2.13) and (2.11) implies the inequality

$$b_{1/n}(H) \geqq k_n. \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$w_n \|b_{1/n}\tilde{b}_{1/n} - e_H\| \geqq w_n k_n \|b_{1/n} - e_H\|. \tag{2.17}$$

Relations (2.10), (2.12), and (2.17) imply $\|b_{1/n} - e_H\| \to 0$ as $n \to \infty$. Now, it is enough to apply Lemma 1.10.

The following result can be proven in a similar manner.

THEOREM 2.3. *Assume b belongs to $Z(G)$ and is infinitely divisible. Suppose there exist a compact subgroup H of G and, for every $n \geqq 1$, an* nth *root $b_{1/n}$ of b with $H_{b_{1/n}} \supseteqq H$. Furthermore, suppose there exists a homomorphism φ from the additive semigroup M_+ of the positive rational numbers into Z with $\varphi(1) = b_1 = b$*

and $\varphi(1/n) = b_{1/n}$ *for* $n \geqq 1$. *Assume, finally, that*

$$\sup_n \|b_{1/n} - e_H\| < 2, \tag{2.18}$$

and that there exists a natural number n_0 *such that*

$$\|b_{1/n_0}\tilde{b}_{1/n_0} - e_H\| < 1. \tag{2.19}$$

Then b is a Poisson law.

These conditions are obviously also necessary.

Let us mention that M_+ is a real submonogeneous semigroup and the conditions concerning φ can be weakened by considering any other submonogeneous subsemigroup of $\mathcal{Z}$ generated by infinitely many roots $b_{1/n}$.

Theorems 2.2 and 2.3 cannot be improved in a certain sense. Even for the case of an abelian compact group it is not possible to weaken condition (2.18) without violating the statement of Theorem 2.3. (See Example 4.1.)

On the other hand, the existence of submonogeneous semigroups generated by the roots of an infinitely divisible law can be replaced in the formulation of Theorem 2.3 by the assumption that G is compact. The following result holds.

THEOREM 2.4 (Carnal [3]). *Let G be a compact group. An infinitely divisible law b, which belongs to* $Z(G)$, *is a Poisson law if and only if there exist for* $n \geqq 1$ *roots* $b_{1/n}$ *of b (with* $b_1 = b$*), which satisfy the following properties. There exists a compact subgroup H of G such that* $H_{b_{1/n}} \supseteqq H$ *for* $n \geqq 1$. *Furthermore,*

$$\liminf_{n \to \infty} \|b_{1/n} - e_H\| < 2. \tag{2.20}$$

There exists a natural number n_0 *such that*

$$\|b_{1/n_0}\tilde{b}_{1/n_0} - e_H\| < 1. \tag{2.21}$$

Before giving a proof, we formulate the following.

DEFINITION 2.1. *Let* μ *be a measure from* $\mathscr{R}(G)$, *and let*

$$M = \begin{pmatrix} m_{1,1} & \cdots & m_{1,r} \\ \vdots & & \vdots \\ m_{r,1} & \cdots & m_{r,r} \end{pmatrix}, \tag{2.22}$$

be a bounded complex representation of the (compact) group G. Then we write

$$M(\mu) = \begin{pmatrix} \int_G m_{1,1}(g)\, d\mu(g) & \cdots & \int_G m_{1,r}(g)\, d\mu(g) \\ \vdots & & \vdots \\ \int_G m_{r,1}(g)\, d\mu(g) & \cdots & \int_G m_{r,r}(g)\, d\mu(g) \end{pmatrix}. \tag{2.23}$$

If $d\mu = f de_G$ *with* $f \in L_1(G)$, *then we write* $M(f)$ *instead of* $M(\mu)$.

Of course, $L_p(G)$ denotes the set of all functions f on G such that $|f|^p$ is integrable with respect to the normed Haar measure of G.

PROOF OF THEOREM 2.4. If μ is an arbitrary, symmetrical measure which belongs to $Z(G)$ (that is, $\mu = \tilde{\mu}$), then there exists a unitary representation M in every class $\mathscr{M}$ of irreducible, bounded representations of G such that $M(\mu)$ is a diagonal matrix. There exists also a unitary $M \in \mathscr{M}$ such that

$$(2.24)\qquad M(e_H) = \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & 1 & & \cdots & & 0 \\ & \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & & \cdots & & 0 \\ & \vdots & & & & \vdots \\ 0 & 0 & & \cdots & & 0 \end{pmatrix}$$

The number of elements of $M(e_H)$ which are different from 0 will be denoted by ℓ_H.

Our first aim is to show that

$$(2.25)\qquad \sup_{n \geqq 1} \sup_{M \in \mathscr{U}_H} n\left(1 - \frac{\operatorname{tr} M(b_{1/n}\tilde{b}_{1/n})}{\ell_H}\right) < \infty,$$

where $\mathscr{U}_H$ is the set of all irreducible unitary representations of G for which $\ell_H > 0$ and where $\operatorname{tr} M(b_{1/n}\tilde{b}_{1/n})$ denotes, of course, the trace of $M(b_{1/n}\tilde{b}_{1/n})$.

To prove (2.25), let us observe that $M(e_H)$ together with $M(b_{1/n_0}b_{1/n_0})$ can be transformed into a diagonal matrix whenever $M \in \mathscr{U}_H$. Yet, it may be necessary to replace M by an equivalent representation. This follows from $b_{1/n_0} \in Z_H$. Denote by $\lambda_{M,n,j}$ for $1 \leqq j \leqq \ell_H$, the eigenvalues of $M(b_{1/n}\tilde{b}_{1/n})$ which are different from 0. Relation (2.21) implies the existence of a nonnegative $\gamma < 1$ which satisfies the condition $0 \leqq 1 - \lambda_{M,n_0,j} \leqq \gamma < 1$. It follows that

$$(2.26)\qquad \lambda_{M,1,j} \geqq (1-\gamma)^{n_0} > 0, \qquad 1 \leqq j \leqq \ell_H.$$

From (2.26), one obtains

$$(2.27)\qquad \left(\frac{\operatorname{tr} M(b_{1/n}\tilde{b}_{1/n})}{\ell_H}\right)^{\ell_H n} \geqq \left(\prod_{j=1}^{\ell_H} \lambda_{M,n,j}\right)^n = \prod_{j=1}^{\ell_H} \lambda_{M,1,j} \geqq (1-\gamma)^{n_0 \ell_H}.$$

This in turn yields

$$(2.28)\qquad n\left[1 - \frac{1}{\ell_H}\operatorname{tr} M(b_{1/n}\tilde{b}_{1/n})\right] \leqq n[1 - (1-\gamma)^{n_0/n}] \to -\log(1-\gamma)^{n_0} \quad \text{as} \quad n \to \infty$$

uniformly for $M \in \mathscr{U}_H$. The inequality (2.25) follows.

Next to be proved is that

$$(2.29)\qquad \sup_{n \geqq 1} n b_{1/n}\tilde{b}_{1/n}(G \backslash H) < \infty.$$

Let VV^{-1} be a neighborhood of the neutral element e of G where V is an open set. There exist $f_V \in C(G)$ which satisfy the following conditions:

(a) $f_V \geqq 0$;

(b) $f_V(G\backslash V) = 0$;

(c) $f_V(g_1g_2) = f_V(g_2g_1), \qquad g_1, g_2 \in G$;

(d) $\int_G f_V de_G = 1$.

Let M be an m dimensional, irreducible representation of G and let E_m be the m dimensional unit matrix. Condition (c) and Schur's lemma imply

$$M(f_V) = c_M E_m, \tag{2.30}$$

where $c_M \neq 0$. For $x, z \in G$, let

$$g_V(x) = \int_G f_V(y) f_V(yx^{-1})\, de_G(y), \tag{2.31}$$

and

$$h_V(z) = \int_G g_V(y^{-1}z)\, de_H(y), \tag{2.32}$$

It may be seen without difficulty that

$$h_V(x) = 0 \qquad \text{when} \quad x \in G\backslash VV^{-1}HVV^{-1} = U. \tag{2.33}$$

Equations (2.30) and (2.31) imply that $M(g_V) = |c_M|^2 E_m$. Taking into account definition (2.32), one obtains

$$M(h_V) = M(e_H)|c_M|^2. \tag{2.34}$$

One can again assume that $M(e_H)$ satisfies equation (2.24). It follows easily from (2.31) and (2.32) that h_V is the convolution of two functions which belong to $L_2(G)$. Therefore, h_V can be expanded in a uniformly convergent series with respect to the elements $m_{i,j}$ of the unitary matrices $M = (m_{i,j})$, $1 \leqq i, j \leqq m$, chosen from every class of irreducible representations. Accordingly,

$$h_V(x) = \sum_M m \sum_{1 \leqq i, j \leqq m} m_{i,j}(x) \int_G \bar{m}_{i,j}(y) h_V(y)\, de_G(y), \qquad x \in G. \tag{2.35}$$

Using (2.34), equation (2.35) leads to

$$h_V(x) = \sum_M m \sum_{i=1}^{\ell_H} |c_M|^2\, m_{i,i}(x). \tag{2.36}$$

Choosing $x = e$, one obtains from (2.36)

$$h_V(e) = \sum_M m\ell_H |c_M|^2 > 0. \tag{2.37}$$

We deduce from (2.25) the existence of a $K > 0$ such that

$$n\left(1 - \frac{\operatorname{tr} M(b_{1/n}\tilde{b}_{1/n})}{\ell_H}\right) \leqq K \tag{2.38}$$

for all $n \geqq 1$ and all $M \in \mathscr{U}_H$. Multiplying (2.38) by $h_V(e)$ and using (2.37), (2.36), and (2.33), we get the inequality

$$Kh_V(e) \geqq nh_V(e)b_{1/n}\tilde{b}_{1/n}(U). \tag{2.39}$$

Inequality (2.39) implies

$$\sup_{n \geqq 1} nb_{1/n}\tilde{b}_{1/n}(G \backslash VV^{-1}HVV^{-1}) \leqq K \tag{2.40}$$

for every neighborhood V of e, and statement (2.29) follows. Condition (2.20) together with (2.29) imply the existence of a sequence $\{n_i\}$ of natural numbers such that

$$\sup_{n_i \geqq 1} n_i \| b_{1/n_i} - e_H \| < \infty. \tag{2.41}$$

By application of Lemma 1.11, the proof is complete.

3. Study of some homomorphisms from M_+ into Z

We turn our attention to the remark which was made at the beginning of the proof of Lemma 1.10. The following problem arises. Let φ be an (algebraic) homomorphism from M_+ into Z. Suppose that there exist at least two different elements $r, s \in M_+$ such that $\varphi(r) = \varphi(s)$. What can be said about $\varphi(1)$? This problem can be generalized somewhat by considering any submonogeneous semigroup contained in Z instead of the homomorphic image of M_+ in Z. But we restrict ourselves to the former, more special problem, and show the following result.

LEMMA 3.1. *Let G be an arbitrary locally compact group and let φ be a homomorphism from M_+ into $Z(G)$. Denote $\varphi(r)$ by b_r whenever $r \in M_+$, and assume that there exist two different elements $r_1, r_2 \in M_+$ such that $b_{r_1} = b_{r_2}$. Then there exists a compact subgroup H of G and an element $x_r \in G$, which belongs to the normalizer of H, such that $S(b_r) \subset x_r H$ for every $r \in M_+$.*

PROOF. The assumptions imply that $\varphi(M_+)$ contains a group. (See [6].) Therefore, according to Lemma 1.4, there exists an $r_0 \in M_+$ and a compact subgroup H of G such that $b_{r_0} = e_H$. Writing $r_0 = p/q$ where p, q are integers, one obtains

$$b_p = b_1^p = e_H. \tag{3.1}$$

Moreover, the relation

$$e_H b_r = b_r e_H. \tag{3.2}$$

holds for every $r \in M_+$. Let us introduce the notation $b_r^* = e_H b_r$. It follows from (3.2) that $\{b_r^*\}$, $r \in M_+$ is a semigroup.

Equation (3.1) implies

$$b_{kp}^* = e_H, \qquad k = 1, 2, \cdots, \tag{3.3}$$

Equation (3.3) together with Lemma 1.6 imply that $H_{b_r^*} \subseteqq H$ for every $r \in M_+$.

On the other hand, the relation $e_H b_r^* = b_r^* e_H = b_r^*$ follows from (3.2), and so $H_{b_r^*} = H$ for every $r \in M_+$. Let $r = t/s$ be any element of M_+, where s and t are integers. Then $b_r^{*sp} = b_{tp}^* = e_H$, and an application of Lemma 1.7 yields $S(b_r^*) = x_r H$, where x_r is some element from G which belongs to the normalizer of H. The statement of the lemma follows immediately.

THEOREM 3.1 (Schmetterer [11]). *Let G be an abelian, locally compact group. Then the assumptions of Lemma 3.1 imply $b_1 = \delta_{x_1} e_H$, where x_1 belongs to the normalizer of H. A similar statement holds for every $r \in M_+$.*

PROOF. Taking into account equation (3.1), it is enough to show that $H_{b_1} \supseteqq H$. If $H = \{e\}$, then no proof is needed. If $H \neq \{e\}$, the relation $[(\delta_e - e_H)b_1]^p = 0$ follows. The algebra $\mathscr{R}(G)$ does not contain nilpotent elements (see [9]) and so it follows that $b_1 = e_H b_1$.

This proof cannot be applied to the case of a nonabelian group. Yet, Theorem 3.1 remains true for a very large class of nonabelian groups (as has been shown very recently by W. Hazod). It is not true for arbitrary nonabelian groups. We shall treat a special case only.

LEMMA 3.2. *Suppose that the assumptions of Lemma 3.1 are satisfied. Furthermore, assume that*

$$e \in S(b_r), \qquad r \in M_+. \tag{3.4}$$

Then the conclusion of Theorem 3.1 holds.

PROOF. It is enough to consider the case where $H \neq \{e\}$. Let $\lambda_r = (\delta_e - e_H)b_r$ for $r \in M_+$. Equation (3.2) implies that $\{\lambda_r\}$, $r \in M_+$ is a semigroup. Using the notation introduced in the proof of Lemma 3.1, we get

$$\lambda_{r_0} = 0. \tag{3.5}$$

It follows from Lemma 3.1 and equation (3.4) that $S(b_r) \subseteqq H$, and so $S(\lambda_r) \subseteqq H$. Let ψ be an irreducible bounded representation of H of dimension m. Equation (3.5) implies $\lambda_{r_0/n}^n = 0$ for every $n \geqq 1$, and this in turn implies $[\psi(\lambda_{r_0/n})_H]^n = 0$, where $(\lambda_r)_H$ is the restriction of λ_r to the Borel sets of H. An application of the Hamilton-Cayley theorem yields $[\psi(\lambda_{r_0/n})_H]^m = 0$, $m \geqq 1$, for every irreducible representation ψ of dimension m. Taking into account that $\lambda_{r_0/n} = \lambda_{r_0/mn}^m$, one obtains $\psi(\lambda_{r_0/n})_H = 0$ for every $n \geqq 1$ and every irreducible representation ψ of H, and so $\lambda_{r_0/n} = 0$. This implies that $\lambda_r = 0$ for every $r \in M_+$.

Another conclusion may be drawn from Lemma 3.1 and the following may be shown.

COROLLARY 3.1. *Let G be an arbitrary locally compact group and let $\exp_H v$ with $v \in \mathscr{K}_H$ be an arbitrary Poisson law. Then the mapping $r \twoheadrightarrow \exp_H \{rv\}$ for $r \in M_+$ is either an isomorphism or a (trivial) homomorphism from M_+ onto e_H.*

PROOF. Let $\varphi: r \twoheadrightarrow \exp_H rv$ be a homomorphism, but not an isomorphism. Then there exists an $r_0 \in M_+$ with $\exp_H r_0 v = e_H$. Lemma 1.8 implies that $H_{\exp_H rv} = H$ for every $r \in M_+$. Therefore, the statement of Theorem 3.1 holds when b_r is replaced by $\exp_H rv$. Clearly, the inequality $\|\exp_H rv - e_H\| < 2$ holds for every $r \in M_+$, and therefore, $\exp_H rv = e_H$ for every $r \in M_+$.

The problems treated here are obviously related to the important question of whether an infinitely divisible law b can always be imbedded in a homomorphism φ from M_+ into Z such that $\varphi(1) = b$. This question admits an affirmative answer if some compactness conditions are satisfied, but in general the answer is negative.

THEOREM 3.2. *Let G be an arbitrary locally compact group, and let b be an infinitely divisible law which belongs to $Z(G)$. For every $n \geqq 1$ let $W_{1/n}(b)$ be the set of all nth roots of b. Suppose that the elements of $W_{1/n}(b)$ are uniformly tight. Then b can be imbedded in a semigroup of probability measures over M_+.*

PROOF. Clearly, $W_{1/n}(b) \neq \emptyset$ for every $n \geqq 1$. But for every integer $m > 0$ the elements of $W_{m/n}(b) = \{a^m : a \in W_{1/n}(b)\}$ are uniformly tight and therefore relatively compact in the weak topology. The convolution is (weakly) continuous on the closure of $W_{m/n}(b)$, and so $W_{m/n}(b)$ is compact. Let $K_n = \{m/n!, m = 1, 2, \cdots\}$ for $n \geqq 1$. Obviously, $K_n \subseteqq K_{n+1}$ and $\cup_{i=1}^{\infty} K_i = M_+$. The Cartesian product $W = \Pi W_{m/n!}(b)$, where $(m/n!) \in K_n$ and $n \geqq 1$, is compact in the product topology. There exists for every $k \geqq 1$ a $b_{1/k} \in Z(G)$ with $b_{1/k}^k = b$. Therefore, $a_{n!,m} = b_{1/n!}^m \in W_{m/n!}(b)$ and

$$a_{n!,n!} = b. \tag{3.6}$$

Furthermore,

$$a_{n!,m_1}a_{n!,m_2} = a_{n!,m_1+m_2}, \qquad n \geqq 1, m_1, m_2 = 1, 2, \cdots, \tag{3.7}$$

Let k be a natural number and let A_k be the set of all those elements $\{c_{m/n!}; (m/n!) \in K_n, n \geqq 1\} \in W$ which satisfy the condition $c_{m/\ell!} = a_{k!,m(\ell+1)\cdots k}$ for $m \geqq 1$ and $1 \leqq \ell \leqq k$. Define $E_n = \cup_{k \geqq n} A_k$ for $n \geqq 1$. The set of all E_n defines a filter base in W and has an accumulation point $\{d_{m/n!}\} \in W$ such that $d_{m_1/\ell_1!} = d_{m_2/\ell_2!}$ whenever $m_1/\ell_1! = m_2/\ell_2!$. Let r be an arbitrary element of M_+ and consider the (well-defined) mapping φ given by $r \twoheadrightarrow d_r$. Taking into account conditions (3.6) and (3.7), it is easy to see that the mapping $r \twoheadrightarrow d_r$ is a homomorphism with $\varphi(1) = b$.

The statement of Theorem 3.2 holds in two important special cases: (a) if b has only finitely many nth roots for every $n \geqq 1$, and (b) if G is compact. But the conditions of Theorem 3.2 are also satisfied for a much broader class of groups. Indeed, the following result holds.

LEMMA 3.3 (Böge [2]). *Let G be a locally compact group which is the union of all its compact and open invariant subgroups N. Suppose that for every N the factor group G/N contains only finitely many elements of every (finite) order (and no other elements). Then the assumptions of Theorem 3.2 are satisfied.*

PROOF. First we show that, for every $n \geqq 1$ and every compact subset C of G, there exists a compact set $C_n \subseteqq G$ with the following property. If $x_1, \cdots, x_{n-1}, x_n = e$ is a sequence of elements satisfying

$$Cx_iCx_k \cap Cx_{i+k} \neq \emptyset \tag{3.8}$$

whenever $i + k \leqq n$, then this sequence belongs to C_n. Let N_0 be a compact and open invariant subgroup (cois) of G, and let C be an arbitrary compact subset of G. There exist finitely many $y_j \in G$, say $y_1, \cdots, y_r$, such that $C \subseteqq \bigcup_{j=1}^{r} N_0 y_j$. Moreover, for every j there exists a cois N_j with $y_j \in N_j$, and so $C \subseteqq N_0 N_1 \cdots N_r$. The set $N = N_0 N_1 \cdots N_r$ is a cois. It follows that C_n can be chosen as the union of all elements of order n (or of a fixed order $\geqq n$) of the group G/N. Now it remains to show that, for every $m \geqq 1$, $W_{1/m}(b)$ is a uniformly tight set of measures. For every ε such that

$$0 < \varepsilon \leqq \tfrac{1}{3} \tag{3.9}$$

there exists a compact subset C of G such that

$$b(C) \geqq 1 - \varepsilon. \tag{3.10}$$

Let c be any measure from $W_{1/m}(b)$. Then (3.10) implies

$$\int_G \int_G I_C(yx)\, dc^k(y)\, dc^{m-k}(x) \geqq 1 - \varepsilon \tag{3.11}$$

for every natural number $k < m$, where I_C is the indicator function of the set C. It follows from (3.11) that there exists an $x_k \in G$ with

$$c^k(Cx_k) \geqq 1 - \varepsilon. \tag{3.12}$$

Define $x_m = e$. Then, whenever $k + \ell \leqq m$, one obtains from (3.12)

$$c^{k+\ell}(Cx_k Cx_\ell) \geqq (1 - \varepsilon)^2. \tag{3.13}$$

It is easy to see that

$$c^{k+\ell}(Cx_k Cx_\ell \cap Cx_{k+\ell}) \geqq c^{k+\ell}(Cx_k Cx_\ell) - c^{k+\ell}[(G\backslash C)x_{k+\ell}]. \tag{3.14}$$

It also follows from (3.12) that if $k + \ell < m$, or from (3.10) if $k + \ell = m$, that

$$c^{k+\ell}[(G\backslash C)x_{k+\ell}] \leqq \varepsilon. \tag{3.15}$$

Inequalities (3.13), (3.14), (3.15), and (3.9) imply

$$c^{k+\ell}(Cx_k Cx_\ell \cap Cx_{k+\ell}) > 0. \tag{3.16}$$

We deduce immediately from (3.16) that $Cx_k Cx_\ell \cap Cx_{k+\ell} \neq \emptyset$ for $k + \ell \leqq m$. Therefore, there exists a compact subset C_m of G which contains all the elements of the sequence $x_1, \cdots, x_m$, and so choosing $k = 1$, equation (3.12) implies $c(CC_m) \geqq 1 - \varepsilon$. The compact set CC_m does not depend on c and so the lemma is proven.

4. Examples

In this section examples are collected which illustrate some results of the previous section and also point out some new features.

EXAMPLE 4.1. Let G be the two dimensional torus and let $\{\alpha_n\}$ be a sequence of positive real numbers such that $\Sigma_{n=1}^{\infty} \alpha_n$ converges. Let $x_n = \exp\{2\pi i/n\}$ for

$n \geqq 1$. It is obvious that $v = \Sigma_{n=1}^{\infty} \alpha_n \delta_{x_n} - (\Sigma_{n=1}^{\infty} \alpha_n)\delta_{x_1}$ belongs to $\mathscr{K}_{\{x_1\}}$. It follows that $\mu_t = \exp_{\{x_1\}} \{tv\}$ is a Poisson law for every real $t > 0$. Let $y_t = \exp\{-2\pi i t\}$ for $t > 0$, and $w_t = \delta_{y_t}\mu_t$. It is easy to see that w_t is not a Poisson process. But $t \to w_t$ is a homomorphism from R_+ into Z. Moreover, one finds by an easy calculation that $\|w_{1/n} - \delta_{x_t}\| < 2$ when $n \geqq 1$. Furthermore, the fact that $w_t\tilde{w}_t$ is a Poisson process implies the existence of a natural number n_0 with $\|w_{1/n_0}\tilde{w}_{1/n_0} - \delta_{x_1}\| < 1$; that is, w_t satisfies condition (2.19).

Another example is concerned with the problem that the representation of a Poisson law b is not necessarily unique. More precisely, it is possible to find a Poisson law b such that $b = \exp_H v_1$ and $b = \exp_H v_2$ with $v_i \in \mathscr{K}_H$ for $i = 1, 2$ and $v_1 \neq v_2$. The construction of such a Poisson law which admits (at least) two different representations can be carried out with the help of the following argument. Let G, for simplicity, be an abelian group. Suppose that there exists a $u \in \mathscr{R}_H(G)$ with $u \neq 0$ such that

$$\exp_H u = e_H. \tag{4.1}$$

Let $u = u_1 - u_2$, where the $u_i \in \mathscr{R}$ are nonnegative measures for $i = 1, 2$ with

$$u_1 \neq u_2. \tag{4.2}$$

It follows that $\exp_H \{u_1 - u_2\} = \exp\{e_H(u_1 - u_2)\} = e_H$, and so

$$\exp_H \{e_H u_1\} = \exp_H \{e_H u_2\}. \tag{4.3}$$

Write $e_H u_i = w_i$ for $i = 1, 2$. Relation (4.2) implies

$$w_1 \neq w_2. \tag{4.4}$$

Moreover, the measures w_i are nonnegative, belong to $\mathscr{R}_H$, and satisfy

$$w_1 - w_2 = u. \tag{4.5}$$

Relation (4.5) implies that $(w_1 - w_2)(1) = u(1) = 0$, and so

$$\|w_1\| = \|w_2\|. \tag{4.6}$$

Obviously, $v_i = w_i - \|w_i\| e_H$ belongs to $\mathscr{K}_H$ for $i = 1, 2$, and it follows from (4.3) and (4.6) that $\exp_H v_1 = \exp_H v_2$. Moreover, (4.4) and (4.6) imply that $v_1 \neq v_2$.

It is easy, even for finite groups, to find examples which realize condition (4.1).

EXAMPLE 4.2 (Böge [1]). Let G be a finite group of order n which contains at least one element g_0 of order > 2. Denote by G' the character group of G. For every $a \in \mathscr{R}$ of the form $\Sigma_{g \in G} a_g g$ (see the remark at the beginning of Section 2) define $\chi(a)$ by $\Sigma_{g \in G} a_g \chi(g)$ for every $\chi \in G'$. It follows that $a_g = (1/n) \Sigma_{\chi \in G'} \chi^{-1}(g)\chi(a)$ and a_g is real if and only if

$$\bar{\chi}(a) = \chi^{-1}(a) \tag{4.7}$$

for every $\chi \in G'$.

Let e be the neutral element of the group G. There exists a $u \in \mathscr{R}$ satisfying

$$\exp_{\{e\}} u = \delta_e, \tag{4.8}$$

if and only if $\exp\{\chi(u)\} = 1$ for every $\chi \in G'$, where exp denotes the usual exponential function. To show that (4.8) has a real solution, $u \neq 0$, take into account that $g_0^2 \neq e$ and find a character χ_0 such that

$$\chi_0(g_0) \neq \chi_0^{-1}(g_0). \tag{4.9}$$

Define $u = \Sigma_{g \in G} u_g g$, where $u_g = (1/n)[\chi_0(g)2\pi i - \chi_0^{-1}(g)2\pi i]$ for every $g \in G$. Equation (4.7) is satisfied for every $\chi \in G'$. Moreover, it follows from (4.9) that $u_{g_0} \neq 0$ and so $u \neq 0$, but u satisfies equation (4.8).

The following example shows that the product of two Poisson laws need not be infinitely divisible.

EXAMPLE 4.3 (Böge [1]). Let G be a finite group with neutral element e, and let x, y be two elements of G such that

$$xy \neq yx. \tag{4.10}$$

Let

$$u = \delta_x - \delta_e \tag{4.11}$$

and

$$v = \delta_y - \delta_e. \tag{4.12}$$

It is trivial to show that $u, v \in \mathscr{K}_{\{e\}}$. On the other hand,

$$uv - vu = \delta_x \delta_y - \delta_y \delta_x \notin \mathscr{K}_{\{e\}}. \tag{4.13}$$

Take $\varepsilon > 0$ and define $a = \exp_{\{e\}}\{\varepsilon u\}$ and $b = \exp_{\{e\}}\{\varepsilon v\}$. Clearly, a and b are Poisson laws. If ε is sufficiently small, then $ab = \exp_{\{e\}}\{\log_{\{e\}} ab\}$, according to Lemma 1.1, and moreover, $\|ab - \delta_e\| < 1$. But $\log_{\{e\}} ab = \varepsilon(u + v) + \frac{1}{2}\varepsilon^2[uv - vu + O(\varepsilon)]$ and it follows from (4.11), (4.12), and (4.13) that $\log_{\{e\}}(ab) \notin \mathscr{K}_{\{e\}}$ if $\varepsilon > 0$ is sufficiently small. Therefore ab is not a Poisson law, and thus, not infinitely divisible, according to Theorem 2.1.

Another example shows that the well-known classical theorem of Raikov for R_1 does not necessarily hold for arbitrary (locally compact) groups.

EXAMPLE 4.4. Let q be a prime number > 2 and let $M_q = \{0, 1, \cdots, q - 1\}$ be the group of residues mod q. Let γ be a positive real number and let $x \in M_q$ with $x \neq 0$. Then we can show that the Poisson law $b = \exp_{\{0\}}\{\gamma(\delta_x - \delta_0)\}$ can contain factors which are not of this form. Define $\nu_1 = t\delta_0 + (1 - t)e_{M_q}$, for $0 < t < 1$, and $\nu_2 = t^{-1}b + (1 - t^{-1})e_{M_q}$. Clearly, $\nu_1 \in Z$ and $\nu_2 \in \mathscr{R}$. Moreover,

$$\nu_1 \nu_2 = b. \tag{4.14}$$

We want to show that $\nu_2 \in Z$ can be proven, if t is appropriately chosen. For this purpose we observe that $b \neq e_{M_q}$ and so $\min_{k \in M_q} b(k) < 1/q$. Choose t in the interval

$$(4.15) \qquad 1 > t \geqq 1 - q \min_{k \in M_q} b(k).$$

It follows from (4.15) that for every $\ell \in M_q$

$$(4.16) \qquad \begin{aligned} \nu_2(\ell) &\geqq t^{-1} \min_{k \in M_q} b(k) + (1 - t^{-1}) q^{-1} \\ &\geqq \frac{1}{1 - q \min_{k \in M_q} b(k)} (q \min_{k \in M_q} b(k) - 1) q^{-1} + q^{-1} \geqq 0. \end{aligned}$$

Equation (4.16) together with $\nu_2(M_q) = 1$ implies that $\nu_2 \in Z$ if (4.15) is fulfilled. It is easy to see that ν_1 is infinitely divisible, and so it is a Poisson law (see Theorem 2.1). But the Fourier transform of ν_1 is identically equal to t, while the Fourier transform of b is given by $j \rightsquigarrow \exp\{\gamma(\exp\{2\pi i x_j/q - 1\})\}$, for $0 < j \leqq q - 1$.

Note that Raikov's theorem holds for infinitely cyclic groups [10].

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